# INDEFINITE METRIC SPACES IN ESTIMATION, CONTROL AND ADAPTIVE FILTERING

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# Abstract

The goal of this thesis is two-fold: first, to present a unified mathematical framework (based upon optimization in indefinite metric spaces) for a wide range of problems in estimation and control, and second, to motivate and introduce the problem of robust estimation and control, and to study its implications to the area of adaptive signal processing.

Robust estimation (and control) is concerned with the design of estimators (and controllers) that have acceptable performance in the face of model uncertainties and lack of statistical information, and can be considered an outgrowth and extension of (the now classical) LQG theory, developed in the 1950's and 1960's, which assumed perfect models and complete statistical knowledge. It has particular significance in adaptive signal processing where one needs to cope with time-variations of system parameters and to compensate for lack of a priori knowledge of the statistics of the input data and disturbances. One method of addressing the above problem is the so-called $H^\infty$ approach, which was introduced by G. Zames in 1980 and that has been recently solved by various authors.

Despite the "fundamental differences" between the philosophies of the $H^\infty$ and LQG approaches to control and estimation, there are striking "formal similarities" between the controllers and estimators obtained from these two methodologies. In an attempt to explain these similarities, we shall describe a new approach to $H^\infty$ estimation (and control), different from the existing (e.g., interpolation-theoretic-based, game-theoretic-based, etc.) approaches, that is based upon setting up estimation (and control) problems, not in the usual Hilbert space of random variables, but in an indefinite (so-called Krein) space.

The Krein space formulation provides a unified approach for problems in LQG, $H^\infty$, risk-sensitive, and game-theoretic, estimation and control, and, most importantly, allows one to use the insight obtained from over three decades of work in traditional LQG theory to obtain new results in these other areas. Proceeding in this spirit, we demonstrate how to generalize the (possibly) numerically superior square-root algorithms and the fast Chandrasekhar algorithms to the $H^\infty$ setting, and embark on some new investigations on the asymptotic behaviour of $H^\infty$ filters and controllers, and on the existence and properties of solutions of Riccati equations with (possibly) indefinite coefficient matrices.

We also study adaptive filtering using the $H^\infty$ approach to robust estimation and show that the celebrated LMS (least-mean squares) adaptive algorithm is $H^\infty$-optimal. This result solves the long standing issue of finding a rigorous basis for LMS (which was long thought to be an approximate least-squares solution). It also suggests some further ramifications, such as the design of robust adaptive filters with more desirable tracking properties, as well as some directions for further research, such as the mixed $H^2/H^\infty$ problem.

# Acknowledgements

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I have had the great fortune of being reunited at Stanford during the past two years, to my brother Arjang, and to the memory of my late grandfather, Kazem Hassibi, who has been the greatest inspiration in education for us all. My deep gratitude also goes to my uncles, Drs. Hossein Naraghipour, Houshang Hassibi and Khosrow Hassibi, who have helped to make possible my education in the US. And for my dear wife, Faranak, whose love, care and patience have inspired and supported me over the years, both when we were on opposite sides of the globe, and now that we here are together, my gratitude is beyond words.

Finally, I should mention that, since childhood, I was always impressed by the dedication on the first page of my father's 1975 PhD thesis, and so am honored to follow in his footsteps, and to dedicate this thesis to "those who support education at all levels for all".

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# Chapter 1

# Introduction

Our primary interest in this thesis (and through the study of which we will later consider related problems in control and adaptive filtering) is estimation theory. Broadly speaking, in estimation theory one is confronted with the following problem: given the values of an observable signal (often called the measurement signal) one would like to estimate (or to predict) the values of another (so-called desired) signal that is not directly observable. Examples of where such problems may arise are numerous and are not difficult to conceive: in weather forecasting one has access to satellite measurements of current and past atmospheric pressure and humidity and would like to predict the future values of these quantities; in communications one typically observes (at the receiver) the output of a communication channel and would like to estimate the values of the bit stream sent (by the transmitter); and so on. In many applications, however, the estimation problem occurs in a more indirect fashion. One such application is in control theory where one observes the output of a dynamical system (the so-called plant) and where the goal is to influence the dynamics of the plant, through a control signal, so that the plant yields some desired behaviour. It turns out that it is often convenient to solve the control problem via a two-step procedure: one first uses the observed output of the plant to estimate the value of certain unobservable signals (that are internal to the plant), and then uses these estimates to construct the required control signals.

In view of the above, it is quite obvious that the solution to the problem of estimating an unobservable signal given an observable one depends on the relationship between the two signals (*i.e.*, on the model describing them) and on the criterion (*i.e.*, on the optimality principle) that one uses to determine the desired estimates. Of course, the above two issues are interrelated: one would like to choose a criterion that is compatible with the model, and vice-versa. However, what influence the choice of model and criterion most significantly are: the underlying problem that we are actually trying to solve (be it weather forecasting, control, or adaptive filtering), and the possibility of actually obtaining a solution to the formulated problem that is easily implementable. In other words, physical significance and mathematical tractability.

The subject of estimation theory, as covered by all possible choices of signal models and optimality criteria, is indeed a vast (and developing) one and is well beyond what can be studied in this thesis (or any other, for that matter). Therefore here (with some minor exceptions) our attention will be devoted to linear models (*i.e.*, cases where the measurement and desired signals are linearly related) and to quadratic (or quadratically-induced) deterministic and stochastic criteria. The linear models that we shall mostly be concerned with are those that are more relevant to system theory, *i.e.*, finite-dimensional linear state-space models and rational transfer matrices. The quadratic criteria that we shall consider include the (now classical) deterministic least-squares and stochastic least-mean-squares criteria, as well as the (more recent) ones in $H^\infty$ theory, dynamic game theory and risk-sensitive estimation and control.

Indeed the major contribution of this thesis is that these apparently different estimation and control problems (with different deterministic and stochastic criteria) can be solved in a unified geometric framework using the concept of an indefinite inner-product (or indefinite metric) space. These so-called Krein spaces are extensions of Hilbert spaces where the self inner-product of any vector can be positive, negative or zero. The key observation is that many estimation problems can be reduced to a projection in a Krein space. Although Hilbert spaces and Krein spaces share many characteristics, they differ in special ways that turn out to mark the differences between the standard least-mean-squares (LQG or $H^2$) theories and the more recent $H^\infty$ and game theories.

Apart from rather more transparent derivations of existing results, the major bonus of this unified approach is that it allows for many new results to be obtained by trying to extend to the $H^\infty$, game-theoretic and risk-sensitive settings some of the huge body of results and insights developed over the last three decades in the fields of Kalman filtering and LQG control. In this thesis, this claim is backed by showing how to generalize the (possibly) numerically superior square-root algorithms and the (so-called) fast Chandrasekhar algorithms to these new settings, by performing new investigations on the asymptotic behaviour of $H^\infty$ filters and controllers, and on the existence and properties of solutions to Riccati equations with (possibly) indefinite coefficient matrices. Moreover, this framework will be used to study the implications of robust estimation to the vast and highly active field of adaptive signal processing.

In this first introductory chapter we shall outline the major estimation and control problems to be studied and shall overview the scope and contributions of this thesis. The material of this chapter will also serve as a motivation for the study of indefinite metric spaces that will begin in Chapter 2. However, before doing so, it will be useful to present some very brief historical remarks.

## 1.1 Some Historical Remarks

The problem of interpreting observations and making estimates and predictions dates back to antiquity. Neugebauer [Neu57] has noted that the ancient Babylonians used a rudimentary form of the Fourier transform for such purposes. The beginnings of a theory of estimation, in the sense that one attempts to minimize a certain function of the errors, is apparently attributed to Galileo Galilei in 1632 [Gal32], after which one encounters a series of illustrious investigators including Roger Cotes, Euler, Lagrange, Laplace, Bernoulli and others.

The method of least-squares (for solving over-determined systems of linear equations), that chooses estimates that best match the observations in a least-squares sense, was apparently first used by Gauss in 1795 [Gau09], although first published by Legendre in 1805 [Leg10] and, independently, by Adrain in 1808 [Adr08]. Since then a vast literature has been developed both on deterministic least-squares problems (see, *e.g.*, any standard textbook on linear algebra and matrix analysis such as

[GL89], [HJ90] and [Str93]) and on least-squares estimation for random variables (see [Har72] for a comprehensive annotated bibliography on this subject).

The problem of least-mean-squares estimation of stochastic processes was first investigated by Kolmogorov [Kol39], [Kol41], Krein [Kre45a], [Kre45b] and Wiener [Wie49]. Although Kolmogorov's approach was more fundamental, the work of Wiener, especially the Wiener filter for the prediction of stationary stochastic processes, has turned out to be more influential. The most important contribution of the work of Kolmogorov and Wiener has been the introduction of statistical ideas to problems in estimation and control. In this framework the underlying signals, (*i.e.*, the measurement and desired signals) are assumed to be stochastic processes with known statistical properties (in particular, they are taken to be stationary stochastic processes with known first and second order statistics). The criterion for finding the desired estimate is then the least-mean-squares criterion, *i.e.*, the resulting signal estimates yield the smallest average squared estimation errors.

As noted above, the assumption that the underlying measurement and desired signal processes are stationary is crucial to the Wiener and Kolmogorov theory and it was not until the late 1950's and early 1960's that a satisfactory theory was developed, primarily by Kalman, that could treat the nonstationary case [Kal60b], [KB61] and [Kal63b]. The theory arose because of the inadequacy of the Wiener-Kolmogorov theory for coping with certain applications in which nonstationarity of the measurement and desired signals was intrinsic to the problem. The new theory soon acquired the name *Kalman filter* (or Wiener-Kalman filter) *theory*, and since then a vast literature on the topic has been developed. (An excellent survey of the developments up until the mid 1970's is given in [Kai74].)

Concurrent with the development of Kalman filter theory a closely related theory of optimal control was being developed in the United States and the Soviet Union [Kal60a], [KB60], [Kal64], [Pon61], [Yak62], [Pop64] and [Won68b]. As in the Kalman filter theory, the underlying assumptions of this theory were that the plant has a known linear (and possibly time-varying) description, and that the exogenous signals (the noises and disturbances) impinging on the feedback system are stochastic in nature, but have known statistical properties. These assumptions turned out to be very well suited to the problems of guidance and control of space vehicles to which the theory was first applied. This theory is now known as linear-quadratic-Gaussian (LQG) control to reflect the fact that the model and optimal controller are linear, that the cost function is quadratic, and that the disturbances are assumed to be stochastic processes with jointly Gaussian distribution.

As described above, classical methods in estimation theory (such as least-mean-squares, Wiener-Kalman, maximum-likelihood and maximum entropy) assume perfect models and regard the underlying signals as stochastic processes with known statistical properties. In many applications, however, one is faced with modeling errors and lack of statistical information. Therefore the aforementioned methods are not directly applicable since the statistics and distributions of the stochastic processes are not known. Moreover, it is not obvious what the behavior of such estimation schemes will be once the assumptions on the statistics and distributions are not met. This has led researches to consider *robust estimation theory* where the objective is to design estimators that have acceptable performance in the face of such deficiencies.

One approach that has been developed to address the above problem is (so-called) $H^\infty$ estimation theory which has followed some pioneering work by Zames [Zam81] in robust control theory. (Some recent papers on $H^\infty$ estimation include [KN91], [Bas91], [ST92] and [Gri93]). Robust control theory itself grew out of the need for designing controllers that were insensitive to plant modeling errors and to lack of statistical information on the exogenous signals. (In the late 1970's it was observed that LQG controllers could be highly nonrobust with respect to such modeling errors.) The $H^\infty$ approach to robust control was extensively studied in the 1980's and has since been solved by numerous authors using various interpolation-theoretic and game-theoretic techniques [ZF83], [FZ84], [BC87], [Kim87], [DGKF89], [Tad90] and [GGLD90].

The main idea in $H^\infty$ estimation is to come up with estimators that minimize (or in the suboptimal case, bound) the maximum energy gain from the disturbances to the estimation errors. This will guarantee that if the disturbances are small (in energy) then (no matter what the disturbances are) the estimation errors will be as small as possible (in energy). The robustness of $H^\infty$ estimators, with respect to disturbance variation, follows from the fact that they safeguard against the estimators *worst-case* performance and make no assumptions on the statistics or distributions of the disturbance signals. Of course, since they make no such assumptions about the disturbances, they have to accommodate for all conceivable disturbances, and thus may be over-conservative.

Despite their fundamentally different objectives, the controllers and estimators obtained in $H^\infty$ theory bear a striking resemblance to those obtained in LQG and Kalman filter theory. Nevertheless, there are enough significant differences that various ingenious methods have been devised to solve these $H^\infty$ problems. Starting with the next chapter in this thesis, we will show that such very different solution methods need not be necessary; the basic LQG and Kalman filtering arguments can still be used, provided we set up appropriate control and estimation problems with elements not in a Hilbert space, but in an indefinite metric (so-called) Krein space. This observation has several different ramifications which we will also explore.

Finally, we should also mention some related developments that are somewhat to the periphery of what was explained above. Motivated primarily by econometric considerations, a game-theoretic approach to control and estimation was developed in the 1960's and 1970's [Isa65], [BO82] in which the disturbance signals are treated as an adversary player in a non-cooperative game. Also, theories for linear-exponential-quadratic-Gaussian (LEQG) and risk-sensitive optimal control and estimation have been developed in the 1970's and 1980's [Jac73], [SDJ74], [Whi81] that essentially replace the quadratic cost of LQG control with an exponential one. Both these theories turn out to be intimately related to $H^\infty$ control and estimation, as has been noted in [GD88], [GM89], [Bas89] and [LAKG92], and as will be seen later in this thesis.

Estimation theory has, of course, much overlap with the fields of adaptive filtering, adaptive signal processing and adaptive neural networks [WS85], [Hay96], [RM86] and [Hay94]. However, even a brief survey of the developments in these related fields will take us too far from our current objectives. Therefore we shall defer an introduction to these areas until we treat them in Chapters 9 and 11.

## 1.2 A Basic Estimation Problem

A general discrete-time estimation problem is shown in Fig. 1.1.[1] Almost all estimation problems (such as Wiener, Kalman and adaptive filtering) can be cast into this framework. Here we assume that $\mathcal{H}$ and $\mathcal{L}$ are known *causal* linear transfer operators (or causal linear systems) that map the input sequence $\{u_j\}$ to their respective outputs. Although we shall not be specific about $\mathcal{H}$ and $\mathcal{L}$ here, we mention that in the finite horizon case $\mathcal{H}$ and $\mathcal{L}$ can be represented by finite block *lower triangular* matrices, and that in the infinite horizon case they are infinite (or semi-infinite) block lower triangular matrices. Another important instance is the infinite horizon case when $\mathcal{H}$ and $\mathcal{L}$ are time-invariant transfer operators, in which case we can represent them by transfer matrices (or transfer functions, in the scalar case) in the $z$-domain, namely $H(z)$ and $L(z)$. The model considered below is general and applies to all of the above cases.

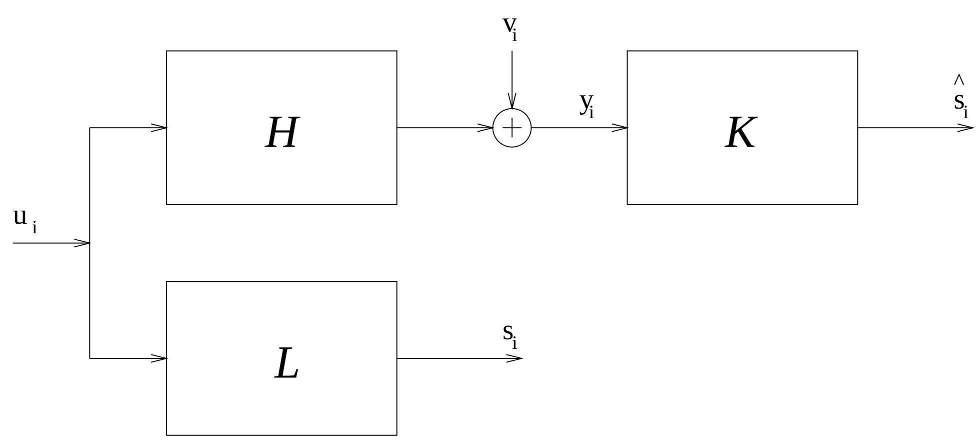


Figure 1.1: A general estimation problem.

In what follows we shall denote sequences such as $\{u_j\}$ by $u$, and simply write

$$s = \mathcal{L}u, \tag{1.2.1}$$

[1] In this thesis we shall, for the most part, be concerned with discrete-time estimation and control problems. Continuous-time counterparts of all the results presented are possible, and in most cases quite straightforward.

to denote that $\mathcal{L}$ maps the input sequence $\{u_j\}$ to the output sequence $\{s_j\}$.

The sequences $\{u_j\}$ and $\{v_j\}$ are assumed to be *unknown*.[2] [$\{u_j\}$ may be considered as a driving disturbance and $\{v_j\}$ as a measurement disturbance. In general, both may include modeling errors resulting from our lack of knowledge of the "true" $\mathcal{H}$ and $\mathcal{L}$.] The goal is to design a *causal* transfer operator (or filter) $\mathcal{K}$ that estimates $s_i$, the *unobservable* output of $\mathcal{L}$, using the *observations* $\{y_j, j \leq i\}$ (which can be regarded as corrupted measurements of the output of $\mathcal{H}$). The estimates we shall denote by $\hat{s}_{i|i}$ and the estimation errors by $\tilde{s}_{i|i} \triangleq s_i - \hat{s}_{i|i}$.

At this point let us note that (roughly speaking) the behavior of any estimator $\mathcal{K}$ can be captured by, $\mathcal{T}_{\mathcal{K}}$, the induced transfer operator that maps the unknown disturbances $\{u_j\}$ and $\{v_j\}$ to the estimation errors $\{\tilde{s}_{j|j}\}$. Thus

$$\mathcal{T}_{\mathcal{K}} : \begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \tilde{s}. \tag{1.2.2}$$

Now using Fig. 1.1, we may write

$$\tilde{s} = s - \hat{s} = (\mathcal{L} - \mathcal{K}\mathcal{H})u - \mathcal{K}v = \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

from which we infer that

$$\mathcal{T}_{\mathcal{K}} = \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix}. \tag{1.2.3}$$

### 1.2.1 Special Cases

We now consider some special cases of the above general formulation.

#### Transfer Matrices

In the infinite-horizon case, when $\mathcal{H}$ and $\mathcal{L}$ are linear time-invariant transfer operators, they can be represented by transfer matrices, $H(z)$ and $L(z)$, of dimensions $p \times m$ and $q \times m$, respectively (assuming that the $u_i$, $y_i$ and $s_i$ are $m$-vectors, $p$-vectors and

[2] For the time being, we are purposefully ambiguous as to whether the $\{u_j\}$ and $\{v_j\}$ are deterministic or stochastic.

$q$-vectors, respectively). In this case we can write

$$\left\{ \begin{array}{rcl} y(z) & = & H(z)u(z) + v(z) \\ s(z) & = & L(z)u(z) \end{array} \right. . \tag{1.2.4}$$

Assuming that the estimator $\mathcal{K}$ has a transfer matrix representation, $K(z)$, (consequently of dimension $q \times p$), then $\mathcal{T}_{\mathcal{K}}$ itself has the following transfer matrix representation

$$T_K(z) = \left[ \begin{array}{cc} L(z) - K(z)H(z) & -K(z) \end{array} \right]. \tag{1.2.5}$$

**State-Space Models**

For a variety of reasons, it is often convenient to represent the relationship between the measurement signal, $y_i$, the desired signal, $s_i$, and the process and measurement noise signals, $u_i$ and $v_i$, via a (possibly time-varying) linear state-space model. In this case, we can write

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F_i x_i + G_i u_i \\ y_i & = & H_i x_i + v_i \\ s_i & = & L_i x_i \end{array} \right. , \tag{1.2.6}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_i \in \mathcal{C}^{n\times m}$, $H_i \in \mathcal{C}^{p\times n}$ and $L_i \in \mathcal{C}^{q\times n}$ are known system matrices, and where $x_i$ is the $n$-dimensional state. Note that we have not specified the range of the time index, $i$, in (1.2.6) since the estimation problem may be finite, semi-infinite, or infinite horizon. [Note, moreover, that according to (1.2.6), since $y_i$ and $s_i$ depend on $\{u_j, j < i\}$, the transfer operators $\mathcal{H}$ and $\mathcal{L}$ are *strictly* causal. It turns out that there is no loss of generality in making this assumption. The benefit is that the algebraic expressions obtained are simpler.]

If we assume that the system matrices in (1.2.6) are time-invariant, *i.e.*,

$$F_i \triangleq F \quad , \quad G_i \triangleq G \quad , \quad H_i \triangleq H \quad , \quad L_i \triangleq L$$

then in the infinite-horizon case we can readily find the transfer matrices $H(z)$ and $L(z)$ from the system matrices via

$$\left\{ \begin{array}{rcl} H(z) & = & H(zI - F)^{-1}G \\ L(z) & = & L(zI - F)^{-1}G \end{array} \right. . \tag{1.2.7}$$

### Adaptive Filtering

In adaptive filtering we observe an output sequence $\{d_i\}$ that obeys the *linear* model

$$d_i = h_i^T w + v_i, \qquad\qquad i \geq 0 \tag{1.2.8}$$

where $h_i^T = \begin{bmatrix} h_{i1} & h_{i2} & \ldots & h_{in} \end{bmatrix}$ is a known input vector, $w = \begin{bmatrix} w_1 & w_2 & \ldots & w_n \end{bmatrix}$ is an unknown weight vector, and $\{v_i\}$ is an unknown disturbance, which may also include modeling errors. The goal is to estimate some linear combination of the unknown weight vector, $L_i w$, (typically either with, $L_i = I$, for estimating $w$ itself, or with, $L_i = h_i^T$, for estimating the uncorrupted output of the filter) using the observations, $\{d_j\}_{j=0}^i$.

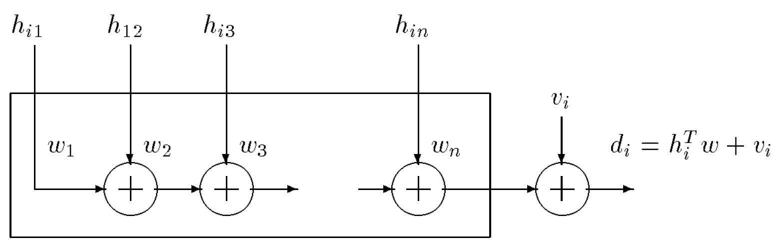


Figure 1.2: The model for adaptive filtering.

Comparing with the general estimation problem considered at the beginning of this section, we see that we can readily identify the observations $\{y_i\}$ with $\{d_i\}$, and that now the sequence $\{u_i\}$ is simply a constant $n$-dimensional vector, $w$. In this case it is straightforward to see that

$$\mathcal{H} = \begin{bmatrix} h_0^T \\ h_1^T \\ h_2^T \\ \vdots \end{bmatrix} \quad \text{and} \quad \mathcal{L} = \begin{bmatrix} L_0 \\ L_1 \\ L_2 \\ \vdots \end{bmatrix}. \tag{1.2.9}$$

A much more useful representation of the adaptive filtering problem is as a special case of a state-space estimation problem. (This point of view has been proposed and pursued in [SK94b] with great effect.) Indeed it is straightforward to see that we may write

$$\begin{cases} x_{i+1} &= x_i \\ d_i &= h_i^T x_i + v_i \\ s_i &= L_i x_i \end{cases}, \quad i \geq 0, \quad x_0 = w. \tag{1.2.10}$$

## 1.3 The $H^2$ Approach

The problem of estimation is to select $\mathcal{K}$, and thereby the estimates $\hat{s}_{i|i}$, based on some performance criterion. The most widely used of such criteria is the $H^2$ norm of the transfer operator, i.e., $\|\mathcal{T}_\mathcal{K}\|_2$.

(i) In the finite horizon case, $\|\mathcal{T}_\mathcal{K}\|_2$ is simply the Frobenius norm of the finite matrix $\mathcal{T}_\mathcal{K}$,

$$\|\mathcal{T}_\mathcal{K}\|_2 \triangleq \|\mathcal{T}_\mathcal{K}\|_F = (\text{trace}\,[\mathcal{T}_\mathcal{K}\mathcal{T}_\mathcal{K}^*])^{1/2} = \left(\sum_{i,j} \|T_{\mathcal{K},ij}\|_F^2\right)^{1/2}, \tag{1.3.1}$$

where $T_{\mathcal{K},ij}$ is the block $(i,j)$-th component of $\mathcal{T}_\mathcal{K}$, *i.e.*, $T_{\mathcal{K},ij}$ maps $\begin{bmatrix} u_j \\ v_j \end{bmatrix}$ to $\tilde{s}_{i|i}$.

(ii) In the infinite-horizon time-invariant case

$$\|\mathcal{T}_\mathcal{K}\|_2 \triangleq \left(\frac{1}{2\pi}\int_0^{2\pi} \|T_K(e^{j\omega})\|_F^2 d\omega\right)^{1/2}, \tag{1.3.2}$$

where $T_K(z)$ is now a transfer matrix.

The widespread use of the $H^2$ theory is mainly due to the facts that the optimal $H^2$ problem has a simple closed-form solution, and that, under certain statistical assumptions on the signals, the solution has several other desirable optimality properties.

### Stochastic Interpretation

$H^2$-optimal estimators have the following two stochastic interpretations.

(a) Assume that the $\{u_j\}$ and $\{v_j\}$ are zero-mean, uncorrelated and temporally white stochastic processes with unit variance, *i.e.*,

$$E\begin{bmatrix} u_i \\ v_i \end{bmatrix}\begin{bmatrix} u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix}\delta_{ij}. \tag{1.3.3}$$

Consider the finite-horizon case (with time index $i$ from 0 to $N$), and compute the estimation error energy,

$$\sum_{i=0}^{N} \tilde{s}_{i|i}^*\tilde{s}_{i|i} = \tilde{s}^*\tilde{s} = \begin{bmatrix} u^* & v^* \end{bmatrix}\mathcal{T}_{\mathcal{K}}^*\mathcal{T}_{\mathcal{K}}\begin{bmatrix} u \\ v \end{bmatrix} = \text{trace}\left(\mathcal{T}_{\mathcal{K}}^*\mathcal{T}_{\mathcal{K}}\begin{bmatrix} u \\ v \end{bmatrix}\begin{bmatrix} u^* & v^* \end{bmatrix}\right). \tag{1.3.4}$$

In view of (1.3.3), we have $E\begin{bmatrix} u \\ v \end{bmatrix}\begin{bmatrix} u^* & v^* \end{bmatrix} = I$, so that taking expectations from both sides of the above equation, the expected estimation error energy becomes

$$E\sum_{i=0}^{N} \tilde{s}_{i|i}^*\tilde{s}_{i|i} = \text{trace}\,(\mathcal{T}_{\mathcal{K}}^*\mathcal{T}_{\mathcal{K}}) = \|\mathcal{T}_{\mathcal{K}}\|_2^2. \tag{1.3.5}$$

But this is simply the cost function that $H^2$-optimal estimators minimize. Therefore, in the finite-horizon case, and under the aforementioned statistical assumptions, $H^2$-optimal estimators minimize the expected estimation error energy. This is why they are also referred to as *linear least-mean-squares estimators.*

Using a similar argument in the infinite-horizon time-invariant case, it is possible to show that

$$E\tilde{s}_{i|i}^*\tilde{s}_{i|i} = \frac{1}{2\pi}\int_0^{2\pi} \|T_K(e^{j\omega})\|_F^2 d\omega = \|\mathcal{T}_{\mathcal{K}}\|_2^2. \tag{1.3.6}$$

Therefore, in the infinite-horizon case, $H^2$-optimal estimators minimize the expected squared estimation error.

(b) If, in addition to the assumptions of part (a), the $\{u_j\}$ and $\{v_j\}$ are assumed to be jointly Gaussian, then the $H^2$-optimal estimator is a *least-mean-squares* estimator (*i.e.*, we do not need to restrict the estimator to being linear) and in addition yields the maximum-likelihood estimate of the $\{s_i\}$.

### 1.3.1 The General Solution

The solution to the $H^2$ estimation problem is well known (see *e.g.*, [Jaz70], [AM79] and [Kai81]) and is given in the following Theorem.

**Theorem 1.3.1 ($H^2$-optimal Estimator)** *The solution to the problem*

$$\min_{\text{causal } \mathcal{K}} \|\mathcal{T}_{\mathcal{K}}\|_2, \tag{1.3.7}$$

*is given by*

$$\mathcal{K} = \left\{\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\right\}_+ (I+\mathcal{H}\mathcal{H}^*)^{-1/2}, \tag{1.3.8}$$

*where $(I+\mathcal{H}\mathcal{H}^*)^{-1/2}$ and $(I+\mathcal{H}\mathcal{H}^*)^{-*/2}$ are found from the canonical (minimum-phase maximum-phase) factorization*

$$I+\mathcal{H}\mathcal{H}^* = (I+\mathcal{H}\mathcal{H}^*)^{1/2}(I+\mathcal{H}\mathcal{H}^*)^{*/2}, \tag{1.3.9}$$

*and where the notation $\{\mathcal{A}\}_+$ denotes the causal part of the transfer operator, $\mathcal{A}$.*

In (1.3.9) the transfer operator $(I+\mathcal{H}\mathcal{H}^*)^{1/2}$ is both causal and causally invertible (hence minimum phase) and the transfer operator $(I+\mathcal{H}\mathcal{H}^*)^{-*/2}$ is both anti-causal and anti-causally invertible (hence maximum phase). Such a factorization of the positive-definite operator $I+\mathcal{H}\mathcal{H}^*$ always exists and is referred to as the *canonical factorization.*

In the finite-horizon case (1.3.9) is the $LL^*$ (block lower-upper triangular) decomposition of the matrix $I+\mathcal{H}\mathcal{H}^*$, and the notation $\{\mathcal{A}\}_+$ denotes the (block) lower triangular part of the matrix $\mathcal{A}$. [Note in this case that the matrix $I+\mathcal{H}\mathcal{H}^*$ is the covariance matrix of the observations signal, $y$, and that therefore (1.3.9) is the canonical factorization of this covariance matrix.] In the time-invariant infinite-horizon case (1.3.9) is the spectral factorization of the $z$-spectral density $I+H(z)H^*(z^{-*})$, and $\{A(z)\}_+$ is the causal part of the function $A(z)$. [Note that in this case $I+\mathcal{H}\mathcal{H}^*$ is the $z$-spectral density function of the observations process, $y$.]

The proof of Theorem 1.3.1 is instructive and is presented below.

**Proof of Theorem 1.3.1:** First note that

$$\|\mathcal{T}_{\mathcal{K}}\|_2^2 = \text{trace}(\mathcal{T}_{\mathcal{K}}^*\mathcal{T}_{\mathcal{K}}) = \text{trace}(\mathcal{T}_{\mathcal{K}}\mathcal{T}_{\mathcal{K}}^*).$$

Using (1.2.3), the expression for $\mathcal{T}_{\mathcal{K}}$, we can write

$$\begin{aligned}
\mathcal{T}_{\mathcal{K}}\mathcal{T}_{\mathcal{K}}^* &= (\mathcal{L}-\mathcal{K}\mathcal{H})(\mathcal{L}-\mathcal{K}\mathcal{H})^* + \mathcal{K}\mathcal{K}^* \\
&= \left[\mathcal{K}-\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-1}\right](I+\mathcal{H}\mathcal{H}^*)\left[\mathcal{K}-\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-1}\right]^* + \mathcal{L}(I+\mathcal{H}^*\mathcal{H})^{-1}\mathcal{L}^*
\end{aligned}$$

where to obtain the second equality we have used a completion of squares argument. Now using the canonical factorization (1.3.9), and the linearity of the trace$(\cdot)$ operator, allows us to conclude

$$\|\mathcal{T}_{\mathcal{K}}\|_2^2 = \|\mathcal{K}(I+\mathcal{H}\mathcal{H}^*)^{1/2} - \mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\|_2^2 + \|\mathcal{L}(I+\mathcal{H}^*\mathcal{H})^{-*/2}\|_2^2.$$

Note that only the first term on the RHS of the above equation depends on $\mathcal{K}$. Therefore it suffices to minimize this first term over causal transfer operators, $\mathcal{K}$. Further inspection of this first term reveals that although $\mathcal{K}(I+\mathcal{H}\mathcal{H}^*)^{1/2}$ is a causal operator, $\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}$ is not. However, using the readily verified identity

$$\|\mathcal{A}\|_2^2 = \|\{\mathcal{A}\}_+\|_2^2 + \|\{\mathcal{A}\}_-\|_2^2\,,$$

where we have denoted the strictly anti-causal part of $\mathcal{A}$ by $\{\mathcal{A}\}_- \triangleq \mathcal{A} - \{\mathcal{A}\}_+$, we may write

$$\begin{gathered}
\|\mathcal{K}(I+\mathcal{H}\mathcal{H}^*)^{1/2} - \mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\|_2^2 = \\
\left\|\mathcal{K}(I+\mathcal{H}\mathcal{H}^*)^{1/2} - \{\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\}_+\right\|_2^2 + \left\|\{\mathcal{L}\mathcal{H}^*(I+\mathcal{H}\mathcal{H}^*)^{-*/2}\}_-\right\|_2^2.
\end{gathered}$$

Note, once more, that the second term on RHS is independent of $\mathcal{K}$. Choosing $\mathcal{K}$ according to (1.3.8) makes the first term vanish and obviously minimizes $\|\mathcal{T}_{\mathcal{K}}\|_2$.

■

**Remark:** The main conclusion to be made from the above proof is that the solution to the $H^2$-optimal estimation problem is obtained from the canonical factorization of a positive definite transfer operator.

### 1.3.2 Special Cases

Depending on the nature of the transfer operators, $\mathcal{H}$ and $\mathcal{L}$, the $H^2$ optimal solution of Theorem 1.3.1 takes on various forms. When $\mathcal{H}$ and $\mathcal{L}$ have state-space structure the solution yields the Kalman filter. When they have transfer function representations, $H(z)$ and $L(z)$, the solution is the Wiener filter, and in the adaptive filtering case the solution corresponds to the recursive-least-squares (RLS) algorithm. We shall now briefly present these.

### The Wiener Filter

As noted in Sec. 1.2.1, in the Wiener filtering problem the model for the measurement and desired signals is given by

$$\begin{cases} y(z) & = & H(z)u(z) + v(z) \\ s(z) & = & L(z)u(z) \end{cases} , \tag{1.3.10}$$

where $\{u_i\}$ and $\{v_i\}$ are assumed to be zero-mean uncorrelated and temporally white stationary processes such that[3]

$$E \begin{bmatrix} u_i \\ v_i \end{bmatrix} \begin{bmatrix} u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}, \tag{1.3.11}$$

or, equivalently,

$$S_u(z) = Q \quad , \quad S_v(z) = R \quad , \quad S_{uv}(z) = 0 \tag{1.3.12}$$

where $S_u(z)$, $S_v(z)$ and $S_{uv}(z)$ are the (obvious) $z$-spectral and cross $z$-spectral densities of the stationary stochastic processes, $\{v_i\}$ and $\{v_i\}$.

The Wiener filter for causally estimating $s_i$, using the $\{y_j, j \leq i\}$, follows straightforwardly from Theorem 1.3.1 and is given by

$$K(z) = \left\{ L(z)QH^*(z^{-*})M^{-*}(z^{-*}) \right\}_+ R_e^{-1} M^{-1}(z), \tag{1.3.13}$$

where $M(z)$, the so-called *modeling filter*, is found from the canonical spectral factorization

$$R + H(z)QH^*(z^{-*}) = M(z)R_e M^*(z^{-*}), \tag{1.3.14}$$

with $M(z)$ causal and causally invertible, and where $R_e$ is a constant matrix chosen such that we have the normalization

$$M(\infty) = I_p. \tag{1.3.15}$$

[3] It is straightforward to consider the case where $u_i$ and $v_i$ are correlated, but for simplicity we shall assume the uncorrelated case here.

Therefore, to find the Wiener filter, all we need to do is perform the canonical spectral factorization (1.3.14). When the $z$-power spectral density function is a scalar, the factorization in (1.3.14) is straightforward, and is obtained by retaining the stable (inside the unit circle) poles and zeros of $R + H(z)QH^*(z^{-*})$ for $M(z)$ (see *e.g.*, [Kai81]). When $R+H(z)QH^*(z^{-*})$ is a matrix, computing the canonical factorization is much more involved and requires the Smith-McMillan form [You61], [Yak70a] and [Kai80].

However, when the transfer matrices have state-space structure, viz.,

$$\begin{cases} H(z) = H(zI-F)^{-1}G \\ L(z) = L(zI-F)^{-1}G \end{cases}, \tag{1.3.16}$$

(recall (1.2.7)), then the canonical factorization can be found via solving a *discrete-time algebraic Riccati equation* (DARE) [Wil71b]. Indeed, if $\{F, GQ^{1/2}\}$ is stabilizable[4] and $\{F, H\}$ is detectable[5], then the modeling filter in (1.3.14) is given by

$$M(z) = I_p + H(zI-F)^{-1}K_p \quad , \quad K_p = FPH^*R_e^{-1} \quad , \quad R_e = R + HPH^* \tag{1.3.17}$$

where $P$ is the unique positive semidefinite solution to the DARE

$$P = FPF^* + GQG^* - K_pR_eK_p^*. \tag{1.3.18}$$

Moreover, $P$ is such that the matrix,

$$F_p \triangleq F - K_pH, \tag{1.3.19}$$

is stable, which is in accordance with the fact that the inverse of the modeling filter,

$$M^{-1}(z) = I - H(zI - F + K_pH)^{-1}K_p = I - H(zI - F_p)^{-1}K_p, \tag{1.3.20}$$

must be causal.

[4]This is a system-theoretic concept. The pair $\{F, GQ^{1/2}\}$ is called stabilizable if there exists a matrix $K$ such that $F - GQ^{1/2}K$ is stable, *i.e.*, if $F$ can be stabilized through state feedback (through the input $Gu$) [Kai80].

[5]Detectability is the dual concept to stabilizability. The pair $\{F, H\}$ is detectable if $\{F^*, H^*\}$ is stabilizable, *i.e.*, if there exists a matrix $K$ such that $F - KH$ is stable.

We are now in a position to give a more explicit formula for the Wiener filter, $K(z)$, of (1.3.13). To this end, note that

$$\begin{aligned} H^*(z^{-*})M^{-*}(z^{-*}) &= G^*(z^{-1}I - F^*)^{-1}H^*\left[I_p - K_p^*(z^{-1}I - F^*)^{-1}H^*\right]^{-1} \\ &= G^*(z^{-1}I - F^* + H^*K_p^*)^{-1}H^* \\ &= G^*(z^{-1}I - F_p^*)^{-1}H^*, \end{aligned}$$

so that we may write

$$\left\{L(z)QH^*(z^{-*})M^{-*}(z^{-*})\right\}_+ = \left\{L(zI-F)^{-1}GQG^*(z^{-1} - F_p^*)^{-1}H^*\right\}_+ .$$

The above expression shows that to find $K(z)$ we need to find the causal part of the transfer matrix, $(zI - F)^{-1}GQG^*(z^{-1} - F_p^*)^{-1}$. But here is a little trick to do so.[6] Using the DARE we may replace $GQG^*$ by

$$P - FPF^* + K_pR_eK_p^* = P - FPF^* + FPH^*K_p^* = P - FP(F - K_pH)^* = P - FPF_p^*,$$

and write

$$(zI - F)^{-1}GQG^*(z^{-1} - F_p^*)^{-1} = (zI - F)^{-1}(P - FPF_p^*)(z^{-1} - F_p^*)^{-1}.$$

Now replacing the center matrix, $P - FPF_p^*$, by

$$P - (zI - (zI - F))\, P\left(z^{-1}I - (z^{-1} - F_p^*)\right),$$

we get, after some algebraic simplifications, that

$$(zI - F)^{-1}GQG^*(z^{-1} - F_p^*)^{-1} = P + (zI - F)^{-1}FP + PF_p^*(z^{-1} - F_p^*)^{-1}. \quad (1.3.21)$$

The above expression is the desired decomposition of $(zI - F)^{-1}GQG^*(z^{-1} - F_p^*)^{-1}$ into its causal and anticausal parts. Indeed since $F$ is stable (by assumption) and $F_p$ is stable (by the solution to the DARE), we have

$$\underbrace{P + \underbrace{(zI - F)^{-1}FP}_{\text{strictly causal}}}_{\text{causal}} + \underbrace{PF_p^*(z^{-1} - F_p^*)^{-1}}_{\text{strictly anticausal}} .$$

[6] We shall see the origin of this trick later in Chapter 7.

This then allows us to write

$$\left\{L(z)QH^*(z^{-*})M^{-*}(z^{-*})\right\}_+ = LPH^* + L(zI-F)^{-1}FPH^*, \tag{1.3.22}$$

so that using (1.3.13) we finally obtain the desired expression for $K(z)$,

$$\begin{aligned} K(z) &= \left(LPH^* + L(zI-F)^{-1}FPH^*\right)R_e^{-1}M^{-1}(z) \\ &= LPH^*R_e^{-1} + L(I-PH^*R_e^{-1}H)(zI-F_p)^{-1}K_p. \end{aligned} \tag{1.3.23}$$

where to obtain the second equality we have used the expression for $M^{-1}(z)$ from (1.3.20).

We can now use the second expression in (1.3.23) to write down a state-space model for the Wiener filter as follows:

$$\left\{\begin{aligned} \hat{x}_{i+1} &= (F-K_pH)\hat{x}_i + K_py_i \\ \hat{s}_{i|i} &= L(I-PH^*R_e^{-1}H)\hat{x}_i + LPH^*R_e^{-1}y_i \end{aligned}\right. , \tag{1.3.24}$$

where the "hat" notation in the state variable $\hat{x}(z) \triangleq (zI-F_p)^{-1}K_py(z)$, has been used since it turns out that $\hat{x}_i$ is indeed the least-means squares prediction of the original state $\hat{x}_i$, given the observations, $\{y_j, j<i\}$.

A further definition that is useful is the so-called *innovations process* [Kai68],

$$e(z) \triangleq M^{-1}(z)y(z) = \left[I_p - H(zI-F_p)^{-1}K_p\right]y(z) = y(z) - H\hat{x}(z), \tag{1.3.25}$$

which, as can be readily verified from the spectral factorization (1.3.14), is a white stationary stochastic process with variance, $R_e$.

It is useful to summarize the results obtained so far in the following Theorem.

**Theorem 1.3.2 (Wiener Filter)** *The solution to the problem*

$$\min_{\text{causal } K(z)} \left\| \begin{bmatrix} (L(z)-K(z)H(z))\,Q^{1/2} & -K(z)R^{1/2} \end{bmatrix} \right\|_2, \tag{1.3.26}$$

*is given by*

$$K(z) = \left\{L(z)QH^*(z^{-*})M^{-*}(z^{-*})\right\}_+ R_e^{-1}M^{-1}(z), \tag{1.3.27}$$

*where $M(z)$ is found from the canonical spectral factorization*

$$R + H(z)QH^*(z^{-*}) = M(z)R_eM^*(z^{-*}),$$

*with $M(z)$ causal and causally invertible, and $M(\infty) = I_p$.*

*When $H(z)$ and $L(z)$ have state-space structure,*

$$\begin{cases} H(z) &= H(zI-F)^{-1}G \\ L(z) &= L(zI-F)^{-1}G \end{cases},$$

*with $\{F, GQ^{1/2}\}$ stabilizable and $\{F, H\}$ detectable, then*

$$K(z) = LPH^*R_e^{-1} + L(I - PH^*R_e^{-1}H)(zI - F_p)^{-1}K_p, \tag{1.3.28}$$

*where $K_p = FPH^*R_e^{-1}$, $R_e = R + HPH^*$, and $P$ is the unique positive semidefinite solution of the DARE,*

$$P = FPF^* + GQG^* - K_pR_eK_p^*.$$

*In this case, a state-space model for $K(z)$ can be given by*

$$\begin{cases} \hat{x}_{i+1} &= (F - K_pH)\hat{x}_i + K_py_i \\ \hat{s}_{i|i} &= L(I - PH^*R_e^{-1}H)\hat{x}_i + LPH^*R_e^{-1}y_i \end{cases}, \tag{1.3.29}$$

*or, defining the innovations, $e_i = y_i - H\hat{x}_i$,*

$$\begin{cases} \hat{x}_{i+1} &= F\hat{x}_i + K_pe_i \\ \hat{s}_{i|i} &= L\hat{x}_i + LPH^*R_e^{-1}e_i \end{cases}. \tag{1.3.30}$$

**Remark:** The filters (1.3.29) and (1.3.29) are the so-called *predicted* form of the Wiener filter (since the state is the predicted estimate, $\hat{x}_i$). Defining $\hat{x}_{i|i} \triangleq \hat{x}_i + PH^*R_e^{-1}e_i$, we can write the following so-called *filtered* form of the Wiener filter

$$\begin{cases} \hat{x}_{i+1|i+1} &= F\hat{x}_{i|i} + PH^*R_e^{-1}(y_{i+1} - HF\hat{x}_{i|i}) \\ \hat{s}_{i|i} &= L\hat{x}_{i|i} \end{cases}. \tag{1.3.31}$$

### The Kalman Filter

As noted in Sec. 1.2.1, in the Kalman filtering problem the model for the measurement and desired signals is given by a (possibly time-variant) linear state-space model

$$\begin{cases} x_{i+1} &= F_ix_i + G_iu_i, \quad i \geq 0, \ x_0 \\ y_i &= H_ix_i + v_i \\ s_i &= L_ix_i \end{cases}. \tag{1.3.32}$$

Moreover, it is assumed that $x_0$ and the $\{u_i\}$ and $\{v_i\}$ are zero-mean uncorrelated random variables with known covariance matrices[7]

$$E\begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix}\begin{bmatrix} x_0^* & u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & 0 \\ 0 & 0 & R_i\delta_{ij} \end{bmatrix}. \tag{1.3.33}$$

In this case, $\hat{s}_{i|i}$, the causal linear least-mean-squares estimate of $s_i$, is given by the Kalman filter recursions [Kal60b],

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & F_i\hat{x}_i + K_{p,i}e_i, \quad \hat{x}_0 = 0 \\ \hat{s}_{i|i} & = & L_i\hat{x}_i + L_iP_iH_i^*R_{e,i}^{-1}e_i \end{array} \right. , \tag{1.3.34}$$

where $e_i \triangleq y_i - H_i\hat{x}_i$ is the (white) innovations process, and where we have defined

$$K_{p,i} = F_iP_iH_i^*R_{e,i}^{-1} \quad \text{and} \quad R_{e,i} = R_i + H_iP_iH_i^* \tag{1.3.35}$$

with $P_i$ the solution to the Riccati recursion

$$P_{i+1} = F_iP_iF_i^* + G_iQ_iG_i^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0 = \Pi_0. \tag{1.3.36}$$

**Remarks:**

(i) Although not shown explicitly here, the Kalman filter recursively performs the (block) triangular decomposition of the output Gramian matrix, $I + \mathcal{H}\mathcal{H}^*$, via the Riccati recursion (1.3.36). (Recall that the Wiener filter performed the canonical factorization of the output $z$-spectral density via the solution of an algebraic Riccati equation). We shall show this result, in fact, in the much more general context of indefinite metric spaces (and indefinite output Gramians) in the next chapter.

(ii) When the $\{F_i, G_i, H_i, Q_i, R_i\}$ are constant matrices, the Kalman filter recursions (1.3.34) bear a striking resemblance to the Wiener filter of Theorem 1.3.2. Indeed it is true that in the time-invariant case, under some rather mild conditions, the Kalman filter recursions converge to the Wiener filter. We will have much more to say about this in Chapter 8.

[7] Once more, for simplicity, we shall assume that the $u_i$ and $v_i$ are uncorrelated.

(iii) The Kalman filter recursions (1.3.34) are in so-called predicted form. Defining the filtered estimates of the states as $\hat{x}_{i|i} \stackrel{\Delta}{=} \hat{x}_i + P_i H_i^* R_{e,i}^{-1} e_i$, we obtain the filtered form of the Kalman filter recursions

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1|i+1} & = & F_i \hat{x}_{i|i} + P_i H_i^* R_{e,i}^{-1}(y_{i+1} - H_{i+1} F_i \hat{x}_{i|i}) \quad \hat{x}_{-1|-1} = 0 \\ \hat{s}_{i|i} & = & L_i \hat{x}_{i|i} \end{array} \right. . \tag{1.3.37}$$

(iv) The Kalman filter solution (as well as the Wiener filter solution) has the interesting property that the structure of the filter does not depend on the linear combination of the state that we intend to estimate (the Riccati recursion and the recursion for $\hat{x}_i$ do not depend on $L_i$). Therefore if one were interested in estimating some other linear combination of the state, say $s_i' = L_i' x_i$, the solution is simply that linear combination of the state estimate, *i.e.*, $\hat{s}_{i|i}' = L_i' \hat{x}_{i|i}$.

(v) There is now a vast literature on the Kalman filter and many variations to the recursions described so far have been developed. We mention in passing the square-root forms of these filters (see *e.g.*, [BG66], [DM69], [KBS71] and the references therein) and the fast (so-called) Chandrasekhar recursions for time-invariant (or structure time-variant) state-space models (see [Kai72], [MSK74] and [SK94a] and the references therein) which we shall encounter in a more general context in Chapter 5.

**The RLS Algorithm**

As noted in Sec. 1.2.1, the adaptive filtering problem can be recast as a special case of a state-space estimation problem where the state-space model has the form,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & x_i \\ d_i & = & h_i^T x_i + v_i \\ s_i & = & L_i x_i \end{array} \right. , \quad i \geq 0, \quad x_0 = w. \tag{1.3.38}$$

*i.e.*, $F_i = I$, $G_i = 0$ and $H_i = h_i^T$. The solution is consequently a special case of the Kalman filter and is given below

$$\left\{ \begin{array}{rcl} \hat{s}_{i|i} & = & L_i \hat{w}_{|i} \\ \hat{w}_{|i} & = & \hat{w}_{|i-1} + \frac{P_i h_i}{1 + h_i^T P_i h_i}(d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{|-1} = 0 \end{array} \right. , \tag{1.3.39}$$

where $\hat{w}_{|i} = \hat{x}_{i|i}$, and where $P_i$ satisfies the Riccati recursion

$$P_{i+1} = P_i - \frac{P_i h_i h_i^T P_i}{1 + h_i^T P_i h_i}, \quad P_0 = \Pi_0. \tag{1.3.40}$$

In the adaptive filtering literature, the algorithm (1.3.39-1.3.40) is known as the *recursive least-squares* (RLS) algorithm [Hay96].

### 1.3.3 The Question of Robustness

We saw that under suitable stochastic assumptions, $H^2$-optimal estimators have certain desirable optimality properties, namely that they minimize the expected estimation error energy and yield maximum-likelihood estimates.

Since, in practice we may not always know the statistics of the disturbances we cannot always guarantee the validity of the assumptions required of $H^2$ estimators. Therefore, the question that begs itself is what the performance of such estimators will be if the assumptions on the disturbances are violated, or if there are modeling errors in our model so that the disturbances must include the modeling errors? In other words

- *is it possible that* **small** *disturbances and modeling errors may lead to* **large** *estimation errors?*

Intuitively, a non-robust algorithm is one for which the above is true, *i.e.*, one for which small disturbances may lead to large estimation errors, and a robust algorithm is one for which small disturbances lead to small estimation errors.

The problem of robust estimation is thus an important one. As we shall presently see, the $H^\infty$ estimation formulation is an *attempt* at addressing this question. It follows from our comments above that any approach to robust estimation requires a *measure* of largeness and smallness (for the signals involved), and in the $H^\infty$ framework this measure is *energy*[8]. The idea is to come up with estimators that minimize (or in the suboptimal case, bound) the maximum energy gain from the disturbances to the estimation errors. This will guarantee that if the disturbances are small (in

[8]Other approaches to robust estimation and control differ in how this measure is defined. For example, in the so-called $l_1$ approach to robust control the measure used is the peak (maximum of the absolute amplitude) of the signals [DP87] and [DDB95].

energy) then the estimation errors will be as small as possible (in energy), *no matter what the disturbances are*. In other words the maximum energy gain is minimized over *all possible* disturbances. The robustness of the $H^\infty$ estimators arises from this fact. However, since they make no assumption about the disturbances and have to accommodate for all conceivable ones, they may be over-conservative.

## 1.4 The $H^\infty$ Approach

In this section we briefly describe the $H^\infty$ approach to robust estimation. [For alternative presentations and derivations see [Kwa86], [DGKF89], [KN91], [Bas91], [LS91], [ST92], [Gri93] and the references therein.]

Returning to the general estimation problem of Sec. 1.2, we recall that a useful representation for any estimation strategy $\mathcal{K}$ is the transfer operator,

$$\mathcal{T}_{\mathcal{K}} = \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix},$$

that maps the disturbance sequences $\{u_j\}$ and $\{v_j\}$ to the estimation error sequence $\{\tilde{s}_{j|j}\}$. Now for any disturbance sequences $u$ and $v$ that yield the estimation error sequence $\tilde{s}$, we may compute the energy gain

$$\frac{\|\tilde{s}\|_2^2}{\|u\|_2^2 + \|v\|_2^2} = \frac{\left\| \mathcal{T}_{\mathcal{K}} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_2^2}{\|u\|_2^2 + \|v\|_2^2}, \tag{1.4.1}$$

where $\|a\|_2^2 = \sum_j a_j^* a_j$ is defined as the energy of the sequence $a = \{a_j\}$.[9] Thus (1.4.1) is a measure of the "amplification" of the noise given our choice of estimator $\mathcal{K}$. Clearly, the ratio in (1.4.1) depends on the particular choice of the input disturbances $u$ and $v$. To remove this dependence we consider the largest energy gain in (1.4.1) over *all* possible disturbance sequences $u$ and $v$, i.e., the $H^\infty$ norm of a transfer operator $\mathcal{T}_{\mathcal{K}}$, as defined below.

[9] Note that we are using the same notation, $\|\cdot\|_2$, for the two-norm of a sequence (the square-root of the energy) and the two norm of an operator. Which one will be meant will be obvious from the context throughout.

**Definition 1.4.1 (The $H^\infty$ Norm)** *The $H^\infty$ norm of a transfer operator $\mathcal{T}$ is defined as*

$$\|\mathcal{T}\|_\infty = \sup_{x\in h^2, x\neq 0} \frac{\|\mathcal{T}x\|_2}{\|x\|_2} \tag{1.4.2}$$

*where $h^2$ denotes the space of all square-summable causal sequences.*

(i) In the finite horizon case, $\|\mathcal{T}\|_\infty$ is simply $\bar{\sigma}(\mathcal{T})$, the maximum singular value of $\mathcal{T}$.

(ii) In the infinite-horizon time-invariant case, we have

$$\|\mathcal{T}\|_\infty = \sup_{0\leq\omega\leq 2\pi} \bar{\sigma}(T(e^{j\omega})), \tag{1.4.3}$$

which is really the origin of the name $H^\infty$.

In $H^\infty$ estimation one seeks the causal estimator $\mathcal{K}$ that minimizes the $H^\infty$ norm of $\mathcal{T}_\mathcal{K}$. The precise statement of the problem follows.

**Problem 1.4.1 (Optimal $H^\infty$ Estimation Problem)** *Find a causal estimator $\mathcal{K}$ that minimizes the $H^\infty$ norm of the transfer operator $\mathcal{T}_\mathcal{K}$ that maps the disturbances $\{u_j\}$ and $\{v_j\}$ to the estimation errors $\{\tilde{s}_{j|j}\}$,* i.e. , *find a causal $\mathcal{K}$ that satisfies,*

$$\inf_{\mathcal{K}} \|\mathcal{T}_\mathcal{K}\|_\infty = \inf_{\mathcal{K}} \left\| \begin{bmatrix} \mathcal{L}-\mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix} \right\|_\infty = \inf_{\mathcal{K}} \sup_{u,v\in h^2, u,v\neq 0} \frac{\left\| \mathcal{T}_\mathcal{K} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_2}{(\|u\|_2^2+\|v\|_2^2)^{1/2}}. \tag{1.4.4}$$

*Moreover find the resulting $\gamma_{opt} = \inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}\|_\infty$.*

[Note that in the above problem statement we are deliberately ambiguous as to whether we are considering the finite horizon or (semi)infinite horizon case.]

The *minimax* nature of $H^\infty$ optimal estimators is evident from (1.4.4). The $H^\infty$ estimation problem can thus be regarded as a game problem: nature (the opponent) has access to the unknown disturbance sequences $u$ and $v$ and chooses it to maximize the energy gain in (1.4.1), whereas we have choice of the causal estimator $\mathcal{K}$ and must choose it to minimize the ratio in (1.4.1).

Note that $H^\infty$ optimal estimators safeguard against the worst-case disturbance that maximizes the energy gain to estimation errors. Since this worst-case disturbance is a single event, such estimators do not require any statistical assumptions on the disturbance signals. Moreover, since the minimization in (1.4.4) is taken over all possible disturbances, these algorithms are robust with respect to disturbance variation.

Unlike in $H^2$ estimation, there are very few cases where a closed-form solution to the optimal $H^\infty$ problem of Prob. 1.4.1 can be found,[10] and in general one relaxes the minimization and settles for a suboptimal solution.

**Problem 1.4.2 (Suboptimal $H^\infty$ Estimation Problem)** *Given a $\gamma > 0$, find a causal estimator $\mathcal{K}$ that guarantees*

$$\|\mathcal{T}_\mathcal{K}\|_\infty = \left\| \begin{bmatrix} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix} \right\|_\infty = \sup_{u,v \in h^2, u,v \neq 0} \frac{\left\| \mathcal{T}_\mathcal{K} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_2}{(\|u\|_2^2 + \|v\|_2^2)^{1/2}} < \gamma. \tag{1.4.5}$$

*This clearly requires checking whether $\gamma \geq \gamma_{opt}$.*

### 1.4.1 The General Solution

We now outline how to find suboptimal $H^\infty$ estimators that achieve a certain level $\gamma$. First note that $\|\mathcal{T}_\mathcal{K}\|_\infty < \gamma$ means that

$$\mathcal{T}_\mathcal{K}\mathcal{T}_\mathcal{K}^* < \gamma^2 I, \tag{1.4.6}$$

where $I$ is the identity transfer operator that maps input sequences to themselves. In other words, the transfer operator, $\gamma^2 I - \mathcal{T}_\mathcal{K}\mathcal{T}_\mathcal{K}^*$, must be positive definite. [When $\mathcal{T}_\mathcal{K}$ is a matrix, the inequality in (1.4.6) is understood to be in the sense of the ordering of Hermitian positive semi-definite matrices, and when $\mathcal{T}_\mathcal{K}$ can be represented as a transfer matrix, $T(z)$, it is understood to be in the sense of $T(e^{j\omega})T^*(e^{j\omega}) < \gamma^2 I$, for all $0 \leq \omega \leq 2\pi$.]

[10] One such case is adaptive filtering which we shall study in detail in Chapter 9.

In either case, (1.4.6) can be rewritten as

$$(\mathcal{L}-\mathcal{K}\mathcal{H})(\mathcal{L}-\mathcal{K}\mathcal{H})^*+\mathcal{K}\mathcal{K}^*-\gamma^2 I<0,$$

or equivalently,

$$\begin{bmatrix} \mathcal{K} & I \end{bmatrix} \begin{bmatrix} I+\mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I+\mathcal{L}\mathcal{L}^* \end{bmatrix} \begin{bmatrix} \mathcal{K}^* \\ I \end{bmatrix} < 0. \tag{1.4.7}$$

Now it can be shown (we shall provide the proof shortly) that a causal $\mathcal{K}$ that guarantees the inequality (1.4.7) can be found if, and only if, the center block operator (or matrix, in the finite-horizon case) in (1.4.7) admits the following canonical factorization

$$\begin{bmatrix} I+\mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I+\mathcal{L}\mathcal{L}^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix}, \tag{1.4.8}$$

where $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ is causal and causally invertible, and, in addition, $\mathcal{L}_{11}$ is causal and causally invertible and $\mathcal{L}_{12}$ is strictly causal. In the matrix case, this means that $\mathcal{L}_{11}$, $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$ are lower triangular and that $\mathcal{L}_{12}$ is strictly lower triangular. When the $\mathcal{L}_{ij}$ can be represented by transfer matrices, this means that $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ is minimum phase and proper, and, in addition, that $L_{11}(z)$ is minimum phase and proper and $L_{12}(z)$ is stable and strictly proper.

In practice, the factorization (1.4.8) can be achieved in various ways. In the matrix case it can be obtained via the Krein space Kalman filter (a generalization of the classical Kalman filter to indefinite metric spaces – see Chapter 2), and in the transfer matrix case it can be obtained via $J$-spectral factorization [KS94].

Once the factorization (1.4.8) has been performed we may rewrite (1.4.7) as follows

$$(\mathcal{K}\mathcal{L}_{11}+\mathcal{L}_{21})(\mathcal{K}\mathcal{L}_{11}+\mathcal{L}_{21})^*-(\mathcal{K}\mathcal{L}_{12}+\mathcal{L}_{22})(\mathcal{K}\mathcal{L}_{12}+\mathcal{L}_{22})^*<0. \tag{1.4.9}$$

Since both $(\mathcal{K}\mathcal{L}_{11}+\mathcal{L}_{21})$ and $(\mathcal{K}\mathcal{L}_{12}+\mathcal{L}_{22})$ are causal, this implies that we must have

$$(\mathcal{K}\mathcal{L}_{11}+\mathcal{L}_{21})=(\mathcal{K}\mathcal{L}_{12}+\mathcal{L}_{22})\mathcal{Q}, \tag{1.4.10}$$

for some *causal strictly contractive* $\mathcal{Q}$ (*i.e.* , a causal $\mathcal{Q}$ such that $\mathcal{Q}\mathcal{Q}^*<I$).

We can now solve the above equation for $\mathcal{K}$ and obtain the following result.

**Theorem 1.4.1 ($H^\infty$ Suboptimal Estimators)** *A causal estimator, $\mathcal{K}$ that achieves*

$$\left\| \begin{bmatrix} \mathcal{L}-\mathcal{K}\mathcal{H} & -\mathcal{K} \end{bmatrix} \right\|_\infty < \gamma,$$

*exists if, and only if, there exists a canonical factorization of the form*

$$\begin{bmatrix} I+\mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I+\mathcal{L}\mathcal{L}^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix},$$

*with* $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ *and $\mathcal{L}_{11}$ causal and causally invertible, and $\mathcal{L}_{12}$ strictly causal. If this is the case, then all possible $H^\infty$ estimators of level $\gamma$ are given by*

$$\mathcal{K} = (\mathcal{L}_{22}\mathcal{Q}-\mathcal{L}_{21})(\mathcal{L}_{11}-\mathcal{L}_{12}\mathcal{Q})^{-1}, \tag{1.4.11}$$

*where $\mathcal{Q}$ is any causal and strictly contractive operator. An important choice results from taking $\mathcal{Q}=0$, so that*

$$\mathcal{K}_{cen} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1}, \tag{1.4.12}$$

*which is the so-called "central" filter.*

**Proof:** We have practically shown sufficiency already. Solving for $\mathcal{K}$ in (1.4.10), yields (1.4.11). What of course remains to be shown is that for any causal contractive $\mathcal{Q}$ the operator, $\mathcal{K}$, is causal.[11] To show this, let us write

$$\mathcal{K} = (\mathcal{L}_{22}\mathcal{Q}-\mathcal{L}_{21})\mathcal{L}_{11}(I-\mathcal{L}_{11}^{-1}\mathcal{L}_{12}\mathcal{Q})^{-1}, \tag{1.4.13}$$

and note that the causal operator $\mathcal{A}=\mathcal{L}_{11}^{-1}\mathcal{L}_{12}\mathcal{Q}$ is strictly contractive since,

$$\begin{aligned} \mathcal{A}\mathcal{A}^* &= \mathcal{L}_{11}^{-1}\mathcal{L}_{12}\mathcal{Q}\mathcal{Q}^*\mathcal{L}_{12}^*\mathcal{L}_{11}^{-*} \\ &< \mathcal{L}_{11}^{-1}\mathcal{L}_{12}\mathcal{L}_{12}^*\mathcal{L}_{11}^{-*}, && \text{since } \mathcal{Q}\mathcal{Q}^* < I \\ &= \mathcal{L}_{11}^{-1}(\mathcal{L}_{11}\mathcal{L}_{11}^* - I - \mathcal{H}\mathcal{H}^*)\mathcal{L}_{11}^{-*}, && \text{equating the } (1,1) \text{ entries in (1.4.8)} \\ &< \mathcal{L}_{11}^{-1}\mathcal{L}_{11}\mathcal{L}_{11}^*\mathcal{L}_{11}^{-*} \\ &= I. \end{aligned}$$

[11] We could lose the causality of $\mathcal{K}$ if $\mathcal{L}_{11}-\mathcal{L}_{12}\mathcal{Q}$ is not causally invertible.

This means that we can expand the inverse in (1.4.13) to write

$$\mathcal{K} = (\mathcal{L}_{22}\mathcal{Q} - \mathcal{L}_{21})\mathcal{L}_{11}(I + \mathcal{A} + \mathcal{A}^2 + \ldots), \tag{1.4.14}$$

where the infinite series converges absolutely since $\mathcal{A}$ is a strict contraction (see [Hir62], [GK69], [SNF70]). Finally, $\mathcal{K}$ is causal since all the operators appearing in (1.4.14) are causal.

The proof of necessity is slightly more detailed. The proof presented here is of independent interest since it also establishes the maximum entropy property of the central solution, (1.4.12), (see also [GM89] and [MG90]).

To this end, suppose that a causal $\mathcal{K}$ exists that solves the $H^\infty$ estimation problem of level $\gamma$. In other words, there exists a causal $\mathcal{K}$ such that (1.4.7) is satisfied, which after a completion of squares argument can be written as

$$\begin{gathered}\gamma^2 I - \mathcal{L}(I + \mathcal{H}^*\mathcal{H})^{-1}\mathcal{L}^* - \\ \left(\mathcal{K}(I + \mathcal{H}\mathcal{H}^*)^{1/2} - \mathcal{L}\mathcal{H}^*(I + \mathcal{H}\mathcal{H}^*)^{-*/2}\right)\left(\mathcal{K}(I + \mathcal{H}\mathcal{H}^*)^{1/2} - \mathcal{L}\mathcal{H}^*(I + \mathcal{H}\mathcal{H}^*)^{-*/2}\right)^* > 0.\end{gathered} \tag{1.4.15}$$

Defining, the causal operator, $\mathcal{F} \triangleq \mathcal{K}(I + \mathcal{H}\mathcal{H}^*)^{1/2}$, and the operator $\mathcal{T} \triangleq \mathcal{L}\mathcal{H}^*(I + \mathcal{H}\mathcal{H}^*)^{-*/2}$, we may write the above equation as

$$\gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F} - \mathcal{T})(\mathcal{F} - \mathcal{T})^* > 0. \tag{1.4.16}$$

Consider now cost function

$$\varepsilon_\gamma(\mathcal{F}) = \log\ \det\left[\gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F} - \mathcal{T})(\mathcal{F} - \mathcal{T})^*\right], \tag{1.4.17}$$

subject to the positivity constraint (1.4.16). The above cost function is referred to as the *entropy cost* (due to the similarity of its form to information theoretic entropy[12] [MG90], [CT91]). Note that, over the set (1.4.16), $\varepsilon_\gamma(\mathcal{F})$ is a concave function and that its value approaches negative infinity as $\mathcal{F}$ approaches the boundary of (1.4.16). Therefore as the set defined via (1.4.16) is convex and nonempty (we have assumed that a causal $H^\infty$ estimator of level $\gamma$ exists), there will exist a unique causal operator, which we denote by $\mathcal{F}_{opt}$, that maximizes $\varepsilon_\gamma(\mathcal{F})$ subject to the constraint (1.4.16) (see

[12] The entropy of a stationary zero-mean Gaussian process with variance, $R_x$, is given by $\log\det R_x$.

*e.g.*, [Gol72], [NN93], [BGFB94] on convex optimization theory). In other words, there exists a solution to the following convex optimization problem,

$$\left\{ \begin{array}{cc} \max_{\text{causal } \mathcal{F}} & \varepsilon_\gamma(\mathcal{F}) \\ \text{subject to} & (1.4.16) \end{array} \right. \tag{1.4.18}$$

Now since $\mathcal{F}_{opt}$ lies inside the constraint set, it is straightforward to see that it should make the derivative of $\varepsilon_\gamma(\mathcal{F})$ with respect to causal $\mathcal{F}$ equal to zero. Computing this derivative shows that $\mathcal{F}_{opt}$ satisfies the equation

$$\left\{ \left[\gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F}_{opt} - \mathcal{T})(\mathcal{F}_{opt} - \mathcal{T})^*\right]^{-1} (\mathcal{F}_{opt} - \mathcal{T}) \right\}_+ = 0. \tag{1.4.19}$$

To summarize what has been shown so far, if the $H^\infty$ estimation problem has a solution then (1.4.19) has a solution that satisfies the constraint (1.4.16). Solving Eq. (1.4.19) seems a formidable task, but we shall again show that certain factorizations provide the trick.

Due to (1.4.19) we readily see that

$$\gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F}_{opt} - \mathcal{T})(\mathcal{F}_{opt} - \mathcal{T})^* > 0,$$

so that we can introduce the canonical spectral factorization

$$\mathcal{L}_\Delta \mathcal{L}_\Delta^* = \gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F}_{opt} - \mathcal{T})(\mathcal{F}_{opt} - \mathcal{T})^*, \tag{1.4.20}$$

with $\mathcal{L}_\Delta$ causal and causally invertible. With this definition (1.4.19) can be written as

$$\left\{ \mathcal{L}_\Delta^{-*} \mathcal{L}_\Delta^{-1} (\mathcal{F}_{opt} - \mathcal{T}) \right\}_+ = 0. \tag{1.4.21}$$

Note that the above equation implies that the operator appearing within the $\{\cdot\}_+$ must be strictly anticausal. Since $\mathcal{L}_\Delta^{-*}$ is anticausal, this implies that $\mathcal{L}_\Delta^{-1}(\mathcal{F}_{opt} - \mathcal{T})$ must be strictly anticausal.[13] Denoting this strictly anticausal operator by $-\bar{\mathcal{L}}_{12}^*$, we have

$$\mathcal{F}_{opt} - \mathcal{T} = -\mathcal{L}_\Delta \bar{\mathcal{L}}_{12}^*. \tag{1.4.22}$$

[13] Denote the strictly anticausal operator appearing within the $\{\cdot\}_+$ by $\mathcal{S} \triangleq \mathcal{L}_\Delta^{-*}\mathcal{L}_\Delta^{-1}(\mathcal{F}_{opt} - \mathcal{T})$. Then $\mathcal{L}_\Delta^{-1}(\mathcal{F}_{opt} - \mathcal{T}) = \mathcal{L}_\Delta^* \mathcal{S}$ must be strictly anticausal since it is the product of an anticausal operator (*i.e.*, $\mathcal{L}_\Delta^*$) and a strictly anticausal operator (*i.e.*, $\mathcal{S}$).

Likewise, let us define the causal and causally invertible transfer operator $\bar{\mathcal{L}}_{11}$ via the canonical factorization

$$\bar{\mathcal{L}}_{11}\bar{\mathcal{L}}_{11}^* = I + \bar{\mathcal{L}}_{12}\bar{\mathcal{L}}_{12}^*. \tag{1.4.23}$$

Finally, let us define the causal operators $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$ via

$$\mathcal{L}_{21} = -\mathcal{F}_{opt}\bar{\mathcal{L}}_{11} \quad \text{and} \quad \mathcal{L}_{22} = \mathcal{L}_\Delta + \mathcal{L}_{21}\bar{\mathcal{L}}_{11}^{-1}\bar{\mathcal{L}}_{12}. \tag{1.4.24}$$

With the above two definitions it is not hard to show that

$$-\mathcal{T} = \mathcal{L}_{21}\bar{\mathcal{L}}_{11}^* - \mathcal{L}_{22}\bar{\mathcal{L}}_{12}^*, \tag{1.4.25}$$

and

$$-\gamma^2 I + \mathcal{L}\mathcal{L}^* = \mathcal{L}_{21}\mathcal{L}_{21}^* - \mathcal{L}_{22}\mathcal{L}_{22}^*. \tag{1.4.26}$$

Combining (1.4.23), (1.4.25) and (1.4.26) shows that the causal and causally invertible $\bar{\mathcal{L}}_{11}$, the strictly causal $\bar{\mathcal{L}}_{12}$, and the causal $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$ can be found from the factorization

$$\begin{bmatrix} I & -\mathcal{T}^* \\ -\mathcal{T} & -\gamma^2 I + \mathcal{L}\mathcal{L}^* \end{bmatrix} = \begin{bmatrix} \bar{\mathcal{L}}_{11} & \bar{\mathcal{L}}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \bar{\mathcal{L}}_{11}^* & \mathcal{L}_{21}^* \\ \bar{\mathcal{L}}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix}. \tag{1.4.27}$$

Note that, from the definition (1.4.24), we can readily find the desired solution to Eq. (1.4.19) using the entries of the above factorization via

$$\mathcal{F}_{opt} = -\mathcal{L}_{21}\bar{\mathcal{L}}_{11}^{-1}. \tag{1.4.28}$$

Finally defining $\mathcal{L}_{11} \triangleq \bar{\mathcal{L}}_{11}(I + \mathcal{H}\mathcal{H}^*)^{1/2}$ and $\mathcal{L}_{12} \triangleq \bar{\mathcal{L}}_{12}(I + \mathcal{H}\mathcal{H}^*)^{1/2}$ and using the definition of $\mathcal{T}$ shows that we have the following factorization

$$\begin{bmatrix} I + \mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I + \mathcal{L}\mathcal{L}^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix},$$

with $\mathcal{L}_{11}$ causal and causally invertible, $\mathcal{L}_{21}$ and $\mathcal{L}_{22}$ causal, and $\mathcal{L}_{12}$ strictly causal. Therefore all that remains to show is that $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ is causally invertible. But this readily follows from the block triangular factorization

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ \mathcal{L}_{21}\mathcal{L}_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & 0 \\ 0 & \underbrace{\mathcal{L}_{22} - \mathcal{L}_{21}\mathcal{L}_{11}^{-1}\mathcal{L}_{12}}_{\mathcal{L}_\Delta} \end{bmatrix} \begin{bmatrix} I & \mathcal{L}_{11}^{-1}\mathcal{L}_{12} \\ 0 & I \end{bmatrix}$$

and the facts that $\mathcal{L}_{11}$ and $\mathcal{L}_\Delta$ are causally invertible.

∎

Note that the above proof establishes the following Corollary.

**Corollary 1.4.1 (Maximum Entropy Property of Central Solution)** *The solution to the problem*

$$\max_{\text{causal } \mathcal{K}} \log \det \left[\gamma^2 I - \mathcal{T}_\mathcal{K}\mathcal{T}_\mathcal{K}^*\right] \tag{1.4.29}$$

*is given by the central estimator*

$$\mathcal{K}_{cen} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1} \tag{1.4.30}$$

**Proof:** The proof follows from the facts that

$$\gamma^2 I - \mathcal{T}_\mathcal{K}\mathcal{T}_\mathcal{K}^* = \gamma^2 I - \mathcal{L}\mathcal{L}^* + \mathcal{T}\mathcal{T}^* - (\mathcal{F} - \mathcal{T})(\mathcal{F} - \mathcal{T})^*,$$

so that the cost function in (1.4.29) is simply $\varepsilon_\gamma(\mathcal{F})$, and that $\mathcal{K} = \mathcal{F}(I + \mathcal{H}\mathcal{H}^*)^{-1/2}$ and

$$\mathcal{F}_{opt} = -\mathcal{L}_{21}\bar{\mathcal{L}}_{11}^{-1} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1}(I + \mathcal{H}\mathcal{H}^*)^{1/2}.$$

∎

**Remark:** The major conclusion to be made from above discussions is that the solution to the $H^\infty$ estimation problem can be obtained by a suitable canonical spectral factorization of an *indefinite* transfer operator. In this sense, the $H^\infty$ solution is a certain generalization of the $H^2$ solution. We will have much more to say about this in the next chapter.

### 1.4.2 Special Cases

The general solution presented in the previous section subsumes as special cases the finite and infinite-horizon $H^\infty$ estimation problems and the $H^\infty$ adaptive filtering problem. We shall now briefly present these.

### The Infinite Horizon Case

In the infinite-horizon time-invariant case the relationship between the measurement and desired signals is given by the transfer matrices

$$\left\{ \begin{array}{rcl} y(z) & = & H(z)u(z)+v(z) \\ s(z) & = & L(z)u(z) \end{array} \right. . \tag{1.4.31}$$

In particular, when $H(z)$ and $L(z)$ have finite-dimensional state-space descriptions, we may write

$$\left\{ \begin{array}{rcl} H(z) & = & H(zI-F)^{-1}G \\ L(z) & = & L(zI-F)^{-1}G \end{array} \right. . \tag{1.4.32}$$

Under the above assumptions, Theorem 1.4.1 takes the following form.

**Theorem 1.4.2 (Infinite Horizon $H^\infty$ Suboptimal Estimators)** *(a) A causal estimator, $K(z)$, that achieves*

$$\left\| \left[ \begin{array}{cc} L(z)-K(z)H(z) & -K(z) \end{array} \right] \right\|_\infty < \gamma,$$

*exists if, and only if, there exists a factorization of the form*

$$\begin{array}{c} \left[ \begin{array}{cc} I_p + H(z)H^*(z^{-*}) & -H(z)L^*(z^{-*}) \\ -L(z)H^*(z^{-*}) & -\gamma^2 I_q + L(z)L^*(z^{-*}) \end{array} \right] = \\ \left[ \begin{array}{cc} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{array} \right] \left[ \begin{array}{cc} I_p & 0 \\ 0 & -I_q \end{array} \right] \left[ \begin{array}{cc} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ L_{12}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{array} \right], \end{array} \tag{1.4.33}$$

*with* $\left[ \begin{array}{cc} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{array} \right]$ *and $L_{11}(z)$ minimum phase and proper, and $L_{12}(z)$ causal and strictly proper. If this is the case, then all possible $H^\infty$ estimators of level $\gamma$ are given by*

$$K(z) = \left(L_{22}(z)Q(z) - L_{21}(z)\right)\left(L_{11}(z) - L_{12}(z)Q(z)\right)^{-1}, \tag{1.4.34}$$

*where $Q(z)$ is any causal and strictly contractive transfer matrix,* i.e., *$Q(e^{j\omega})Q^*(e^{j\omega}) < I$, $\forall \omega \in [0, 2\pi]$. The central filter results from the choice $Q(z)=0$, so that*

$$K_{cen}(z) = -L_{21}(z)L_{11}^{-1}(z). \tag{1.4.35}$$

*(b) Assume that $H(z)$ and $L(z)$ have state-space structure,*

$$\begin{cases} H(z) &= H(zI-F)^{-1}G \\ L(z) &= L(zI-F)^{-1}G \end{cases},$$

*with $\{F,G\}$ stabilizable and $\{F,H\}$ detectable. Then a solution to the $H^\infty$ estimation problem with level $\gamma$ exists if, and only if, there exists a solution to the DARE*

$$P = FPF^* + GG^* - K_p R_e K_p^*, \tag{1.4.36}$$

*with*

$$K_p = FP\begin{bmatrix} H^* & L^* \end{bmatrix} R_e^{-1} \quad \text{and} \quad R_e = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} P \begin{bmatrix} H^* & L^* \end{bmatrix} \tag{1.4.37}$$

*such that*

*(i)* $F_p \triangleq F - K_p \begin{bmatrix} H \\ L \end{bmatrix}$ *is stable.*

*(ii)* $R_e$ *and* $\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}$ *have the same inertia*[14].

*If this is the case, then the $L_{ij}(z)$ in the canonical factorization (1.4.33) are given by*

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \left( \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} + \begin{bmatrix} H \\ -L \end{bmatrix} (zI-F)^{-1} K_p \right) \times \tag{1.4.38}$$
$$\begin{bmatrix} (I_p + HPH^*)^{1/2} & 0 \\ -LPH^*(I_p+HPH^*)^{-*/2} & (\gamma^2 I_q - LP(I+H^*HP)^{-1}L^*)^{1/2} \end{bmatrix}.$$

[14]By the inertia of a Hermitian matrix, we mean the number of its positive, negative and zero eigenvalues. A simple way of calculating the inertia of a strongly regular Hermitian matrix $R$ (*i.e.* one whose leading minors are all nonzero), is by computing its $LDU$ decomposition

$$R = LDL^*,$$

where $L$ is a lower triangular matrix with unit diagonal and $D$ is a diagonal matrix: the number of positive and negative elements of $D$ give the number of positive and negative eigenvalues of $R$ and hence the inertia.

*In particular, defining* $R_{He} \triangleq I_p + HPH^*$, *we have*

$$\left\{ \begin{array}{rcl} L_{11}(z) & = & \left[I_p + H(zI-F)^{-1}FPH^*R_{He}^{-1}\right] R_{He}^{1/2} \\ L_{21}(z) & = & -\left[LPH^*R_{He}^{-1} + L(zI-F)^{-1}FPH^*R_{He}^{-1}\right] R_{He}^{1/2} \end{array} \right. , \tag{1.4.39}$$

*so that*

$$K_{cen}(z) = LPH^*R_{He}^{-1} + L\left(I - PH^*R_{He}^{-1}H\right)(zI-F_1)^{-1}FPH^*R_{He}^{-1}, \tag{1.4.40}$$

*where* $F_1 \triangleq F - FPH^*R_{He}^{-1}H$.

**Proof:** Part (a) is simply a restatement of Theorem 1.4.1 when we have transfer matrices.

In part (b), the relation between canonical spectral factorizations of rational matrix functions and the solution to certain algebraic Riccati equations is wellknown (see *e.g.*, [BLW91], [LR95]) and will be studied in detail in Chapter 7. There we will prove that a canonical factorization of the desired form exists if, and only if, a solution to the DARE (1.4.36) with the aforementioned properties exists.

Here we shall confine ourselves to showing that if a solution to the DARE with properties (i) and (ii) exists, then the $L_{ij}(z)$ have all the required properties. To this end, first note that from (1.4.38) it follows that the $L_{ij}(z)$ are stable (since $F$ is stable[15]) and in particular, that $L_{12}(z)$ is strictly proper. Moreover,

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}^{-1} \propto \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} - \begin{bmatrix} H \\ -L \end{bmatrix} (zI-F_p)^{-1}K_p,$$

where the notation '$\propto$ means that the two matrices are related by a constant invertible matrix. Since $F_p$ is stable the causality of $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}^{-1}$ is established. Finally, note that

$$L_{11}(z)^{-1} \propto I_p - H(zI - F + K_1H)^{-1}K_1,$$

where we have defined $K_1 = FPH^*(I_p + HPH^*)^{-1}$. Therefore to show the causality of $L_{11}(z)^{-1}$ we need to demonstrate that $F - K_1H$ is stable. To do so, we proceed

[15] We have actually assumed instead that $\{F, H\}$ is detectable. In Chapter 7 we shall show that there is no loss in making this weaker assumption.

as follows. First, in Chapter 7 we shall show that if $P$ is a stabilizing solution to the DARE (1.4.36) then $P \geq 0$. This then implies that $I_p + HPH^* > 0$, so that for the inertia condition (ii) to hold the Schur complement of $R_e$ with respect to its $(1,1)$ block entry must be negative definite, *i.e.*,

$$-\gamma^2 I_q + LPL^* - LPH^*(I_p + HPH^*)^{-1}HPL^* = -\gamma^2 I_q + LP(I + H^*HP)^{-1}L^* < 0.$$

Now we can rewrite the DARE (1.4.36) as

$$P = (F - K_1H)P(F - K_1H)^* + GG^* + K_1K_1^* + \underbrace{FP(I + H^*HP)^{-1}L^*\left(\gamma^2 I_q - LP(I + H^*HP)^{-1}L^*\right)^{-1} L(I + PH^*H)^{-1}PF^*}_{\geq 0}.$$

Now since $P \geq 0$ and $\{F - K_1H, \begin{bmatrix} G & K_1 \end{bmatrix}\}$ is stabilizable (since $\{F, G\}$ is stabilizable), Lyapunov theory proves that $F - K_1H$ is stable[16] [Lya67], [GQ95].

■

**Remarks:**

(i) Despite the fundamental differences in their approaches, the above $H^\infty$ filter and the Wiener filter of Theorem 1.3.2 bear striking resemblances. Indeed the only difference between the central $H^\infty$ filter (1.4.40) and the Wiener filter (1.3.28) is that the Riccati variables, $P$, satisfy different DARE's. We will have more to say about this in the next subsection and in Chapter 3.

(ii) It is straightforward to give state-space representations for the central filter (1.4.40). Indeed we have

$$\begin{cases} \hat{x}_{i+1} = (F - K_1H)\hat{x}_i + K_1 y_i \\ \hat{s}_{i|i} = L(I - PH^*R_{He}^{-1}H)\hat{x}_i + LPH^*R_{He}^{-1}y_i \end{cases}, \qquad (1.4.41)$$

with $K_1 = FPH^*R_{He}^{-1}$ and $R_{He} = I_p + HPH^*$, which is referred to as the predicted form of the central estimator. Defining $\hat{x}_{i|i} \triangleq \hat{x}_i + PH^*R_{He}^{-1}(y_i - H\hat{x}_i)$, we can write the following so-called filtered form of the central estimator

$$\begin{cases} \hat{x}_{i+1|i+1} = F\hat{x}_{i|i} + PH^*R_{He}^{-1}(y_{i+1} - HF\hat{x}_{i|i}) \\ \hat{s}_{i|i} = L\hat{x}_{i|i} \end{cases}. \qquad (1.4.42)$$

[16] According to a theorem in stability theory, if the Lyapunov equation $P = FPF^* + Q$, where $Q \geq 0$ and $\{F, Q^{1/2}\}$ is stabilizable, has a solution, $P \geq 0$, then $F$ is a stable matrix.

### The Finite Horizon Case

When the measurement and observations processes have a (possibly time-variant) state-space model, we may write

$$\begin{cases} x_{i+1} &= F_i x_i + G_i u_i, \quad N \geq i \geq 0, \; x_0 \\ y_i &= H_i x_i + v_i \\ s_i &= L_i x_i \end{cases} . \tag{1.4.43}$$

In this case, the disturbances are expanded to include the unknown initial state, $x_0$, so that the transfer operator in question becomes,

$$\mathcal{T}_{\mathcal{K}} : \; \left\{ \Pi_0^{-1/2} x_0, \{u_i\}_{i=0}^N, \{v_i\}_{i=0}^N \right\} \rightarrow \{s_i - \hat{s}_{i|i}\}_{i=0}^N. \tag{1.4.44}$$

where $\Pi_0 > 0$ reflects the weight that is given to uncertainty in the initial condition relative to uncertainty in the $u_i$'s and $v_i$'s. Therefore the finite-horizon suboptimal $H^\infty$ estimation problem can be formulated as follows.

**Problem 1.4.3 (Sub-optimal Finite Horizon $H^\infty$ Estimation Problem)** *Given a scalar $\gamma > 0$ and a final time, $N$, find an $H^\infty$ sub-optimal estimator $\hat{s}_{j|j} = \mathcal{K}(y_0, \ldots, y_j)$ that achieves $\|\mathcal{T}_{\mathcal{K}}\|_\infty < \gamma$. In other words,*

$$\sup_{x_0, u \in h_2, v \in h_2} \frac{\sum_{j=0}^N \tilde{s}_{j|j}^* \tilde{s}_{j|j}}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^N u_j^* u_j + \sum_{j=0}^N v_j^* v_j} < \gamma^2. \tag{1.4.45}$$

*This clearly requires checking whether $\gamma > \gamma_{opt}$.*

The solution to the above problem is given below (see [KN91], [ST92], [HSK96b], [HSK93b], [GL95]).

**Theorem 1.4.3 (Finite Horizon $H^\infty$ Filter)** *An $H^\infty$ filter of level $\gamma$ exists if, and only if, the matrices*

$$R_i = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \;\; \textit{and} \;\; R_{e,i} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H_i \\ L_i \end{bmatrix} P_i \begin{bmatrix} H_i^* & L_i^* \end{bmatrix} \tag{1.4.46}$$

*have the same inertia for all* $0 \le i \le N$, *where* $P_0 = \Pi_0$ *and* $P_i$ *satisfies the Riccati recursion*

$$P_{i+1} = F_i P_i F_i^* + G_i G_i^* - F_i P_i \begin{bmatrix} H_i^* & L_i^* \end{bmatrix} R_{e,i}^{-1} \begin{bmatrix} H_i \\ L_i \end{bmatrix} P_i F_i^*. \tag{1.4.47}$$

*If this is the case, then the central* $H^\infty$ *estimator is given by*

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & F_i \hat{x}_i + K_{1,i}(y_i - H_i \hat{x}_i), \quad \hat{x}_0 = 0 \\ \hat{s}_{i|i} & = & L_i \hat{x}_i + L_i P_i H_i^* R_{He,i}^{-1}(y_i - H_i \hat{x}_i) \end{array} \right. , \tag{1.4.48}$$

*with* $K_{1,i} = F_i P_i H_i^* R_{He,i}^{-1}$ *and* $R_{He,i} = I_p + H_i P_i H_i^*$. *If we define* $\hat{x}_{i|i} \triangleq \hat{x}_i + P_i H_i^* R_{He,i}^{-1}(y_i - H_i \hat{x}_i)$, *we can alternatively write the central estimator as*

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1|i+1} & = & F_i \hat{x}_{i|i} + P_i H_i^* R_{He,i}^{-1}(y_{i+1} - H_{i+1} F_i \hat{x}_{i|i}), \quad \hat{x}_{-1|-1} = 0 \\ \hat{s}_{i|i} & = & L_i \hat{x}_{i|i} \end{array} \right. . \tag{1.4.49}$$

**Proof:** The proof will be given in Chapter 3.

■

Note that the filter of Theorem 1.4.3 looks very much like a Kalman filter solution, except that the Riccati recursion differs from that of the Kalman filter, since

(i) we have indefinite *covariance* matrices, $\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}$.

(ii) the $L_i$ (of the quantity to be estimated) enters the Riccati equation.

(iii) we have an additional condition, (1.4.46), that must be satisfied for the filter to exist; in the Kalman filter problem the $L_i$ would not appear, and the $P_i$ would be positive definite, so that (1.4.46) is immediate.

The appearance of the $L_i$ means that the $H^\infty$ estimate of say, the first component of the state vector $x_i$, is not the first component of the $H^\infty$ estimate of the whole state vector (because in the first case $L_i = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}$, and in the second case $L_i = I$). This is very different from the situation in the $H^2$ case, where the estimate of any linear combination of the state is simply given by that linear combination of the state estimate.

Despite these differences, we will see in Chapter 2 (see also [HSK96c], [HSK96b], [HSK93c],[HSK93b]) that the filter of Theorem 1.4.3 can in fact be obtained as a certain Kalman filter, not in an $H^2$ (Hilbert) space, but in a certain indefinite metric space, called a Krein space. The indefinite covariances and the appearance of $L_i$ in the Riccati equation are all easily explained in this framework. The additional condition (1.4.46) will be seen to arise from the fact that in Krein space, unlike as in the usual Hilbert space context, quadratic forms need not always have minima or maxima, unless certain additional conditions are met.

We finally mention that when the system matrices are constant, under some mild conditions, the above finite-horizon $H^\infty$ filter converges to the infinite-horizon $H^\infty$ of Theorem 1.4.2, as time progresses to infinity. The asymptotic behaviour of $H^\infty$ and $H^2$ filters will be dealt with in Chapter 8.

### The LMS Algorithm

As noted in Sec. 1.2.1, the adaptive filtering problem can be recast as a special case of a state-space estimation problem where the state-space model has the form,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & x_i \\ d_i & = & h_i^T x_i + v_i \\ s_i & = & L_i x_i \end{array} \right. , \quad i \geq 0, \quad x_0 = w. \tag{1.4.50}$$

*i.e.*, $F_i = I$, $G_i = 0$ and $H_i = h_i^T$. Therefore to obtain $H^\infty$ adaptive filters all one needs to do is to apply the above state-space model to the filter of Theorem 1.4.3. However, since in adaptive filtering one does not usually know the input vector $\{h_i\}$ beforehand, it is useful to have some method of obtaining a priori bounds on the value of admissible $\gamma$'s. This problem will be dealt with in Chapter 11 (see also [HK94a], [HK95a]).

However, in the important special case of estimating the uncorrupted output, *i.e.*, when $L_i = h_i^T$, it turns out that one can solve the optimal $H^\infty$ adaptive filtering problem! It so happens that in this case, $\gamma_{opt} = 1$, *i.e.*, there is no amplification of the disturbances. Moreover, the central $H^\infty$-optimal adaptive algorithm is the celebrated normalized *least-mean-squares* (LMS) algorithm of Widrow and Hoff [WH60],

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \frac{h_i}{\frac{1}{\mu} + h_i^T h_i}(d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{|-1} = 0. \tag{1.4.51}$$

This result, for the first time, furnished a rigorous basis for the LMS algorithm, which was long regarded as an approximate least-squares solution, and provided theoretical justification for its widely observed robust performance (see [HSK96a], [HSK93a], [HSK95]). We will treat these topics in depth in Chapter 9.

## 1.5 Control Problems

We will now focus our attention on control problems. As mentioned earlier, the main goal in control theory is to influence the dynamical behaviour of a given plant, through a control signal, such that the plant yields some desired performance. For pedagogical reasons it will be more convenient to present two different control problems: first, the full information control problem, where the control signal has access to all the information available up to the current time, and second, the measurement feedback control problem, where the control signal has only access to a certain measurement signal. Although the latter scenario is much more realistic of practical control problems, we shall begin by studying the former problem for two important reasons: first, the full information control problem turns out to be the *dual* of the estimation problem considered so far, and second, the solution to the measurement feedback control problem is simply obtained by estimating (from the measurements) the full information control signal. [This fact is known as the *separation principle* or as the *certainty equivalence principle* in the literature. The first complete proof of this principle in the $H^2$ case is attributed to Wonham [Won68b] and similar results in $H^\infty$ and risk-sensitive control have been obtained in [DGKF89] and [Whi90].]

### 1.5.1 Full Information Control

A general discrete-time full information control problem is shown in Fig. 1.3 which subsumes both the finite and infinite horizon cases. Here we assume that $\mathcal{P}_1$ and $\mathcal{P}_2$

are known *causal* linear transfer operators that map the input sequences $w \triangleq \{w_i\}$ and $u \triangleq \{u_i\}$ to the output sequence $s \triangleq \{s_i\}$, *i.e.*,

$$s = \mathcal{P}_1 w + \mathcal{P}_2 u. \tag{1.5.1}$$

As before, in the finite horizon case the $\mathcal{P}_i$ can be represented by lower triangular matrices, and in the infinite horizon time-invariant case they can be represented by transfer matrices, $P_i(z)$.

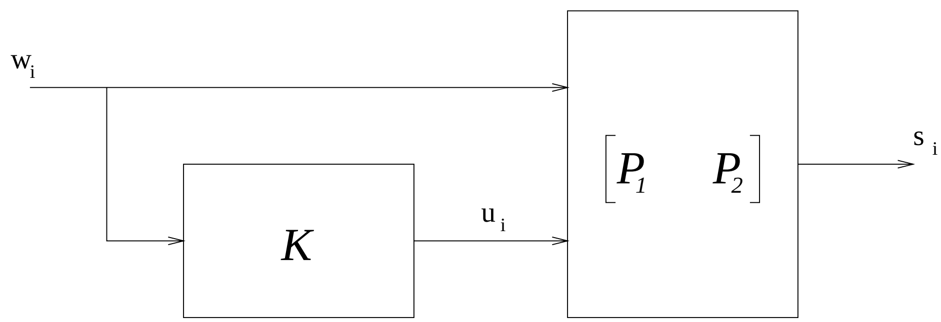


Figure 1.3: The full information control problem.

The signal $w$ is referred to as the *exogenous input*. It can essentially be considered as process noise (or the driving disturbance), and in the finite horizon case it includes the initial state of the plant, $x_0$. The signal $u$ is referred to as the *control input* and is used to influence the dynamical behaviour of the plant, $\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 & \mathcal{P}_2 \end{bmatrix}$. In the full information case, the control signal at a given time, $i$, is allowed to be a function of all the information available up to time $i$, *i.e.*, it is allowed to be a function of $\{w_j, j \leq i\}$ (since all prior information is in the exogenous input, $w$). This means that we can write

$$u = \mathcal{K} w, \tag{1.5.2}$$

where $\mathcal{K}$ is a causal linear operator referred to as the *controller*. Finally, the signal $s$ is referred to as the *regulated output*, and in the finite horizon case it includes the final state of the plant, $x_{N+1}$. Roughly speaking, the goal of control is to keep the regulated signal as 'small' as possible (where, of course, 'small' is meant in a certain sense).

It is of course quite conceivable that if the choice of $u$ were not constrained in any way, then it should be possible to make the regulated signal $s$ arbitrarily small. As this may (and typically will) require an arbitrarily large control signal, in order to guarantee the cost-effectiveness of the final control strategy, it is necessary to try to keep the control signal small as well. Therefore we are left with the twofold objective of designing a causal controller, $\mathcal{K}$, that simultaneously guarantees that the regulated signal, $s$, and the control signal, $u$, be small. (These are obviously two competing objectives.)

In view of the above remarks, we now note that (roughly speaking) the behaviour of any controller $\mathcal{K}$ can be captured by, $\mathcal{T}_{\mathcal{K}}^c$, the transfer operator that maps the exogenous input, $w$, to the regulated and control signals, $\{s, u\}$. Thus

$$\mathcal{T}_{\mathcal{K}}^c : \ w \rightarrow \begin{bmatrix} s \\ u \end{bmatrix}. \tag{1.5.3}$$

From Fig. 1.3 it is straightforward to infer that

$$\mathcal{T}_{\mathcal{K}}^c = \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix}. \tag{1.5.4}$$

**Remark:** Note that if we take the adjoint (conjugate transpose, for matrices) of the operator, $\mathcal{T}_{\mathcal{K}}^c$, we may write

$$\mathcal{T}_{\mathcal{K}}^{c*} = \begin{bmatrix} \mathcal{P}_1^* + \mathcal{K}^*\mathcal{P}_2^* & \mathcal{K}^* \end{bmatrix}. \tag{1.5.5}$$

Comparing the above transfer operator with $\mathcal{T}_{\mathcal{K}}$, the transfer operator under consideration in estimation problems, and given by Eq. (1.2.3), we see that if we make the transformations

$$\mathcal{P}_1^* \leftrightarrow \mathcal{L} \quad , \quad \mathcal{P}_2^* \leftrightarrow \mathcal{H} \quad , \quad \mathcal{K}^* \leftrightarrow -\mathcal{K}$$

then we can obtain one transfer operator from the other. In other words, by replacing causal operators ($\mathcal{L}$ and $\mathcal{H}$) with anticausal operators ($\mathcal{P}_1^*$ and $\mathcal{P}_2^*$) and by insisting on an anticausal estimator ($-\mathcal{K}^*$ instead of $\mathcal{K}$) we see that we can solve the full information control problem by solving a related (so-called *dual*) estimation problem. The above observation is at the heart of the duality between estimation and (full information) control and will be used (to an extent) in the remainder of this chapter to solve the various control problems under consideration. [We shall return to duality in Chapter 6 and study it from a deeper geometric viewpoint. See also the recent paper [Lue92] and the book [Wal89] on duality in mathematical programming.]

### 1.5.2 Measurement Feedback Control

A general discrete-time measurement feedback control problem is shown in Fig. 1.4 which, as before, subsumes both the finite and infinite horizon cases. Here the $\mathcal{P}_{ij}$ are known causal linear transfer operators that map the input sequences $w \triangleq \{w_i\}$ and $u \triangleq \{u_i\}$ to the output sequences $s \triangleq \{s_i\}$ and $t \triangleq \{t_i\}$ according to the formula

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}. \tag{1.5.6}$$

In the finite horizon case the $\mathcal{P}_{ij}$ can be represented by lower triangular matrices, and in the infinite horizon time-invariant case they can be represented by transfer matrices, $P_{ij}(z)$.

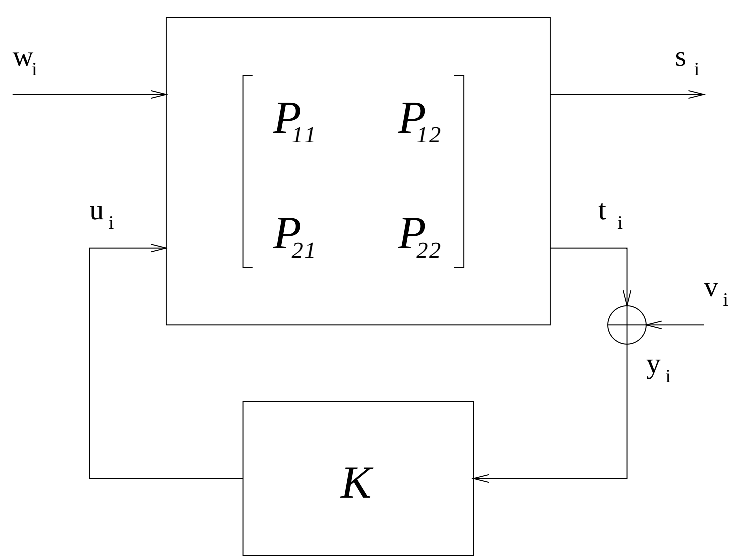


Figure 1.4: The measurement feedback control problem.

Here the sequences $w$ and $v$ are the exogenous signals. The signal $w$ can be considered to be the process noise (or the driving disturbance) and the signal $v$ can be considered to be the measurement noise (which corrupts the output signal, $t$). The signal $u$ is the control input used to influence the dynamical behaviour of the plant. In the measurement feedback case, the control signal at any given time, $i$, is only allowed to be a function of current and past observations of $y$. (The signal $y$ is the so-called measurement signal which can be regarded as a corrupted version of the output signal, $t$.) In other words, contrary to the full-information case where the control signal had access to all the available information, here $u_i$ is only allowed to be a function of $\{y_j, j \leq i\}$. This means that we can write

$$u = \mathcal{K}y, \tag{1.5.7}$$

for some causal linear operator, $\mathcal{K}$, called the controller. As before, $s$ is the regulated output that we intend to keep small.

In the full information problem we noted that in order to keep the control law cost-effective, it is necessary to ensure that the control signal be not too large. This observation also holds for the measurement feedback problem, so that (as in the previous section) we are confronted with the two-fold task of keeping both the regulated signal, $s$, as well as the control signal, $u$, small (in some pre-defined sense).

In view of the above, the behaviour of any controller, $\mathcal{K}$, can be captured by the $\mathcal{T}_{\mathcal{K}}^c$, the transfer operator that maps the exogenous inputs, $\{w, v\}$, to the regulated and control signals, $\{s, u\}$, *i.e.*,

$$\mathcal{T}_{\mathcal{K}}^c : \begin{bmatrix} w \\ v \end{bmatrix} \to \begin{bmatrix} s \\ u \end{bmatrix}. \tag{1.5.8}$$

Using the model presented so far we can write

$$\left\{ \begin{array}{rcl} s & = & \mathcal{P}_{11}w + \mathcal{P}_{12}u \\ y & = & \mathcal{P}_{21}w + \mathcal{P}_{22}u + v \\ u & = & \mathcal{K}y \end{array} \right. ,$$

from which, after some algebra, yields

$$\mathcal{T}_{\mathcal{K}}^c = \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix}. \tag{1.5.9}$$

The above expression shows an apparently complicated relationship between the transfer operator $\mathcal{T}_{\mathcal{K}}^c$ and the controller, $\mathcal{K}$. Indeed referring back to Eqs. (1.2.3) and (1.5.4), we see that in the estimation problem and in the full information control problem, the transfer operators, $\mathcal{T}_{\mathcal{K}}$ and $\mathcal{T}_{\mathcal{K}}^c$ were *affine* in the respective estimator and controller, $\mathcal{K}$. Here $\mathcal{T}_{\mathcal{K}}^c$ is obviously not affine in $\mathcal{K}$. However, if we define the transfer operator

$$\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}, \tag{1.5.10}$$

(which is a bilinear transformation of the controller) we immediately see that

$$\mathcal{T}_{\mathcal{K}}^c = \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{Q} \\ \mathcal{Q}\mathcal{P}_{21} & \mathcal{Q} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{P}_{12} \\ I \end{bmatrix} \mathcal{Q} \begin{bmatrix} \mathcal{P}_{21} & I \end{bmatrix}, \tag{1.5.11}$$

which is now affine in $\mathcal{Q}$. The above expression for $\mathcal{T}_{\mathcal{K}}^c$ is the starting point for the Youla-Jabr-Bongiorno-Kucera (YBJK) parametrization of all controllers that stabilize $\mathcal{T}_{\mathcal{K}}^c$, [Kuc74], [Kuc75], [YBJ76a], [YBJ76b]. [Indeed when the original plant is stable, *i.e.*, when the $P_{ij}$ are all stable, it is straightforward to see that $\mathcal{T}_{\mathcal{K}}^c$ is stable if, and only if, $\mathcal{Q}$ is stable. Therefore the set of all stabilizing controllers are those $\mathcal{K}$ that yield a stable $\mathcal{Q}$. The parametrization in the case of an unstable plant is slightly more involved and requires certain interpolation conditions on $\mathcal{Q}$.]

The above formula for $\mathcal{T}_{\mathcal{K}}^c$ is quite useful and may be used for controller design (see *e.g.*, [BB91] and the references therein). We shall make some (although not very explicit) use of it in the solutions to the $H^2$ and $H^\infty$ measurement feedback control problems. However, for the most part, our solutions will take a somewhat different route.

### 1.5.3 Special Cases

We now consider some special cases of the above general formulations.

#### Transfer Matrices

In the infinite-horizon case, when the $\mathcal{P}_i$ (for the full information problem) and the $\mathcal{P}_{ij}$ (for the measurement feedback problem) are linear time-invariant transfer operators, they can be represented by transfer matrices, $P_i(z)$ and $P_{ij}(z)$, respectively. [In particular, given that the $w_i$, $u_i$, $y_i$ and $s_i$ are $m_1$-vectors, $m_2$-vectors, $p$-vectors and $q$-vectors, respectively, the dimensions of the corresponding transfer matrices can be immediately found.] In this case, for the full information problem, we can write

$$\left\{ \begin{array}{rcl} s(z) & = & P_1(z)w(z) + P_2(z)u(z) \\ u(z) & = & K(z)w(z) \end{array} \right. , \tag{1.5.12}$$

and for the measurement feedback problem,

$$\left\{ \begin{array}{rcl} s(z) & = & P_{11}(z)w(z) + P_{12}(z)u(z) \\ y(z) & = & P_{21}(z)w(z) + P_{22}(z)u(z) + v(z) \\ u(z) & = & K(z)y(z) \end{array} \right. . \tag{1.5.13}$$

The transfer operator $\mathcal{T}_K^c$ itself will also have a transfer matrix representation. For the full information problem,

$$T_K^c = \begin{bmatrix} P_1(z) + P_2(z)K(z) \\ K(z) \end{bmatrix}, \tag{1.5.14}$$

and for the measurement feedback problem,

$$T_K^c = \begin{bmatrix} P_{11}(z) + P_{12}(z)K(z)\left(I - P_{22}(z)K(z)\right)^{-1} P_{21}(z) & P_{12}(z)K(z)\left(I - P_{22}(z)K(z)\right)^{-1} \\ K(z)\left(I - P_{22}(z)K(z)\right)^{-1} P_{21}(z) & K(z)\left(I - P_{22}(z)K(z)\right)^{-1} \end{bmatrix}. \tag{1.5.15}$$

### State-Space Models

For a variety of reasons, it is often convenient to represent the relationship between the exogenous signals, $w_i$ and $v_i$, the control signal, $u_i$, the measurement signal, $y_i$, and the regulated signal, $s_i$, via a (possibly time-varying) linear state-space model. In this case, for the full information problem we can write, write

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \end{array} \right. , \tag{1.5.16}$$

and for the measurement feedback problem,

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \\ y_i & = & H_i x_i + v_i \end{array}\right. , \tag{1.5.17}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_{1,i} \in \mathcal{C}^{n\times m_1}$, $G_{1,2} \in \mathcal{C}^{n\times m_2}$, $H_i \in \mathcal{C}^{p\times n}$ and $L_i \in \mathcal{C}^{q\times n}$ are known system matrices, and where $x_i$ is the $n$-dimensional state. Note that we have not specified the range of the time index, $i$, in either of the models (1.5.16) or (1.5.17) since the control problem may be finite, semi-infinite, or infinite horizon. [Note, moreover, that according to (1.5.16) and (1.5.16), since $y_i$ and $s_i$ depend on $\{w_j, j < i\}$ and $\{u_j, j < i\}$, the transfer operators $\mathcal{P}_i$ and $\mathcal{P}_{ij}$ are strictly causal. It turns out that there is no loss of generality in making this assumption. The benefit is that the algebraic expressions obtained are simpler.]

If we assume that the system matrices in (1.5.16) and (1.5.17) are time-invariant, *i.e.*,

$$F_i \triangleq F \quad , \quad G_{1,i} \triangleq G_1 \quad , \quad G_{2,i} \triangleq G_2 \quad , \quad H_i \triangleq H \quad , \quad L_i \triangleq L$$

then in the infinite-horizon case we can readily find the transfer matrices $P_i(z)$ and $P_{ij}(z)$. For the full information problem we have,

$$\left[\begin{array}{cc} P_1(z) & P_2(z) \end{array}\right] = L(zI - F)^{-1} \left[\begin{array}{cc} G_1 & G_2 \end{array}\right], \tag{1.5.18}$$

and for the measurement feedback problem,

$$\left[\begin{array}{cc} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{array}\right] = \left[\begin{array}{c} L \\ H \end{array}\right] (zI - F)^{-1} \left[\begin{array}{cc} G_1 & G_2 \end{array}\right]. \tag{1.5.19}$$

## 1.6 The $H^2$ Approach

The problem of control (be it full information or measurement feedback) is to select the controller $\mathcal{K}$ so that the transfer operator $\mathcal{T}_{\mathcal{K}}^c$ is small in some sense. This will (roughly speaking) guarantee that the regulated signal, $s$, and the control signal, $u$, are simultaneously small. The most widely used criterion for this purpose is the $H^2$ norm of $\mathcal{T}_{\mathcal{K}}^c$, *i.e.*, $\|\mathcal{T}_{\mathcal{K}}^c\|_2$. The remarks given at the beginning of Sec. 1.3 (on the definition of the $H^2$ norm in the finite and infinite-horizon cases, and on the reasons for the pervasive use of the $H^2$ criterion) all apply here as well and will therefore not be repeated.[17] Instead we shall comment on the connections with *linear-quadratic-regulator* (LQR) and *linear-quadratic-Gaussian* (LQG) control.

### 1.6.1 Connections to LQR and LQG Control

Consider the finite-horizon case and the quadratic cost function,

$$J_N^c = \sum_{i=0}^{N} s_i^* s_i + \sum_{i=0}^{N} u_i^* u_i. \tag{1.6.1}$$

The above cost function is obviously a measure of how large the regulated and control signals are and is an indication of how well our control strategy performs. We can, of course, also write

$$J_N^c = \|s\|_2^2 + \|u\|_2^2. \tag{1.6.2}$$

Assume now that the $\{w_j\}$ and $\{v_j\}$ are zero-mean, uncorrelated and temporally white stochastic processes with unit variance, *i.e.*,

$$E \begin{bmatrix} w_i \\ v_i \end{bmatrix} \begin{bmatrix} w_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_p \end{bmatrix} \delta_{ij}. \tag{1.6.3}$$

Then $J_N^c$ itself will be a random variable. Computing the mean value of $J_N^c$ yields, in the full information case,

$$EJ_N^c = E \begin{bmatrix} s^* & u^* \end{bmatrix} \begin{bmatrix} s \\ u \end{bmatrix} = E\ w^* \mathcal{T}_{\mathcal{K}}^{c*} \mathcal{T}_{\mathcal{K}}^{c} w = E\ \mathrm{trace}\left( \mathcal{T}_{\mathcal{K}}^{c*} \mathcal{T}_{\mathcal{K}}^{c} w w^* \right), \tag{1.6.4}$$

and in the measurement feedback case,

$$\begin{aligned} EJ_N^c = E \begin{bmatrix} s^* & u^* \end{bmatrix} \begin{bmatrix} s \\ u \end{bmatrix} &= E \begin{bmatrix} w^* & v^* \end{bmatrix} \mathcal{T}_{\mathcal{K}}^{c*} \mathcal{T}_{\mathcal{K}}^{c} \begin{bmatrix} w \\ v \end{bmatrix} \\ &= E\ \mathrm{trace} \left( \mathcal{T}_{\mathcal{K}}^{c*} \mathcal{T}_{\mathcal{K}}^{c} \begin{bmatrix} w \\ v \end{bmatrix} \begin{bmatrix} w^* & v^* \end{bmatrix} \right). \end{aligned} \tag{1.6.5}$$

[17] An excellent bibliography on developments in $H^2$ control until 1971 has been complied in [MG71]. See also the 1971 special issue of the IEEE Transactions on Automatic Control devoted to LQG control [Ath71].

In view of (1.6.3), we have $E\begin{bmatrix} w \\ v \end{bmatrix}\begin{bmatrix} w^* & v^* \end{bmatrix} = I$, so that in both the full information and measurement feedback cases the expected quadratic cost functions become,

$$E\left(\sum_{i=0}^{N} s_i^* s_i + \sum_{i=0}^{N} u_i^* u_i\right) = \operatorname{trace}\left(\mathcal{T}_{\mathcal{K}}^{c*}\mathcal{T}_{\mathcal{K}}^{c}\right) = \|\mathcal{T}_{\mathcal{K}}^{c}\|_2^2, \tag{1.6.6}$$

which is the cost function that $H^2$-optimal controllers minimize. Therefore, in the finite-horizon case, and under the aforementioned statistical assumptions, $H^2$-optimal controllers minimize an expected quadratic cost function. This is why they are also referred to as linear least-mean-squares controllers.

Using a similar argument in the infinite-horizon time-invariant case, it is possible to show that

$$E\left(s_i^* s_i + u_i^* u_i\right) = \frac{1}{2\pi}\int_0^{2\pi} \|T_K(e^{j\omega})\|_F^2 d\omega = \|\mathcal{T}_{\mathcal{K}}\|_2^2. \tag{1.6.7}$$

Therefore, in the infinite-horizon case, $H^2$-optimal controllers also minimize an expected quadratic cost function.

**Remarks:**

(a) If, in addition to the aforementioned statistical assumptions, the $\{w_j\}$ and $\{v_j\}$ are assumed to be jointly Gaussian, then the $H^2$-optimal controller is a *least-mean-squares* controller (*i.e.*, we do not need to restrict the controller to being linear). $H^2$-optimal controllers are therefore often called LQG controllers to emphasize that (L), they are linear, (Q), the cost function is quadratic, and (G), the exogenous inputs are Gaussian.

(b) Consider the full information problem where we have state-space structure. When the driving disturbance is assumed zero, so that the only exogenous variable is the initial state, $x_0$, it is very easy to see that $H^2$-optimal controllers solve the following finite and infinite horizon optimization problems,

$$\min_{\{u_i\}} \sum_{i=0}^{N} s_i^* s_i + \sum_{i=0}^{N} u_i^* u_i \quad \text{and} \quad \min_{\{u_i\}} \sum_{i=0}^{\infty} s_i^* s_i + \sum_{i=0}^{\infty} u_i^* u_i. \tag{1.6.8}$$

The above problems are referred to as linear quadratic regulator problems. We shall study them in more detail in Chapter 6.

## 1.6.2 The Full Information Solution

The solution to the $H^2$ full information control problem is well known (see *e.g.*, [Bro70], [AM71] and [KS72]) and is given in the following Theorem.

**Theorem 1.6.1 ($H^2$-optimal Full Information Controller)** *The solution to the problem*

$$\min_{\text{causal } \mathcal{K}} \left\| \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix} \right\|_2 \tag{1.6.9}$$

*is given by*

$$\mathcal{K} = -(I + \mathcal{P}_2^*\mathcal{P}_2)^{-1/2} \left\{ (I + \mathcal{P}_2^*\mathcal{P}_2)^{-*/2}\mathcal{P}_2^*\mathcal{P}_1 \right\}_+ \tag{1.6.10}$$

*where $(I+\mathcal{P}_2^*\mathcal{P}_2)^{-1/2}$ and $(I+\mathcal{P}_2^*\mathcal{P}_2)^{-*/2}$ are found from the dual canonical (maximum-phase minimum-phase) factorization*

$$I + \mathcal{P}_2^*\mathcal{P}_2 = (I + \mathcal{P}_2^*\mathcal{P}_2)^{*/2}(I + \mathcal{P}_2^*\mathcal{P}_2)^{1/2}, \tag{1.6.11}$$

*and where the notation $\{\mathcal{A}\}_+$ denotes the causal part of the transfer operator, $\mathcal{A}$.*

In (1.6.11) the transfer operator $(I+\mathcal{P}_2^*\mathcal{P}_2)^{1/2}$ is both causal and causally invertible (hence minimum phase) and the transfer operator $(I + \mathcal{P}_2^*\mathcal{P}_2)^{*/2}$ is both anti-causal and anti-causally invertible (hence maximum phase). Note, however, that the factorization (1.6.11) is different from the factorization (1.3.9), required in the solution of the $H^2$ estimation problem, in the sense that the maximum-phase factor appears on the left and the minimum-phase factor on the right, rather than vice-versa. For this reason, we have called Eq. (1.6.11) the *dual* canonical factorization. It should be remarked that, as with the usual canonical factorization, the dual canonical factorization of a positive-definite operator (such as $I + \mathcal{P}_2^*\mathcal{P}_2$) always exists (see *e.g.* [Spe85],[GK86].) [In the finite-horizon case (1.6.11) is simply the $UU^*$ (block upper-lower triangular) decomposition of the matrix $I + \mathcal{P}_2^*\mathcal{P}_2$.]

As mentioned earlier, the full information control problem can be considered to be the dual of the estimation problem. Although the final solution of Eq. (1.6.10) already shows some of this duality, in order to display the duality in a more revealing fashion let us take the negative conjugate transpose of (1.6.10) to write

$$-\mathcal{K}^* = \left\{ \mathcal{P}_1^*\mathcal{P}_2(I + \mathcal{P}_2^*\mathcal{P}_2)^{-1/2} \right\}_{\text{a.c.}} (I + \mathcal{P}_2^*\mathcal{P}_2)^{*/2}, \tag{1.6.12}$$

where $\{\mathcal{A}\}_{\text{a.c.}}$ denotes the anticausal part of the operator $\mathcal{A}$, and where we have used the readily verified identity $\{\mathcal{A}\}_+^* = \{\mathcal{A}^*\}_{\text{a.c.}}$. Inspection of (1.6.12) reveals that by replacing the anticausal operators $-\mathcal{K}^*$, $\mathcal{P}_1^*$ and $\mathcal{P}_2^*$ with the causal operators $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{H}$, replacing $\{\cdot\}_{\text{a.c.}}$ with $\{\cdot\}_+$, and replacing the dual canonical factorization (1.6.11) with the usual factorization, one readily obtains the $H^2$ estimator of Eq. (1.3.8). Thus the solutions of Theorems 1.3.1 and 1.6.1 are truly the dual of one another.

**Proof of Theorem 1.6.1:** The proof is the dual to the proof of Theorem 1.3.1 and will not be repeated here.

■

### 1.6.3 The Measurement Feedback Solution

In the estimation problems of Secs. 1.3 and 1.4 and in the full information control problem considered so far, the estimators and controllers obtained were both causal and stable.[18] This was true even when the underlying transfer operators ($\mathcal{L}$ and $\mathcal{H}$ in estimation, $\mathcal{P}_1$ and $\mathcal{P}_2$ in control) where causal but unstable. The resulting $H^2$ and $H^\infty$ estimators and full information controllers also lead to causal and stable $\mathcal{T}_\mathcal{K}$ and $\mathcal{T}_\mathcal{K}^c$. However, as we shall presently see, in the measurement feedback problem we can no longer guarantee the stability of the causal controller, $\mathcal{K}$. Despite this, the stability of $\mathcal{T}_\mathcal{K}^c$ can still ge guaranteed.

The solution to the $H^2$ measurement feedback control problem is given below and, as mentioned earlier, involves a certain two-step procedure known as the separation principle.

[18]Until the present we have deliberately not been very specific about the distinction between causal and stable operators. A causal operator is one for which the mapping from future inputs to past and current outputs is zero, *i.e.*, the output at any given time is only a function of past and current inputs. On the other hand, there are many different notions of stability for linear transfer operators (such as $L^2$, $L^\infty$ and exponential stability), but they all essentially amount to the fact that if the inputs are bounded (in some sense) then the outputs are also bounded (in some sense). For example, causal operators with transfer matrix representations can formally be written as $H(z) = H_0 + H_1 z^{-1} + \ldots$. However, such operators need not be stable since in such a formal expansion the $H_i$ need not be bounded. $L^2$ (or $H^2$) stability, for example, requires that $\sum_j^\infty \text{trace}(H_j^* H_j) < \infty$.

**Theorem 1.6.2** **($H^2$-optimal Measurement Feedback Controller)** *The solution to the problem*

$$\min_{\text{causal } \mathcal{K}} \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix} \right\|_2 \tag{1.6.13}$$

*is given by*

$$\mathcal{K} = \left((I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2} + \mathcal{K}_e\mathcal{P}_{22}\right)^{-1}\mathcal{K}_e, \tag{1.6.14}$$

*where* $\mathcal{K}_e = \left\{(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f\mathcal{P}_{21}^*(I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{-*/2}\right\}_+ (I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{-1/2}$ *is the solution to the estimation problem,*

$$\min_{\text{causal } \mathcal{K}_e} \left\| \begin{bmatrix} (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f - \mathcal{K}_e\mathcal{P}_{21} & -\mathcal{K}_e \end{bmatrix} \right\|_2, \tag{1.6.15}$$

*and* $\mathcal{K}_f = -(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{-1/2}\left\{(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{-*/2}\mathcal{P}_{12}^*\mathcal{P}_{11}\right\}_+$ *is the solution to the full information control problem,*

$$\min_{\text{causal } \mathcal{K}_f} \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}_f \\ \mathcal{K}_f \end{bmatrix} \right\|_2, \tag{1.6.16}$$

*with* $(I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{1/2}$ *and* $(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}$ *given by the canonical and dual canonical factorizations,*

$$\left\{ \begin{array}{rcl} I+\mathcal{P}_{21}\mathcal{P}_{21}^* &=& (I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{1/2}(I+\mathcal{P}_{21}\mathcal{P}_{21}^*)^{*/2} \\ I+\mathcal{P}_{12}^*\mathcal{P}_{12} &=& (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{*/2}(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2} \end{array} \right. . \tag{1.6.17}$$

**Proof:** Let us (primarily to simplify the notation in the cost function (1.6.13)) define $\mathcal{Q} = \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}$. Note that we can now write the cost function as

$$\left\| \begin{bmatrix} \mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{Q} \\ \mathcal{Q}\mathcal{P}_{21} & \mathcal{Q} \end{bmatrix} \right\|_2^2 = \left\| \begin{bmatrix} \mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} \\ \mathcal{Q}\mathcal{P}_{21} \end{bmatrix} \right\|_2^2 + \left\| \begin{bmatrix} \mathcal{P}_{12}\mathcal{Q} \\ \mathcal{Q} \end{bmatrix} \right\|_2^2. \tag{1.6.18}$$

Now using the solution to the full information control problem (1.6.16) it is straightforward to see that we may write (see *e.g.*, the proofs of Theorems 1.3.1 and 1.4.1)

$$\begin{aligned} \left\| \begin{bmatrix} \mathcal{P}_{11}+\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21} \\ \mathcal{Q}\mathcal{P}_{21} \end{bmatrix} \right\|_2^2 &= \left\|(I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f - (I+\mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q}\mathcal{P}_{21}\right\|_2^2 \\ &\quad + \text{terms independent of } \mathcal{Q}. \end{aligned}$$

Moreover, we have

$$\left\| \begin{bmatrix} \mathcal{P}_{12}\mathcal{Q} \\ \mathcal{Q} \end{bmatrix} \right\|_2^2 = \left\| (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q} \right\|_2^2. \tag{1.6.19}$$

Combining these last two expressions shows that we need to minimize the cost function,

$$\left\| \begin{bmatrix} (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f - (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q}\mathcal{P}_{21} & -(I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q} \end{bmatrix} \right\|_2^2. \tag{1.6.20}$$

But defining, $\mathcal{K}_e = (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{Q}$, yields the estimation problem (1.6.15). Solving $\mathcal{K}_e = (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$ for the desired controller $\mathcal{K}$ gives our final result (1.6.14).

■

Although we have obtained the separation principle and the solution to the measurement feedback control problem through a purely algebraic route, the solution has certain physical significance that should be noted. In the first step we need to find the full information controller that satisfies

$$\min_{\text{causal } \mathcal{K}_f} \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}_f \\ \mathcal{K}_f \end{bmatrix} \right\|_2 .$$

The above controller would have been the $H^2$-optimal controller if we had access to the exogenous input, $w$. (Recall that $s = \mathcal{P}_{11}w + \mathcal{P}_{12}u$.) Since we do not have access to $w$, we cannot construct the full information control signal, $u_f = \mathcal{K}_f w$. However, what the separation principle says is that we should estimate the signal $u_f$ using the observations, $y = \mathcal{P}_{21}w + \mathcal{P}_{22}u + v$. (Note here that, since at each time instant the previous values of $u$ are known, this is a standard estimation problem.) The exact statement of the resulting $H^2$ estimation problem is,

$$\min_{\text{causal } \mathcal{K}_e} \left\| \begin{bmatrix} (I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f - \mathcal{K}_e\mathcal{P}_{21} & -\mathcal{K}_e \end{bmatrix} \right\|_2 ,$$

which means that we are estimation the signal $(I + \mathcal{P}_{12}^*\mathcal{P}_{12})^{1/2}\mathcal{K}_f w$ using the observation, $y - \mathcal{P}_{22}u = \mathcal{P}_{21} + v$. [The last step (1.6.14) simply means that we should unwind our estimator, which was in terms of previous $u$ and current and previous $y$, to make it an estimator in terms of $y$ alone.]

As a matter of fact, we shall later see that in actually trying to find the $H^2$-optimal measurement feedback controller (say, for systems described by state-space models) it is easier to keep the physical meaning of the separation in mind, rather than to explicitly use the solution of Theorem 1.6.2.

Finally, we remark once more that the key to the solution of the $H^2$ full information and measurement feedback control problems is obtained through the canonical factorization of certain positive definite transfer operators.

### 1.6.4 Special Cases

We will now consider the solutions of Theorems 1.6.1 and 1.6.2 in the cases where the underlying transfer operators have transfer matrix representations and state-space structure. We shall, in particular, see that the in the full information problem the solution is given by a (well known) state feedback law, and that in the measurement feedback problem one instead uses a feedback based on estimates of the state.

#### Transfer Matrices - Full Information Case

As noted in Sec. 1.5.3, when the underlying transfer operators have transfer matrix representations we can write

$$\left\{ \begin{array}{lcl} s(z) & = & P_1(z)w(z) + P_2(z)u(z) \\ u(z) & = & K(z)w(z) \end{array} \right. . \tag{1.6.21}$$

Suppose, moreover, that the quadratic cost whose expected value we seek to minimize is given by

$$s_i^* R^c s_i + u_i^* Q^c u_i, \tag{1.6.22}$$

where $R_c \geq 0$ and $Q^c \geq 0$ are given weighting matrices.[19]

Using Theorem 1.6.1 the full information controller is given by

$$K(z) = -N^{-1}(z)R_c^{-1}\left\{N^{-*}(z^{-*})P_2^*(z^{-*})R^c P_1(z)\right\}_+, \tag{1.6.23}$$

[19]This slight generalization of the problem considered so far is of no significant consequence. Indeed we can go back to the standard problem by the transformations, $(R^c)^{*/2}s_i \to s_i$, $(Q^c)^{*/2}u_i \to u_i$, $(R^c)^{*/2}P_2(z) \to P_2(z)$ and $(R^c)^{*/2}P_1(z) \to P_1(z)$.

where $N(z)$ is found from the dual canonical factorization,

$$Q^c + P_2^*(z^{-*})R^c P_2(z) = N^*(z^{-*})R_c N(z), \tag{1.6.24}$$

with $N(z)$ causal and causally invertible, and where $R_c$ is such that we have the normalization

$$N(\infty) = I_{m_1}. \tag{1.6.25}$$

As with the $H^2$ estimation problem, the solution requires a (here dual) canonical spectral factorization. As also was the case there, when the transfer matrices have state-space structure, viz.,

$$\begin{bmatrix} P_1(z) & P_2(z) \end{bmatrix} = L(zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix}, \tag{1.6.26}$$

then the canonical factorization can be explicitly obtained via the solution of a DARE (discrete-time algebraic Riccati recursion). Indeed, if $\{F, G_2(Q^c)^{1/2}\}$ is stabilizable and $\{F, (R^c)^{*/2}L\}$ is detectable, then $N(z)$ in (1.6.24) is given by

$$N(z) = I_{m_2} + K_c(zI - F)^{-1}G_2 \quad , \quad K_c = R_c^{-1}G_2^* P^c F \quad , \quad R_c = Q^c + G_2^* P^c G_2 \tag{1.6.27}$$

where $P^c$ is the unique positive semidefinite solution to the (dual) DARE,[20]

$$P^c = F^* P^c F + L^* R^c L - K_c^* R_c K_c. \tag{1.6.28}$$

Moreover, $P^c$ is such that the matrix,

$$F_c \triangleq F - G_2 K_c, \tag{1.6.29}$$

is stable, which is in accordance with the fact that the inverse of $N(z)$,

$$N^{-1}(z) = I_{m_2} - K_c(zI - F + G_2K_c)^{-1}G_2 = I - K_c(zI - F_c)^{-1}G_2, \tag{1.6.30}$$

must be causal.

[20]Note that this DARE is the dual to the DARE (1.3.18) given in the solution to the $H^2$ estimation problem.

We are now in a position to give a more explicit formula for the $H^2$-optimal full information controller $K(z)$, of (1.6.23). To this end, note that

$$\begin{aligned} N^{-*}(z^{-*})P_2^*(z^{-*}) &= \left[I_{m_2} + G_2^*(z^{-1}I - F^*)^{-1}K_c^*\right]^{-1} G_2^*(z^{-1}I - F^*)^{-1}L^* \\ &= G_2^*(z^{-1}I - F^* + K_c^*G_2^*)^{-1}L^* \\ &= G_2^*(z^{-1}I - F_c^*)^{-1}L^*, \end{aligned}$$

so that we may write

$$\left\{N^{-*}(z^{-*})P_2^*(z^{-*})R^cP_1(z)\right\}_+ = \left\{G_2^*(z^{-1}I - F_c^*)^{-1}L^*R^cL(zI - F)^{-1}G_1\right\}_+ .$$

The causal part of the above expression can be found in a manner similar to what was done in the Wiener filtering setting. Using an argument analogous to the one used in proving Eq. (1.3.21), allows us to write

$$(z^{-1}I - F_c^*)^{-1}L^*R^cL(zI - F)^{-1} = P^c + P^cF(zI - F)^{-1} + (z^{-1}I - F_c^*)^{-1}F_c^*P^c, \quad (1.6.31)$$

which is the desired decomposition of $(z^{-1}I - F_c^*)^{-1}L^*R^cL(zI - F)^{-1}$ into its causal and anticausal parts. This then allows us to write

$$\left\{G_2^*(z^{-1}I - F_c^*)^{-1}L^*R^cL(zI - F)^{-1}G_1\right\}_+ = G_2^*P^cG_1 + G_2^*P^cF(zI - F)^{-1}G_1, \quad (1.6.32)$$

so that using (1.6.23) we finally obtain the desired expression for $K(z)$,

$$\begin{aligned} K(z) &= -N^{-1}(z)R_c^{-1}\left(G_2^*P^cG_1 + G_2^*P^cF(zI - F)^{-1}G_1\right)^{-1} \\ &= -R_c^{-1}G_2^*P^cG_1 - K_c(zI - F_c)^{-1}(I - G_2R_c^{-1}G_2^*P^c)G_1. \end{aligned} \quad (1.6.33)$$

where to obtain the second equality we have used the expression for $N^{-1}(z)$ from (1.6.30).

We can now use the above transfer matrix representation to write a state-space model for the controller as follows,

$$\left\{\begin{aligned} x_{i+1} &= (F - G_2K_c)x_i + (I - G_2R_c^{-1}G_2^*P^c)G_1w_i \\ u_i &= -K_cx_i - R_c^{-1}G_2^*P^cG_1w_i \end{aligned}\right. . \quad (1.6.34)$$

The reason why we have used the notation $x_i$ for the state variable in (1.6.34) is that it really is the state vector for the plant. Indeed using the expression for $u_i = -K_c x_i - R_c^{-1} G_2^* P^c G_1 w_i$ in the state equation of (1.6.34) allows us to write,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & Fx_i + G_1 w_i + G_2 u_i \\ u_i & = & -K_c x_i - R_c^{-1} G_2^* P^c G_1 w_i \end{array} \right. , \tag{1.6.35}$$

which shows that $x_i$ is the plant's state vector.

The above solution has an interesting structure and shows that the control signal, $u_i$, is a function *only* of the current state, $x_i$, and the current driving disturbance, $w_i$. This is slightly different from the (now famous) state feedback solution of continuous-time $H^2$ optimal control where the control signal depends only on the current state. The reason why this difference occurs is that (from the outset) we have insisted on causal controllers. Had we insisted on *strictly* causal controllers, *i.e.*, controllers that only have access to *past* values of the exogenous input, $w$, then we would have obtained a state feedback controller.

Indeed in that case, following similar arguments, one can show that the strictly causal $H^2$-optimal full information controller is given by,

$$K(z) = -N^{-1}(z) R_c^{-1} \left\{ N^{-*}(z^{-*}) P_2^*(z^{-*}) R^c P_1(z) \right\}_{s.c.}, \tag{1.6.36}$$

where the notation $\{A(z)\}_{s.c.}$ denotes the strictly causal part of the transfer matrix $A(z)$. In this case,

$$\left\{ G_2^*(z^{-1}I - F_c^*)^{-1} L^* R^c L (zI - F)^{-1} G_1 \right\}_{s.c.} = G_2^* P^c F (zI - F)^{-1} G_1,$$

from which we conclude

$$K(z) = K_c (zI - F_c)^{-1} G_1, \tag{1.6.37}$$

which has state-space representation,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & Fx_i + G_1 w_i + G_2 u_i \\ u_i & = & -K_c x_i \end{array} \right. . \tag{1.6.38}$$

This last expression is the wellknown state feedback law of $H^2$-optimal control.

It is useful to summarize the results obtained so far in the following Theorem.

**Theorem 1.6.3 (Infinite-Horizon $H^2$-optimal Full Information Controller)** *(a) The solution to the problem*

$$\min_{\text{causal } K(z)} \left\| \begin{bmatrix} R^{*/2}\left(P_1(z)+P_2(z)K(z)\right) \\ Q^{*/2}K(z) \end{bmatrix} \right\|_2, \tag{1.6.39}$$

*is given by*

$$K(z) = -N^{-1}(z)R_c^{-1}\left\{N^{-*}(z^{-*})P_2^*(z^{-*})R^cP_1(z)\right\}_+, \tag{1.6.40}$$

*where $N(z)$ is found from the dual canonical spectral factorization*

$$Q^c + P_2^*(z^{-*})R^cP_2(z) = N^*(z^{-*})R_cN(z),$$

*with $N(z)$ causal and causally invertible, and $N(\infty) = I_{m_1}$.*

*When $P_1(z)$ and $P_2(z)$ have state-space structure,*

$$\begin{bmatrix} P_1(z) & P_2(z) \end{bmatrix} = L(zI-F)^{-1}\begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

*with $\{F, G_2(Q^c)^{1/2}\}$ stabilizable and $\{F,(R^c)^{*/2}L\}$ detectable, then*

$$K(z) = -R_c^{-1}G_2^*P^cG_1 - K_c(zI-F_c)^{-1}(I-G_2R_c^{-1}G_2^*P^c)G_1, \tag{1.6.41}$$

*where $K_c = R_c^{-1}G_2^*P^cF$, $R_c = Q^c + G_2^*P^cG_2$, and $P^c$ the unique positive semidefinite solution of the DARE,*

$$P^c = F^*PF + L^*R^cL - K_c^*R_cK_c.$$

*In this case, the control signal, $u_i$, is given by*

$$u_i = -K_cx_i - R_c^{-1}G_2^*P^cG_1w_i, \tag{1.6.42}$$

*where $x_i$ is the state variable satisfying, $x_{i+1} = Fx_i + G_1w_i + G_2u_i$.*
*(b) The solution to the problem*

$$\min_{\text{strictly causal } K(z)} \left\| \begin{bmatrix} R^{*/2}\left(P_1(z)+P_2(z)K(z)\right) \\ Q^{*/2}K(z) \end{bmatrix} \right\|_2, \tag{1.6.43}$$

*is given by*

$$K(z) = N^{-1}(z)R_c^{-1}\left\{N^{-*}(z^{-*})P_2^*(z^{-*})R^cP_1(z)\right\}_{s.c.}. \tag{1.6.44}$$

*When $P_1(z)$ and $P_2(z)$ have the state-space structure of part (a), then the $H^2$-optimal control signal is given by the state feedback law,*

$$u_i = -K_c x_i, \tag{1.6.45}$$

*where $K_c$ is as in part (a).*

### Transfer Matrices - Measurement Feedback Case

In the measurement feedback case, the underlying transfer operators have transfer matrix representations of the form,

$$\begin{cases} s(z) &= P_{11}(z)w(z) + P_{12}(z)u(z) \\ y(z) &= P_{21}(z)w(z) + P_{22}(z)u(z) + v(z) \\ u(z) &= K(z)y(z) \end{cases} . \tag{1.6.46}$$

We shall once more assume that the cost function (whose expected value is to be minimized) is given by

$$s_i^* R^c s_i + u_i^* Q^c u, \tag{1.6.47}$$

and that the $\{w_i, v_i\}$ are zero-mean stationary stochastic processes such that

$$E \begin{bmatrix} w_i \\ v_i \end{bmatrix} \begin{bmatrix} w_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \delta_{ij}. \tag{1.6.48}$$

The $H^2$-optimal measurement feedback controller can now be found using the recipe of Theorem 1.6.2. However, for brevity we shall not do so here, since the solution is simply the restatement of Theorem 1.6.2 when transfer operators are replaced by transfer matrices. Instead we shall focus on the case where the $P_{ij}(z)$ have state-space structure of the form

$$\begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} L \\ H \end{bmatrix} (zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix}. \tag{1.6.49}$$

As we have just seen, the full information controller is given by

$$u_{f,i} = -K_c x_i - R_c^{-1} G_2^* P^c G_1 w_i.$$

Now according to Theorem 1.6.2 the $H^2$-optimal measurement feedback control signal is simply the $H^2$ estimate of $u_{f,i}$ given the observations, $\{y_j, j \leq i\}$. Thus,

$$u_i = -K_c \hat{x}_{i|i} - R_c^{-1} G_2^* P^c G_1 \hat{w}_{i|i}, \tag{1.6.50}$$

where $\hat{x}_{i|i}$ and $\hat{w}_{i|i}$ denote the linear least-mean-squares estimates of $x_i$ and $w_i$ given $\{y_j, j \leq i\}$.[21] But since the $\{y_j, j \leq i\}$ are independent of $w_i$ (since $P_{21}(z) = H(zI - F)^{-1}G_1$ is strictly causal and the $\{w_j\}$ are white), we have $\hat{w}_{i|i} = 0$, so that

$$u_i = -K_c \hat{x}_{i|i}. \tag{1.6.51}$$

Since the observations, $y_i$, satisfy the state-space model,

$$\begin{cases} x_{i+1} &= F x_i + G_1 w_i + G_2 u_i \\ y_i &= H x_i + v_i \end{cases} \tag{1.6.52}$$

the state estimates can be readily found from the Wiener filter recursions (see Theorem 1.3.2),

$$\begin{cases} \hat{x}_{i+1} &= (F - K_p H)\hat{x}_i + K_p y_i + G_2 u_i \\ \hat{x}_{i|i} &= (I - PH^* R_e^{-1} H)\hat{x}_i + PH^* R_e^{-1} y_i \end{cases}, \tag{1.6.53}$$

where $K_p = FPH^* R_e^{-1}$, $R_e = R + HPH^*$, $P$ is the unique positive semidefinite solution to the DARE,

$$P = FPF^* + G_1 Q G_1^* - K_p R_e K_p^*, \tag{1.6.54}$$

and where we have used the fact that at time $i$, the $\{u_j, j < i\}$ are known. Using, $u_i = -K_c \hat{x}_{i|i}$ The above state-space model may be rewritten as

$$\begin{cases} \hat{x}_{i+1} &= \left(F - K_p H - G_2 K_c (I - PH^* R_e^{-1} H)\right) \hat{x}_i + (K_p - G_2 K_c P H^* R_e^{-1}) y_i \\ u_i &= -K_c (I - PH^* R_e^{-1} H)\hat{x}_i - K_c P H^* R_e^{-1} y_i \end{cases}, \tag{1.6.55}$$

from which we infer that optimal controller is

$$K(z) = -K_c \left[ PH^* R_e^{-1} + (I - PH^* R_e^{-1} H)(zI - F_m)^{-1}(K_p - G_2 K_c P H^* R_e^{-1}) \right], \tag{1.6.56}$$

[21]Note that we have used the fact that in $H^2$ estimation the best estimate of any linear combination of the state is simply that linear combination of the best state estimate. We saw that this was not the case in $H^\infty$ estimation: a fact that will have certain ramifications for $H^\infty$ control.

where we have defined, $F_m \triangleq F - K_pH - G_2K_c(I - PH^*R_e^{-1}H)$.

A simpler expression for the controller results if we insist on a strictly causal $K(z)$, in which case,

$$u_i = -K_c\hat{x}_i, \tag{1.6.57}$$

where $\hat{x}_i$ is the $H^2$-optimal estimate of $x_i$ given $\{y_j, j < i\}$ and satisfies the predicted form of the Wiener filter,

$$\hat{x}_{i+1} = (F - K_pH)\hat{x}_i + K_py_i + G_2u_i = (F - K_pH - G_2K_c)\hat{x}_i + K_py_i. \tag{1.6.58}$$

In this case, $K(z)$, is given by

$$K(z) = -K_c(zI - F + K_pH + G_2K_c)^{-1}K_p. \tag{1.6.59}$$

We summarize the results obtained in the following theorem.

**Theorem 1.6.4 (Infinite-Horizon $H^2$-optimal Measurement Feedback Controller)** *Consider the state-space model*

$$\begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} L \\ H \end{bmatrix} (zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

*with $\{F, G_1Q^{1/2}\}$ and $\{F, G_2(Q^c)^{1/2}\}$ stabilizable and $\{F, H\}$ and $\{F, (R^c)^{*/2}L\}$ detectable, and consider the transfer matrix,*

$$T_K = \begin{bmatrix} (R^c)^{*/2}\left(P_{11} + P_{12}K\left(I - P_{22}K\right)^{-1}P_{21}\right)Q^{1/2} & (R^c)^{*/2}\left(P_{12}K\left(I - P_{22}K\right)^{-1}\right)R^{1/2} \\ (Q^c)^{*/2}\left(K\left(I - P_{22}K\right)^{-1}P_{21}\right)Q^{1/2} & (Q^c)^{*/2}\left(K\left(I - P_{22}K\right)^{-1}\right)R^{1/2} \end{bmatrix},$$

*where for notational simplicity we have suppressed the dependence of the transfer matrices on $z$.*

*(a) The solution to the problem*

$$\min_{\text{causal } K(z)} \|T_K(z)\|_2$$

*is given by*

$$K(z) = -K_c\left[PH^*R_e^{-1} + (I - PH^*R_e^{-1}H)(zI - F_m)^{-1}(K_p - G_2K_cPH^*R_e^{-1})\right],$$

*where* $F_m = F - K_pH - G_2K_c(I - PH^*R_e^{-1}H)$, $K_p = FPH^*R_e^{-1}$, $R_e = R + HPH^*$, $K_c = R_c^{-1}G_2^*P^cF$, $R_c = Q^c + G_2^*P^cG_2$, *and where* $P$ *and* $P^c$ *are the unique positive semidefinite solutions to the DARE's,*

$$\left\{\begin{array}{rcl} P & = & FPF^* + G_1QG_1^* - K_pR_eK_p^* \\ P^c & = & F^*P^cF + L^*R^cL - K_c^*R_cK_c \end{array}\right. .$$

*In particular, the control signal is given by*

$$u_i = -K_c\left[(I - PH^*R_e^{-1}H)\hat{x}_i + PH^*R_e^{-1}y_i\right],$$

*where*

$$\hat{x}_{i+1} = (F - K_pH)\hat{x}_i + K_py_i + G_2u_i.$$

*(b) The solution to the problem*

$$\min_{\text{strictly causal } K(z)} \|T_K(z)\|_2$$

*is given by*

$$K(z) = -K_c(zI - F + K_pH + G_2K_c)^{-1}K_p,$$

*where* $K_p$ *and* $K_c$ *are as in part (a). In particular, the control signal is given by*

$$u_i = -K_c\hat{x}_i,$$

*where*

$$\hat{x}_{i+1} = (F - K_pH)\hat{x}_i + K_py_i + G_2u_i.$$

### State-Space Models

Consider now the finite horizon case and suppose that the transfer operators have (possibly time-varying) state-space models which for the full information problem we may write as

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_ix_i + G_{1,i}w_i + G_{2,1}u_i \\ s_i & = & L_ix_i \end{array}\right. , x_0, \quad 0 \le i \le N \tag{1.6.60}$$

and for the measurement feedback problem,

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \\ y_i & = & H_i x_i + v_i \end{array}\right. , x_0, \quad 0 \leq i \leq N. \tag{1.6.61}$$

Moreover, it is assumed that $x_0$ and the $\{u_i\}$ and $\{v_i\}$ are zero-mean uncorrelated random variables with known covariance matrices

$$E \begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix} \begin{bmatrix} x_0^* & u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i \delta_{ij} & 0 \\ 0 & 0 & R_i \delta_{ij} \end{bmatrix}, \tag{1.6.62}$$

and that the cost function whose expected value is to be minimized is given by

$$\sum_{i=0}^{N} s_i^* R_i^c s_i + \sum_{i=0}^{N} u_i^* Q_i^c u_i + x_{N+1}^* P_{N+1}^c x_{N+1}, \tag{1.6.63}$$

where $R_i^c \geq 0$, $Q_i^c \geq 0$ and $P_{N+1}^c \geq 0$ are given weighting matrices.

We shall not give the finite horizon full information and measurement feedback controllers here since they are very similar to the solutions given by Theorems 1.6.3 and 1.6.4. Essentially, the only difference is that the DARE's for $P$ and $P^c$ are replaced by Riccati recursions for the time-varying matrices, $P_i$ and $P_i^c$, *i.e.*,

$$P_{i+1} = F_i P_i F_i^* + G_{1,i} Q_i G_{1.i}^* - K_{p,i} R_{e,i} K_{p,i}^*, \quad P_0 = \Pi_0 \tag{1.6.64}$$

where $K_{p,i} = F_i P_i H_i^* R_{e,i}^{-1}$ and $R_{e,i} = R_i + H_i P_i H_i^*$, and

$$P_i^c = F_i^* P_{i+1}^c F_i + L_i^* R_i^c L_i - K_{c,i}^* R_{c,i} K_{c,i}, \quad P_{N+1}^c \tag{1.6.65}$$

where $K_{c,i} = R_{c,i}^{-1} G_{2,i}^* P_{i+1}^c F$ and $R_{c,i} = Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i}$.

We end this section with two comments. First, as we shall in Chapter 2, the above Riccati recursions recursively perform the canonical and dual canonical factorizations of the positive definite Gramians, $I + \mathcal{P}_{21}\mathcal{P}_{21}^*$ and $I + \mathcal{P}_{12}^*\mathcal{P}_{12}$. Second, when the system matrices are constant and as time progresses to infinity, it is an issue whether or not the Riccati recursion solutions converge to the unique positive semidefinite solutions of their corresponding DARE's (in other words, whether or not the finite horizon controllers converge to their infinite horizon counterparts). This issue will be taken up in Chapter 8.

### 1.6.5 The Question of Robustness

As we have just seen, under suitable stochastic assumptions, such as (jointly Gaussian) uncorrelated disturbances with known first and second order statistics, $H^2$-optimal controllers have certain desirable optimality properties, namely that they minimize the expected value of a certain quadratic cost.

In practice, however, we may not always know the statistics of the disturbances. We also often do not know the exact models that generate our signals of interest. Therefore, as was the case with $H^2$ estimation, we cannot always guarantee the validity of the assumptions required of $H^2$ controllers. The natural question that arises in this regard is what the performance of such controllers will be if the assumptions on the disturbances are violated, or if there are modeling errors in our model so that the disturbances must include the modeling errors? In other words,

- *is it possible that* **small** *disturbances and modeling errors may lead to* **large** *costs?*

Intuitively, a non-robust controller is one for which the above is true, *i.e.*, one for which small disturbances may force us to incur large costs, whereas a robust controller is one for which small disturbances always result in small costs.

In the $H^\infty$ framework to robust control the costs (both the objective cost and the cost associated with the disturbances) are quadratic[22] forms constructed from the signals, and the goal is to come up with controllers that minimize (or in the suboptimal case, bound) the maximum gain from the disturbance cost to the objective cost. This will guarantee that if the disturbances are small, *i.e.*, if their associated cost is small, then the objective cost will be as small as possible, *no matter what the disturbances are.* The robustness of $H^\infty$ controllers, with respect to modeling errors and disturbance variation, follows from the fact that the maximum gain is minimized over *all possible* disturbances. However, since they make no assumption about the disturbances and have to accommodate for all conceivable ones, they may be over-conservative.

Although the aforementioned motivation for $H^\infty$ control is an important one,

[22]Other approaches to robust control differ in the definition of the cost. In $l_1$ control the cost used is related to the peak value of the signals, [DP87] and [DDB95].

historically, the main motivation for its introduction was somewhat different and is related to what is now called the robust stabilization problem. [See the pioneering paper [Zam81] for the initial motivation of $H^\infty$ control and the textbooks [Fra87], [GL95] and [ZDG96] for further properties and related topics.] To understand the robust stabilization problem[23] consider the feedback structure of Fig. 1.5.

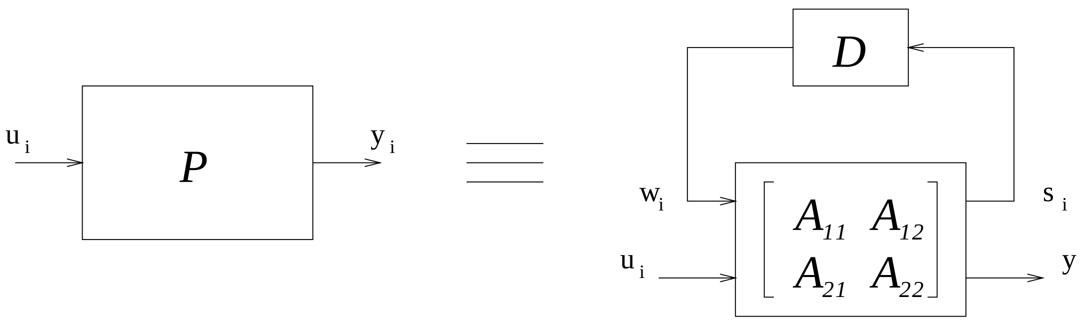


Figure 1.5: A plant with modeling error.

Suppose that we have a plant, $\mathcal{P}$, that can be represented by a known nominal plant, $\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix}$, along with an *unknown* plant, $D$, that are connected in a feedback form. In this setting $\mathcal{A}$ represents that part of the plant model that we are aware of, and $\mathcal{D}$ represents the modeling errors. Although the uncertainty model, $\mathcal{D}$, is unknown we shall assume that it is 'small' (in a sense to be defined shortly).

As depicted in Fig. 1.6, the objective here is to design a controller, $\mathcal{K}$, that stabilizes $\mathcal{P}$ (for all small uncertainty $\mathcal{D}$). Since we know the nominal plant, we can always design a controller that stabilizes it (indeed our ultimate design must also stabilize $\mathcal{A}$ since it corresponds to the special case, $\mathcal{D} = 0$), say, an $H^2$ controller. However, it is not in general clear how robust such a control strategy would be to the uncertainty, $\mathcal{D}$. In other words, is it possible that 'small' modeling errors (which in practice are inevitable) may unstabilize the composite plant?

To answer this question, let us persist with some choice of controller, $\mathcal{K}$, and represent the composite plant as the feedback connection of the uncertainty, $\mathcal{D}$, and

[23]See also [San78] for what seems to be the first analysis of this problem.

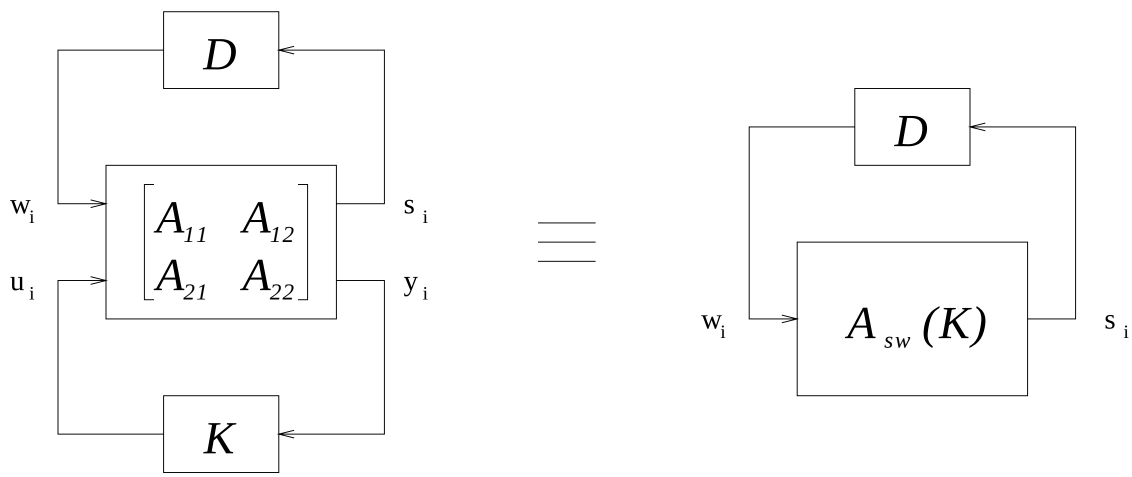


Figure 1.6: The robust stabilization problem.

the known plant,

$$\mathcal{A}_{sw}(\mathcal{K}) = \mathcal{A}_{11} + \mathcal{A}_{12}\mathcal{K}(I - \mathcal{A}_{22}\mathcal{K})^{-1}\mathcal{A}_{21}. \tag{1.6.66}$$

Now the composite plant will be input-output stable in some given norm, $\|\cdot\|$, if, according to the *small gain theorem* (see *e.g.*, [Wil71a], [DV75]), we have

$$\|\mathcal{D}\mathcal{A}_{sw}(\mathcal{K})\| < 1. \tag{1.6.67}$$

But if we choose a submultiplicative norm, such as the $H^\infty$ norm,[24] $\|\cdot\|_\infty$, the composite plant will be stable if,

$$\|\mathcal{D}\|_\infty \cdot \|\mathcal{A}_{sw}(\mathcal{K})\|_\infty < 1. \tag{1.6.68}$$

Suppose now that we know that $\mathcal{D}$ is small in the sense that

$$\|\mathcal{D}\|_\infty < \epsilon. \tag{1.6.69}$$

Then we will have stability if,

$$\|\mathcal{A}_{sw}(\mathcal{K})\|_\infty < \frac{1}{\epsilon} \triangleq \gamma. \tag{1.6.70}$$

Indeed, we even have the following stronger statement. If (1.6.70) is not satisfied, then there exists some norm-bounded uncertainty, $\mathcal{D}$ with $\|\mathcal{D}\|_\infty < \epsilon$, such that

[24] We could also use other submultiplicative norms such as the $l_1$ norm.

the composite plant is unstable. Therefore condition, (1.6.70) is the necessary and sufficient condition for input-output stability of the composite plant for all model uncertainties, $\|\mathcal{D}\|_\infty < \epsilon$.

Thus we are lead to the design of a controller that bounds the $H^\infty$ norm from the exogenous input $w$ to the regulated output $s$ for the nominal plant. We will study the $H^\infty$ control problem in a more general setting (one which subsumes the above problem) that is motivated by bounding the maximum gain from the disturbance cost to the objective cost.

## 1.7 The $H^\infty$ Approach

In this section we briefly describe the $H^\infty$ approach to robust control. [For alternative presentations and derivations see the textbooks [Fra87], [BB95], [GL95], [ZDG96] and the references therein.]

Returning to the control problems of Sec. 1.5, we recall that a useful representation for any control strategy $\mathcal{K}$ is, in the full information problem, the transfer operator,

$$\mathcal{T}^c_{\mathcal{K}} = \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix},$$

that maps the exogenous input $\{w_j\}$ to the regulated and control signals $\{s_j\}$ and $\{u_j\}$, and, in the measurement feedback problem, the transfer operator,

$$\mathcal{T}^c_{\mathcal{K}} = \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix},$$

that maps the exogenous inputs $\{w_j\}$ and $\{v_j\}$ to the regulated and control signals $\{s_j\}$ and $\{u_j\}$.

In either case, for any choice of controller $\mathcal{K}$, and for any exogenous input sequence, we can readily compute the energy gain from the exogenous inputs to the regulated and control signals. For example in the measurement feedback problem,

$$\frac{\|s\|_2^2 + \|u\|_2^2}{\|w\|_2^2 + \|v\|_2^2} = \frac{\left\| \mathcal{T}^c_{\mathcal{K}} \begin{bmatrix} w \\ v \end{bmatrix} \right\|_2^2}{\|w\|_2^2 + \|v\|_2^2}. \tag{1.7.1}$$

Clearly, the ratio in (1.4.1) depends on the particular choice of the input disturbances $w$ and $v$. As in the $H^\infty$ estimation problem, to remove this dependency, and to ensure robustness with respect to disturbance variation, we consider the largest energy gain in (1.7.1) over all possible $w$ and $v$, *i.e.*, the $H^\infty$ norm of a transfer operator $\mathcal{T}_\mathcal{K}^c$. This leads us to the following problem.

**Problem 1.7.1 (Optimal $H^\infty$ Control Problems)** *(a)* Full Information Problem: *Find a causal controller $\mathcal{K}$ that minimizes the $H^\infty$ norm of the transfer operator $\mathcal{T}_\mathcal{K}^c$ that maps the exogenous input $\{w_j\}$ to the regulated and control signals, $\{s_j\}$ and $\{u_j\}$,* i.e. , *find a causal $\mathcal{K}$ that satisfies,*

$$\inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}^c\|_\infty = \inf_\mathcal{K} \left\| \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix} \right\|_\infty . \tag{1.7.2}$$

*Moreover find the resulting $\gamma_{opt} = \inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}^c\|_\infty$.*

*(b)* Measurement Feedback Problem: *Find a causal controller $\mathcal{K}$ that minimizes the $H^\infty$ norm of the transfer operator $\mathcal{T}_\mathcal{K}^c$ that maps the exogenous inputs $\{w_j\}$ and $\{v_j\}$ to the regulated and control signals, $\{s_j\}$ and $\{u_j\}$,* i.e. , *find a causal $\mathcal{K}$ that satisfies,*

$$\inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}^c\|_\infty = \inf_\mathcal{K} \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I-\mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix} \right\|_\infty . \tag{1.7.3}$$

*Moreover find the resulting $\gamma_{opt} = \inf_\mathcal{K} \|\mathcal{T}_\mathcal{K}^c\|_\infty$.*

[Note, once more, that in the above problem statement we are purposefully ambiguous as to whether we are considering the finite horizon or (semi)infinite horizon case.]

The minimax nature of $H^\infty$ optimal controllers is evident from (1.7.2) and (1.7.3). Thus the $H^\infty$ control problem can be regarded as a game where nature (the opponent) has access to the unknown exogenous inputs $w$ and $v$ and we have choice of the causal controller $\mathcal{K}$. Note, moreover, that although $H^\infty$ optimal controllers are robust with respect to disturbance variation they may be over conservative (as no assumption is made regarding the disturbances).

There are very few case where a closed-form solution to the optimal $H^\infty$ problem of Prob. 1.7.1 can be found, and in general one resorts to the following suboptimal solution.

**Problem 1.7.2 (Suboptimal $H^\infty$ Control Problems)** *(a)* Full Information Problem: *Given a $\gamma > 0$, find a causal estimator $\mathcal{K}$ that guarantees*

$$\|\mathcal{T}^c_{\mathcal{K}}\|_\infty = \left\| \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix} \right\|_\infty < \gamma. \tag{1.7.4}$$

*This clearly requires checking whether $\gamma$ is an achievable bound.*

*(b)* Measurement Feedback Problem: *Given a $\gamma > 0$, find a causal estimator $\mathcal{K}$ that guarantees*

$$\|\mathcal{T}^c_{\mathcal{K}}\|_\infty = \left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix} \right\|_\infty < \gamma. \tag{1.7.5}$$

*This clearly requires checking whether $\gamma$ is an achievable bound.*

### 1.7.1 The Full Information Solution

The solution to the full information $H^\infty$ control problem is essentially the dual to the $H^\infty$ estimation problem of Theorem 1.4.1 and is given below.

**Theorem 1.7.1 ($H^\infty$ Suboptimal Full Information Controllers)** *A causal controller, $\mathcal{K}$ that achieves*

$$\left\| \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2\mathcal{K} \\ \mathcal{K} \end{bmatrix} \right\|_\infty < \gamma$$

*exists if, and only if, there exists a dual canonical factorization of the form*

$$\begin{bmatrix} I + \mathcal{P}_2^*\mathcal{P}_2 & \mathcal{P}_2^*\mathcal{P}_1 \\ \mathcal{P}_1^*\mathcal{P}_2 & -\gamma^2 I + \mathcal{P}_1^*\mathcal{P}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix},$$

*with $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ and $\mathcal{L}_{11}$ causal and causally invertible, and $\mathcal{L}_{21}$ strictly causal. If this is the case, then all possible $H^\infty$ estimators of level $\gamma$ are given by*

$$\mathcal{K} = (\mathcal{L}_{11} - \mathcal{Q}\mathcal{L}_{21})^{-1}(\mathcal{Q}\mathcal{L}_{22} - \mathcal{L}_{12}), \tag{1.7.6}$$

*where $\mathcal{Q}$ is any causal and strictly contractive operator. An important choice results from taking $\mathcal{Q} = 0$, so that*

$$\mathcal{K}_{cen} = -\mathcal{L}_{11}^{-1}\mathcal{L}_{12}, \tag{1.7.7}$$

*which is the so-called "central" controller.*

**Proof:** The proof is the dual to the proof of Theorem 1.4.1 and will not be repeated here.

∎

Comparing the results of Theorems 1.4.1 and 1.7.1 reveals the duality between $H^\infty$ estimation and $H^\infty$ full information control. We shall not comment on this duality here (other than, perhaps, noting the replacement of an indefinite canonical factorization with a dual one), and shall instead close this section with a remark on the maximum entropy property of the central full information controller.

**Corollary 1.7.1 (Maximum Entropy Property of Central Solution)** *The solution to the problem*

$$\max_{\text{causal } \mathcal{K}} \log \det \left[\gamma^2 I - \mathcal{T}_{\mathcal{K}}^{c*}\mathcal{T}_{\mathcal{K}}^{c}\right] \tag{1.7.8}$$

*is given by the central controller,*

$$\mathcal{K}_{cen} = -\mathcal{L}_{11}^{-1}\mathcal{L}_{12}. \tag{1.7.9}$$

### 1.7.2 The Measurement Feedback Solution

The solution to the $H^\infty$ measurement feedback control problem is given below and, as mentioned earlier, involves a certain separation principle. The separation principle for $H^\infty$ control was first derived in [DGKF89] and is essentially the same as the separation principle of risk-sensitive control (see [Whi90]), although the form given below somewhat differs from those of [DGKF89] and [Whi90]. We should also remark that the $H^\infty$ separation principle is different from its $H^2$ counterpart since the $H^\infty$ full information control problem, and the $H^\infty$ estimation problem, that the $H^\infty$ measurement feedback problem breaks into are no longer decoupled. We will have more to say about this in a moment. We also note that, as in the $H^2$ case, we can no longer guarantee the stability of the controller. Moreover, for notational simplicity we have taken $\gamma = 1$.

**Theorem 1.7.2 ($H^\infty$ Suboptimal Measurement Feedback Controllers)** *A causal controller $\mathcal{K}$ that achieves*

$$\left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \\ \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} & \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1} \end{bmatrix} \right\|_\infty < 1 \tag{1.7.10}$$

*exists if, and only if, the following two conditions are satisfied.*

*(i) The full information control problem,*

$$\left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}_f \\ \mathcal{K}_f \end{bmatrix} \right\|_\infty < 1 \tag{1.7.11}$$

*has a causal solution, $\mathcal{K}_f$. In other words, there exists the dual canonical factorization,*

$$\begin{bmatrix} I + \mathcal{P}_{12}^*\mathcal{P}_{12} & \mathcal{P}_{12}^*\mathcal{P}_{11} \\ \mathcal{P}_{11}^*\mathcal{P}_{12} & -I + \mathcal{P}_{11}^*\mathcal{P}_{11} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}, \tag{1.7.12}$$

*with $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ and $\mathcal{L}_{11}$ causal and causally invertible, and $\mathcal{L}_{21}$ strictly causal.*

*(ii) The estimation problem,*

$$\left\| \begin{bmatrix} \mathcal{L}_{12}\mathcal{L}_{22}^{-1} - \mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} & -\mathcal{K}_e \end{bmatrix} \right\|_\infty < 1 \tag{1.7.13}$$

*has a causal solution, $\mathcal{K}_e$, where the $\mathcal{L}_{ij}$ are as in part (i). In other words, there exists the canonical factorization,*

$$\begin{bmatrix} I + \mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{22}^{-*}\mathcal{P}_{21}^* & \mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{22}^{-*}\mathcal{L}_{12}^* \\ \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{22}^{-*}\mathcal{P}_{21}^* & -I + \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{22}^{-*}\mathcal{L}_{12}^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11}^e & \mathcal{L}_{12}^e \\ \mathcal{L}_{21}^e & \mathcal{L}_{22}^e \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^{e*} & \mathcal{L}_{21}^{e*} \\ \mathcal{L}_{12}^{e*} & \mathcal{L}_{22}^{e*} \end{bmatrix} \tag{1.7.14}$$

*with $\begin{bmatrix} \mathcal{L}_{11}^e & \mathcal{L}_{12}^e \\ \mathcal{L}_{21}^e & \mathcal{L}_{22}^e \end{bmatrix}$ and $\mathcal{L}_{11}^e$ causal and causally invertible, and $\mathcal{L}_{12}^e$ strictly causal.*

*If this is the case, then all possible measurement feedback $H^\infty$ controllers are given by,*

$$\mathcal{K} = (I + \mathcal{Q}\mathcal{P}_{22})^{-1}\mathcal{Q}, \tag{1.7.15}$$

*where*

$$\mathcal{Q} = -\mathcal{K}_e \left[\mathcal{L}_{11} - (\mathcal{L}_{12} - \mathcal{K}_e \mathcal{P}_{21})\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\right]^{-1}, \tag{1.7.16}$$

*and $\mathcal{K}_e$ is the solution to the estimation problem (1.7.13) which is parametrized by*

$$\mathcal{K}_e = (\mathcal{L}_{22}^e \mathcal{S} - \mathcal{L}_{21}^e)(\mathcal{L}_{11}^e - \mathcal{L}_{12}^e \mathcal{S})^{-1}, \tag{1.7.17}$$

*with $\mathcal{S}$ any causal and strictly contractive operator. The so-called central controller, $\mathcal{K}_{cen}$ results from the choice $\mathcal{S} = 0$ and corresponds to $\mathcal{K}_{e,cen} = -\mathcal{L}_{21}^e(\mathcal{L}_{11}^e)^{-1}$.*

**Proof:** Recall the defining relations

$$\begin{cases} s &= \mathcal{P}_{11} w + \mathcal{P}_{12} u \\ y &= \mathcal{P}_{21} w + \mathcal{P}_{22} u + v \\ u &= \mathcal{K} y \end{cases}$$

so that we may write, $u = \mathcal{Q}\mathcal{P}_{21} u + \mathcal{Q} v$, with $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$. Now the condition (1.7.10) is satisfied if, and only if, for all $u$ and $v$ in $h^2$,

$$\frac{\|s\|_2^2 + \|u\|_2^2}{\|w\|_2^2 + \|v\|_2^2} < 1 \quad \Longleftrightarrow \quad \|s\|_2^2 + \|u\|_2^2 - \|w\|_2^2 - \|v\|_2^2 < 0. \tag{1.7.18}$$

Defining, $J \triangleq \|s\|_2^2 + \|u\|_2^2 - \|w\|_2^2 - \|v\|_2^2$, we may write,

$$\begin{aligned} J &= \|\mathcal{P}_{11} w + \mathcal{P}_{12} u\|_2^2 + \|u\|_2^2 - \|w\|_2^2 - \|v\|_2^2 \\ &= \begin{bmatrix} u^* & w^* \end{bmatrix} \begin{bmatrix} I + \mathcal{P}_{12}^* \mathcal{P}_{12} & \mathcal{P}_{12}^* \mathcal{P}_{11} \\ \mathcal{P}_{11}^* \mathcal{P}_{12} & -\gamma^2 I + \mathcal{P}_{11}^* \mathcal{P}_{11} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} - \|v\|_2^2 \\ &= \begin{bmatrix} u^* & w^* \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} - \|v\|_2^2 \\ &= \|\mathcal{L}_{11} u + \mathcal{L}_{12} w\|_2^2 - \|\mathcal{L}_{21} u + \mathcal{L}_{22} w\|_2^2 - \|v\|_2^2 < 0, \end{aligned}$$

where in the third step we have used the obvious fact that if a measurement feedback controller exists, if, and only if, a full information one does, in which case dual canonical factorization can be performed. But this last inequality implies,

$$\frac{\|\mathcal{L}_{11} u + \mathcal{L}_{12} w\|_2^2}{\|\mathcal{L}_{21} u + \mathcal{L}_{22} w\|_2^2 + \|v\|_2^2} < 1.$$

We can now conclude that a measurement feedback controller exists if, and only if, $\mathcal{K}$ can be chosen such that,

$$\left\| \mathcal{T}_{\mathcal{K}}^{ba} \right\| < 1, \tag{1.7.19}$$

where $\mathcal{T}_{\mathcal{K}}^{ba}$ is the transfer operator that maps the signals $b$ and $v$ to the signal $a$, *i.e.*,

$$\mathcal{T}_{\mathcal{K}}^{ba} : \begin{bmatrix} b \\ v \end{bmatrix} \to a, \tag{1.7.20}$$

where we have defined,

$$\left\{ \begin{array}{lcl} a & = & \mathcal{L}_{11} u + \mathcal{L}_{12} w \\ b & = & \mathcal{L}_{21} u + \mathcal{L}_{22} w \end{array} \right. \tag{1.7.21}$$

Let us now identify $\mathcal{T}_{\mathcal{K}}^{ba}$. To this end, note that

$$\begin{aligned} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} \mathcal{Q}\mathcal{P}_{21} & \mathcal{Q} \\ I & 0 \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{12} & \mathcal{L}_{11}\mathcal{Q} \\ \mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22} & \mathcal{L}_{21}\mathcal{Q} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}. \end{aligned}$$

Solving for $w$ from the second of the above set of equations yields,

$$w = (\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22})^{-1} b - (\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22})^{-1} \mathcal{L}_{21}\mathcal{Q} v.$$

Plugging the above expression into the equation $a = (\mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{12})w + \mathcal{L}_{11}\mathcal{Q}v$ allows us to obtain

$$\mathcal{T}_{\mathcal{K}}^{ba} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_1 \end{bmatrix}, \tag{1.7.22}$$

where the entries of $\mathcal{T}_{\mathcal{K}}^{ba}$ are given by

$$\left\{ \begin{array}{lcl} \mathcal{A}_1 & = & (\mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{12})(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22})^{-1} \\ \mathcal{A}_2 & = & -(\mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{12})(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21} + \mathcal{L}_{22})^{-1}\mathcal{L}_{21}\mathcal{Q} + \mathcal{L}_{11}\mathcal{Q} \end{array} \right. . \tag{1.7.23}$$

As given above, the dependence of $\mathcal{T}_{\mathcal{K}}^{ba}$ on $\mathcal{Q}$ seems quite complicated and it does not appear clear how to choose $\mathcal{Q}$ (and hence $\mathcal{K}$) to make $\mathcal{T}_{\mathcal{K}}^{ba}$ a strict contraction.

However, let us persist and attempt to simplify the entries of $\mathcal{T}_{\mathcal{K}}^{ba}$. Indeed,

$$\begin{aligned}
\mathcal{A}_2 &= \mathcal{L}_{11}\mathcal{Q}\left[-\mathcal{P}_{21}(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+\mathcal{L}_{22})^{-1}\mathcal{L}_{21}\mathcal{Q}+I\right] - \mathcal{L}_{12}(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+\mathcal{L}_{22})^{-1}\mathcal{L}_{21}\mathcal{Q} \\
&= \mathcal{L}_{11}\mathcal{Q}\left[I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\right]^{-1} - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}(I+\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}\mathcal{L}_{22}^{-1})\mathcal{L}_{21}\mathcal{Q} \\
&= \mathcal{L}_{11}\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1} - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1} \\
&= (\mathcal{L}_{11}-\mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21})\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1} \\
&\triangleq -\mathcal{K}_e.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\mathcal{A}_1 &= \mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21}(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+\mathcal{L}_{22})^{-1} + \mathcal{L}_{12}(\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+\mathcal{L}_{22})^{-1} \\
&= \mathcal{L}_{11}\mathcal{Q}\mathcal{P}_{21}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} + \mathcal{L}_{12}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= \mathcal{L}_{11}\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1}\mathcal{P}_{21}\mathcal{L}_{22}^{-1} + \mathcal{L}_{12}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= -\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} + \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}(I+\mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1}\mathcal{P}_{21}\mathcal{L}_{22}^{-1} + \mathcal{L}_{12}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= -\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} + \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} + \mathcal{L}_{12}(I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= -\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} + \mathcal{L}_{12}\left[\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21}+I\right](I+\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q}\mathcal{P}_{21})^{-1}\mathcal{L}_{22}^{-1} \\
&= -\mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} + \mathcal{L}_{12}\mathcal{L}_{22}^{-1}.
\end{aligned}$$

Thus we have shown,

$$\mathcal{T}_{\mathcal{K}}^{ba} = \left[\begin{array}{cc} \mathcal{L}_{12}\mathcal{L}_{22}^{-1} - \mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} & -\mathcal{K}_e \end{array}\right]. \tag{1.7.24}$$

In other words, the measurement feedback problem has a solution if, and only if, $\mathcal{K}_e$ can be chosen such that $\mathcal{T}_{\mathcal{K}}^{ba}$ is a strict contraction. But this is simply the estimation problem (1.7.13). Finally, we note that solving $\mathcal{K}_e = (\mathcal{L}_{11} - \mathcal{L}_{12}\mathcal{L}_{22}^{-1}\mathcal{L}_{21})\mathcal{Q}(I + \mathcal{P}_{21}\mathcal{L}_{22}^{-1}\mathcal{L}_{21}\mathcal{Q})^{-1}$ for $\mathcal{Q}$, and solving $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$ for $\mathcal{K}$, yields the desired results (1.7.15) and (1.7.16).

■

A few remarks on the above result are in order. Note that the separation principle obtained states that to solve the problem we first need to solve the full information control problem,

$$\left\| \begin{bmatrix} \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}_f \\ \mathcal{K}_f \end{bmatrix} \right\|_\infty < \gamma, \tag{1.7.25}$$

and then the estimation problem,

$$\left\| \begin{bmatrix} \mathcal{L}_{12}\mathcal{L}_{22}^{-1} - \mathcal{K}_e\mathcal{P}_{21}\mathcal{L}_{22}^{-1} & -\mathcal{K}_e \end{bmatrix} \right\|_\infty < \gamma. \tag{1.7.26}$$

Note, however, that, unlike the $H^2$ case, the above two problems are coupled, since the estimation problem involves quantities that have to be found from the full information problem. As a matter of fact, not only does the solution to the estimation problem depend on the full information controller, but the actual condition for the existence of the estimator depends on it as well. [This dependence occurs through the transfer operators $\mathcal{L}_{12}$ and $\mathcal{L}_{22}$ which are in turn determined by $\mathcal{P}_{11}$ and $\mathcal{P}_{12}$.] This is unlike the $H^2$ problem where the separation principle completely decouples the full information and estimation problems. [There it was neither necessary to decide on the noise covariances before designing the full information controller, nor on the control objectives before designing the optimal estimator.]

As we shall shortly see, in the state-space case, the coupling between the full information control and estimation problems is exemplified by the fact that their corresponding DARE's (or Riccati recursions in the finite-horizon case) are coupled. This is as opposed to the $H^2$ case where the two Riccati equations are independent of one another.

Referring to the estimation problem (1.7.26) shows that the estimator $\mathcal{K}_e$ estimates the output of the system, $\mathcal{L}_{12}\mathcal{L}_{22}^{-1}$, using noisy observations of the output of the system $\mathcal{P}_{21}\mathcal{L}_{22}^{-1}$ (see Sec. 1.2). However, we shall later see that in actually trying to find the $H^\infty$ measurement feedback controller (say, for systems described by state-space models) it is much easier to keep the physical meaning of the separation in mind (such as identifying the signals $a$ and $b$ in the above proof), rather than to explicitly use the solution of Theorem 1.7.2.

Finally, it should also be noted that, in the measurement feedback case, the central solution has the maximum entropy property of

$$\max_{\text{causal } \mathcal{K}} \ \log\, \det \left[\gamma^2 I - \mathcal{T}_{\mathcal{K}}^{c*}\mathcal{T}_{\mathcal{K}}^{c}\right]. \tag{1.7.27}$$

### 1.7.3 Special Cases

The general solutions presented so far subsume both the finite and infinite horizon cases. We now briefly present these.

#### The Infinite Horizon Case

Let us first begin with the full information problem for which we have the following result.

**Theorem 1.7.3 (Infinite Horizon $H^\infty$ Full Information Controller)** *(a) A causal controller, $K(z)$, that achieves*

$$\left\| \begin{bmatrix} P_1(z) + P_2(z)K(z) \\ K(z) \end{bmatrix} \right\|_\infty < \gamma,$$

*exists if, and only if, there exists a dual canonical factorization of the form*

$$\begin{bmatrix} I_{m_2} + P_2(z)P_2^*(z^{-*}) & P_2(z)P_1^*(z^{-*}) \\ P_1(z)P_2^*(z^{-*}) & -\gamma^2 I_{m_1} + P_2(z)P_2^*(z^{-*}) \end{bmatrix} = \begin{bmatrix} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ L_{12}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{bmatrix} \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix} \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}, \tag{1.7.28}$$

*with $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ and $L_{11}(z)$ minimum phase and proper, and $L_{21}(z)$ causal and strictly proper. If this is the case, then all possible $H^\infty$ full information controllers of level $\gamma$ are given by*

$$K(z) = \left(L_{11}(z) - Q(z)L_{21}(z)\right)^{-1}\left(Q(z)L_{22}(z) - L_{12}\right)^{-1}, \tag{1.7.29}$$

*where $Q(z)$ is any causal and strictly contractive transfer matrix,* i.e., *$Q(e^{j\omega})Q^*(e^{j\omega}) < I$, $\forall \omega \in [0, 2\pi]$. The central filter results from the choice $Q(z) = 0$, so that*

$$K_{cen}(z) = -L_{11}^{-1}(z)L_{12}(z). \tag{1.7.30}$$

*(b) Assume that $P_1(z)$ and $P_2(z)$ have state-space structure,*

$$\begin{bmatrix} P_1(z) & P_2(z) \end{bmatrix} = L(zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

*with $\{F, G_2\}$ stabilizable and $\{F, L\}$ detectable. Then a solution to the full information $H^\infty$ control problem with level $\gamma$ exists if, and only if, there exists a solution to the (dual) DARE*

$$P^c = F^* P^c F + L^* L - K_c^* R_c K_c, \tag{1.7.31}$$

*with*

$$K_c = R_c^{-1} \begin{bmatrix} G_2^* \\ G_1^* \end{bmatrix} P^c F \quad \text{and} \quad R_c = \begin{bmatrix} I_{m_2} & 0 \\ 0 & -\gamma^2 I_{m_1} \end{bmatrix} + \begin{bmatrix} G_2^* \\ G_1^* \end{bmatrix} P^c \begin{bmatrix} G_2 & G_1 \end{bmatrix} \tag{1.7.32}$$

*such that*

*(i)* $F_c \triangleq F - \begin{bmatrix} G_2 & G_1 \end{bmatrix} K_c$ *is stable.*

*(ii)* $R_c$ *and* $\begin{bmatrix} I_{m_2} & 0 \\ 0 & -\gamma^2 I_{m_1} \end{bmatrix}$ *have the same inertia.*

*If this is the case, then the $L_{ij}(z)$ in the dual canonical factorization (1.7.28) are given by*

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} (I_{m_2} + G_2^* P^c G_2)^{*/2} & (I_{m_2} + G_2^* P^c G_2)^{-1/2} G_2^* P^c G_1 \\ 0 & (\gamma^2 I_{m_1} - G_1^* P^c (I + G_2 G_2^* P^c)^{-1} G_1)^{*/2} \end{bmatrix} \times$$
$$\left( \begin{bmatrix} I_{m_2} & 0 \\ 0 & I_{m_1} \end{bmatrix} + K_c (zI - F)^{-1} \begin{bmatrix} G_2 & G_1 \end{bmatrix} \right). \tag{1.7.33}$$

*In particular, defining $R_{Gc} \triangleq I_{m_2} + G_2^* P^c G_2$, we have*

$$\left\{ \begin{array}{rcl} L_{11}(z) & = & R_{Gc}^{*/2} \left[ I_{m_2} + R_{Gc}^{-1} G_2^* P^c F (zI - F)^{-1} G_2 \right] \\ L_{12}(z) & = & R_{Gc}^{*/2} \left[ R_{Gc}^{-1} G_2^* P^c G_1 + R_{Gc}^{-1} G_2^* P^c F (zI - F)^{-1} G_1 \right] \end{array} \right. , \tag{1.7.34}$$

*so that*

$$K_{cen}(z) = -R_{Gc}^{-1} G_2^* P^c G_1 - R_{Gc}^{-1} G_2^* P^c F (zI - F_2)^{-1} (I - G_2 R_{Gc}^{-1} G_2^* P^c) G_1, \tag{1.7.35}$$

*where $F_2 \triangleq F - G_2 R_{Gc}^{-1} G_2^* P^c F$.*

**Proof:** The proof is the dual of the proof of Theorem 1.4.2 and will not be repeated here.

∎

**Remarks:**

(i) It is straightforward to give state-space representations for the central controller (1.7.35). Indeed we have

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F_2 x_i + (I - G_2 R_{Gc}^{-1} G_2^* P^c) G_1 w_i \\ u_i & = & -R_{Gc}^{-1} G_2^* P^c F x_i - R_{Gc}^{-1} G_2^* P^c G_1 w_i \end{array} \right. , \tag{1.7.36}$$

where we have denoted the state variable by $x_i$, since it really is the state of the original state-space model. To see why, we can use the formulas $F_2 = F - G_2 R_{Gc}^{-1} G_2^* P^c F$ and $u_i = -R_{Gc}^{-1} G_2^* P^c F x_i - R_{Gc}^{-1} G_2^* P^c G_1 w_i$ in the above state equation to write,

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F x_i + G_1 w_i + G_2 u_i \\ u_i & = & -R_{Gc}^{-1} G_2^* P^c F x_i - R_{Gc}^{-1} G_2^* P^c G_1 w_i \end{array} \right. , \tag{1.7.37}$$

which is our desired result.

The above controller shows that the full information control signal is a function only of the current state $x_i$ and the current exogenous input, $w_i$.[25] This is somewhat different from the state-feedback law that occurs in continuous-time $H^\infty$ control. The reason is that we have insisted on causal controllers. Had we sought strictly causal controllers we would have obtained a state feedback law (for the central solution), as we now (very) briefly describe.

(ii) The solution to the problem of finding a *strictly* causal full information controller that achieves,

$$\left\| \begin{bmatrix} P_1(z) + P_2(z) K(z) \\ K(z) \end{bmatrix} \right\|_\infty < \gamma,$$

[25] The above statement is only true of the central controller. Other full information $H^\infty$ controllers, as parametrized by (1.7.29), need not be so.

is exactly the same as the solution of Theorem 1.7.3 part (a) with the exception that, in the dual canonical factorization (1.7.28), instead of having $L_{21}(z)$ being strictly proper we require that $L_{12}(z)$ be strictly proper.

When we have state-space structure, the only difference in the existence condition is that condition (ii) should be replaced by[26]

$$-\gamma^2 I_{m_1} + G_1^* P^c G_1 < 0 \quad \text{and} \quad I_{m_2} + G_2^* P^c (I - \gamma^{-2} G_1 G_1^* P^c)^{-1} G_1 > 0. \quad (1.7.38)$$

In this case, the $L_{ij}(z)$ are given by,

$$\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix} = \begin{bmatrix} (I_{m_2} + G_2^* P^c (I - \gamma^{-2} G_1 G_1^* P^c)^{-1} G_1)^{*/2} & 0 \\ G_1^* P^c G_2 (\gamma^2 I_{m_1} - G_1^* P^c G_1)^{-1/2} & (\gamma^2 I_{m_1} - G_1^* P^c G_1)^{*/2} \end{bmatrix} \times$$
$$\left( \begin{bmatrix} I_{m_2} & 0 \\ 0 & I_{m_1} \end{bmatrix} + K_c (zI - F)^{-1} \begin{bmatrix} G_2 & G_1 \end{bmatrix} \right). \quad (1.7.39)$$

The above expression allows us to find the central solution $K(z) = -L_{11}(z)^{-1} L_{12}(z)$, which turns out to be,

$$K_{cen}(z) = \tilde{K}_c (zI - F + G_2 \tilde{K}_c)^{-1} G_1, \quad (1.7.40)$$

where

$$\tilde{K}_c = (I_{m_2} + G_2^* \tilde{P}^c G_2)^{-1} G_2^* \tilde{P}^c F \quad \text{and} \quad \tilde{P}^c = P^c - P^c G_1 (-\gamma^2 I_{m_1} + G_1^* P^c G_1)^{-1} G_1^* P^c. \quad (1.7.41)$$

The above central solution can be shown to have the following state-space model,

$$\begin{cases} x_{i+1} = F x_i + G_1 w_i + G_2 u_i \\ u_i = \tilde{K}_c x_i \end{cases}, \quad (1.7.42)$$

which is, of course, a state-feedback control law.

(iii) Note, once more, the surprising similarities between the above full information $H^\infty$ controller and the $H^2$ full information controller of Theorem 1.6.3.

[26]Note that this condition is more stringent than (ii). If it is true, then $R_c$ and $I_{m_2} \oplus (-I_{m_1})$ have the same inertia, but not necessarily vice versa.

Let us now consider the measurement feedback problem[27] where the model is given by,

$$\left\{ \begin{array}{rcl} s(z) & = & P_{11}(z)w(z) + P_{12}(z)u(z) \\ y(z) & = & P_{21}(z)w(z) + P_{22}(z)u(z) + v(z) \\ u(z) & = & K(z)y(z) \end{array} \right. . \tag{1.7.43}$$

Since the solution in this case is simply a restatement of Theorem 1.7.2 where one has to replace transfer operators by their corresponding transfer matrices, we shall not repeat it here. Therefore, we shall instead focus on the case where we have state-space structure,

$$\begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} L \\ H \end{bmatrix} (zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix}, \tag{1.7.44}$$

and study the consequences of Theorem 1.7.2.

Recall from the proof (and separation principle) of Theorem 1.7.2 that the solution to the measurement feedback problem is given by that choice of controller that renders the mapping from the signals $\{b_i\}$ and $\{v_i\}$ to the signal $\{a_i\}$ to be a strict contraction, where

$$\left\{ \begin{array}{rcl} a(z) & = & L_{11}(z)u(z) + L_{12}(z)w(z) \\ b(z) & = & L_{21}(z)u(z) + L_{22}(z)w(z) \end{array} \right. , \tag{1.7.45}$$

and where the $L_{ij}(z)$ are found from the dual canonical factorization,

$$\begin{gathered} \begin{bmatrix} I_{m_2} + P_{12}(z)P_{12}^*(z^{-*}) & P_{12}(z)P_{11}^*(z^{-*}) \\ P_{11}(z)P_{12}^*(z^{-*}) & -I_{m_1} + P_{12}(z)P_{12}^*(z^{-*}) \end{bmatrix} = \\ \begin{bmatrix} L_{11}^*(z^{-*}) & L_{21}^*(z^{-*}) \\ L_{12}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{bmatrix} \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix} \begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}, \end{gathered} \tag{1.7.46}$$

with $\begin{bmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{bmatrix}$ and $L_{11}(z)$ minimum phase and proper, and $L_{21}(z)$ causal and strictly proper.

Let us now identify the $L_{ij}(z)$ (and thereby the $a(z)$ and $b(z)$) using the state-space descriptions given. Using Eq. (1.7.33) in Theorem 1.7.3 we can, after some

[27] We have, once more, for simplicity, assumed that $\gamma = 1$.

algebra, write

$$\begin{cases} L_{11}(z) &= R_{Gc}^{*/2} + R_{Gc}^{*/2} K_d (zI - F)^{-1} G_2 \\ L_{12}(z) &= R_{Gc}^{-1/2} G_2^* P^c G_1 + R_{Gc}^{*/2} K_d (zI - F)^{-1} G_1 \\ L_{21}(z) &= \Delta^{*/2} K_w (zI - F)^{-1} G_2 \\ L_{22}(z) &= \Delta^{*/2} + \Delta^{*/2} K_w (zI - F)^{-1} G_2 \end{cases}, \tag{1.7.47}$$

where we have defined

$$K_d = R_{Gc}^{-1} G_2^* P^c F \quad , \quad K_w = \Delta^{-1} G_1^* P^c (I + G_2 G_2^* P^c)^{-1} F \tag{1.7.48}$$

and

$$R_{Gc} = I_{m_2} + G_2^* P^c G_2 > 0 \quad , \quad \Delta = I_{m_1} - G_1^* P^c (I + G_2 G_2^* P^c)^{-1} G_1 > 0 \tag{1.7.49}$$

and where $P^c$ is given by the solution to the DARE (1.7.31) with the aforementioned properties. This allows us to write

$$a(z) = R_{Gc}^{*/2} K_d (zI - F)^{-1} \left[ G_1 w(z) + G_2 u(z) \right] + R_{Gc}^{-1/2} G_2^* P^c G_1 w(z) + R_{Gc}^{*/2} u(z),$$

which has state-space description,

$$\begin{cases} x_{i+1} &= F x_i + G_1 w_i + G_2 u_i \\ a_i &= R_{Gc}^{*/2} K_d x_i + R_{Gc}^{-1/2} G_2^* P^c G_1 w_i + R_{Gc}^{*/2} u_i \end{cases}, \tag{1.7.50}$$

and

$$b(z) = \Delta^{*/2} w(z) + \Delta^{*/2} K_w (zI - F)^{-1} \left[ G_1 u(z) + G_2 u(z) \right],$$

which has state-space description,

$$\begin{cases} x_{i+1} &= F x_i + G_1 w_i + G_2 u_i \\ b_i &= \Delta^{*/2} K_w x_i + \Delta^{*/2} w_i \end{cases}. \tag{1.7.51}$$

Now from the last of the above equations, we conclude that $w_i = \Delta^{-*/2} b_i - K_w x_i$. Inserting this expression into the state-space model (1.7.50) yields,

$$\begin{cases} x_{i+1} &= (F - G_1 K_w) x_i + G_1 \Delta^{-*/2} b_i + G_2 u_i \\ a_i &= (R_{Gc}^{*/2} K_d - R_{Gc}^{-1/2} G_2^* P^c G_1 K_w) x_i + R_{Gc}^{-1/2} G_2^* P^c G_1 \Delta^{-*/2} b_i + R_{Gc}^{*/2} u_i \end{cases}. \tag{1.7.52}$$

Since $a_i$ is the signal that we are trying to make small (through our choice of control signal, $u_i$), we can define the desired signal,

$$d_i \triangleq -R_{Gc}^{*/2}K_u x_i - R_{Gc}^{-1/2}G_2^* P^c G_1 \Delta^{-*/2} b_i \quad , \quad K_u \triangleq (K_d - R_{Gc}^{-1}G_2^* P^c G_1 K_w) \tag{1.7.53}$$

and gather the equations that we have to write,

$$\left\{\begin{array}{rcl} x_{i+1} & = & (F - G_1 K_w)x_i + G_1 \Delta^{-*/2} b_i + G_2 u_i \\ d_i & = & -R_{Gc}^{*/2} K_u x_i - R_{Gc}^{-1/2} G_2^* P^c G_1 \Delta^{-*/2} b_i \\ y_i & = & H x_i + v_i \end{array}\right. . \tag{1.7.54}$$

Therefore the problem of finding a causal controller that makes the mapping from $b$ and $v$ to $a$ strictly contractive is equivalent to the problem of constructing a causal estimate of $d_i$, such that the mapping from the disturbances $\{b_i, v_i\}$ to the estimation errors, $\{\tilde{d}_{i|i} \triangleq d_i - \hat{d}_{i|i}\}$, is strictly contractive. But this is now a standard estimation problem that we know how to solve.[28] Indeed, using a slight generalization of Theorem 1.4.2, the central solution is given by

$$R_{Gc}^{*/2} u_i = \hat{d}_{i|i} = -R_{Gc}^{*/2} K_u \hat{x}_{i|i}, \tag{1.7.55}$$

where $R_{Gc}^{*/2} K_u \hat{x}_{i|i}$ satisfies,

$$\left\{\begin{array}{rcl} \hat{x}_{i+1} & = & (F - G_1 K_w - K_1 H)\hat{x}_i + K_1 y_i + G_2 u_i \\ R_{Gc}^{*/2} K_u \hat{x}_{i|i} & = & R_{Gc}^{*/2} K_u (I - P H^* R_{He}^{-1} H)\hat{x}_i + R_{Gc}^{*/2} K_u P H^* R_{He}^{-1} y_i \end{array}\right. , \tag{1.7.56}$$

with $K_1 = (F - G_1 K_w) P H^* R_{He}^{-1}$ and $R_{He} = I_p + H P H^*$. Moreover, $P$ is that solution of the DARE,

$$P = (F - G_1 K_w) P (F - G_1 K_w)^* + G_1 \Delta^{-1} G_1^* - K_p R_e K_p^*, \tag{1.7.57}$$

with

$$K_p = \left\{ (F - G_1 K_w) P \left[\begin{array}{cc} H^* & K_u^* R_{Gc}^{1/2} \end{array}\right] + \left[\begin{array}{cc} G_1 \Delta^{-1} G_1^* P^c G_2 R_{G^c}^{-*/2} & 0 \end{array}\right] \right\} R_e^{-1},$$

[28] In fact, it is a slight generalization of the standard problem we have been considering so far, since here the disturbance $b_i$ also enters directly into the equation for the desired signal, $d_i = -R_{Gc}^{*/2} K_u x_i - R_{Gc}^{-1/2} G_2^* P^c G_1 \Delta^{-*/2} b_i$. Nonetheless, the problem can still be solved using the same techniques and the only difference turns out to be a small modification in the $K_p$ of the DARE.

and

$$R_e = \begin{bmatrix} I_p & 0 \\ 0 & -I_{m_2} \end{bmatrix} + \begin{bmatrix} H \\ R_{Gc}^{*/2} K_u \end{bmatrix} P \begin{bmatrix} H^* & K_u^* R_{Gc}^{1/2} \end{bmatrix},$$

that satisfies the following conditions,

(i) $F_p \triangleq F - G_1 K_w - K_p \begin{bmatrix} H \\ R_{Gc}^{*/2} K_u \end{bmatrix}$ is stable.

(ii) $R_e$ and $I_p \oplus (-I_{m_2})$ have the same inertia.

Incidentally, from Theorem 1.4.2, the existence of a $P$ with the above properties is necessary and sufficient for the existence of a solution to the measurement feedback problem.

Note also that replacing $u_i = -K_u(I - PH^* R_{He}^{-1} H)\hat{x}_i - K_u P H^* R_{He}^{-1} y_i$ into (1.7.56), we can write

$$\begin{cases} \hat{x}_{i+1} &= F_m \hat{x}_i + (K_1 - G_2 K_u P H^* R_{He}^{-1}) y_i \\ u_i &= K_u(I - PH^* R_{He}^{-1} H)\hat{x}_i - K_u P H^* R_{He}^{-1} y_i \end{cases} . \tag{1.7.58}$$

where $F_m \triangleq F - G_1 K_w - K_1 H - G_2 K_u (I - PH^* R_{He}^{-1} H)$, which is the desired system that maps the observations, $y_i$, to the control signal, $u_i$.

We can now gather the results obtained so far in the following theorem where for simplicity we have taken $\gamma = 1$. [We have also, without loss of generality, taken $Q = I$, $R = I$, $Q^c = I$ and $R^c = I$ since these weighting matrices can be absorbed into the $\{G_1, G_2, H, L\}$.]

**Theorem 1.7.4 (Infinite-Horizon $H^\infty$ Measurement Feedback Controller)** *Consider the state-space model*

$$\begin{bmatrix} P_{11}(z) & P_{12}(z) \\ P_{21}(z) & P_{22}(z) \end{bmatrix} = \begin{bmatrix} L \\ H \end{bmatrix} (zI - F)^{-1} \begin{bmatrix} G_1 & G_2 \end{bmatrix},$$

*with $\{F, G_1\}$ and $\{F, G_2\}$ stabilizable and $\{F, H\}$ and $\{F, L\}$ detectable, and the transfer matrix,*

$$T_K = \begin{bmatrix} P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21} & P_{12} K (I - P_{22} K)^{-1} \\ K (I - P_{22} K)^{-1} P_{21} & K (I - P_{22} K)^{-1} \end{bmatrix},$$

*where for notational simplicity we have suppressed the dependence of the transfer matrices on $z$.*

*Then there exists a causal controller that achieves*

$$\|T_K(z)\|_\infty < 1,$$

*if, and only if, the DARE's,*

$$\begin{cases} P^c &= F^*P^cF + L^*L - K_c^*R_cK_c \\ P &= (F-G_1K_w)P(F-G_1K_w)^* + G_1\Delta^{-1}G_1^* - K_pR_eK_p^* \end{cases}, \tag{1.7.59}$$

*where*

$$\begin{cases} K_c &= R_c^{-1}\begin{bmatrix} G_2^* \\ G_1^* \end{bmatrix} P^cF \\ R_c &= \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix} + \begin{bmatrix} G_2^* \\ G_1^* \end{bmatrix} P^c \begin{bmatrix} G_2 & G_1 \end{bmatrix} \\ K_w &= \Delta^{-1}G_1^*P^c(I+G_2G_2^*P^c)^{-1}F \\ \Delta &= I_{m_1} - G_1^*P^c(I+G_2G_2^*P^c)^{-1}G_1 \\ K_p &= \left\{(F-G_1K_w)P\begin{bmatrix} H^* & K_u^*R_{Gc}^{1/2}\end{bmatrix} + \begin{bmatrix} G_1\Delta^{-1}G_1^*P^cG_2R_{G^c}^{-*/2} & 0 \end{bmatrix}\right\} R_e^{-1} \\ R_e &= \begin{bmatrix} I_p & 0 \\ 0 & -I_{m_2} \end{bmatrix} + \begin{bmatrix} H \\ R_{Gc}^{*/2}K_u \end{bmatrix} P \begin{bmatrix} H^* & K_u^*R_{Gc}^{1/2} \end{bmatrix} \\ \begin{bmatrix} K_w \\ K_u \end{bmatrix} &= K_c \\ R_{Gc} &= I_{m_2} + G_2^*P^cG_2 \end{cases}, \tag{1.7.60}$$

*have solutions, $P^c$ and $P$, such that*

*(i)* $F_c \triangleq F - \begin{bmatrix} G_2 & G_1 \end{bmatrix} K_c$ *is stable.*

*(ii)* $R_c$ *and* $I_{m_2} \oplus (-I_{m_1})$ *have the same inertia.*

*(iii)* $F_p \triangleq F - G_1K_w - K_p \begin{bmatrix} H \\ R_{Gc}^{*/2}K_u \end{bmatrix}$ *is stable.*

*(iv)* $R_e$ *and* $I_p \oplus (-I_{m_2})$ *have the same inertia.*

*If this is the case, then the "so-called" central controller is given by*

$$K_{cen}(z) = -K_u P H^* R_{He}^{-1} - K_u(I - PH^* R_{He}^{-1} H)(zI - F_m)^{-1}(K_1 - G_2 K_u P H^* R_{He}^{-1}), \tag{1.7.61}$$

*where* $F_m = F - G_1 K_w - K_1 H - G_2 K_u(I - PH^* R_{He}^{-1} H)$*, which has the following state-space model,*

$$\left\{ \begin{array}{rcl} \hat{x}_{i+1} & = & (F - G_1 K_w - K_1 H)\hat{x}_i + K_1 y_i + G_2 u_i \\ u_i & = & -K_u(I - PH^* R_{He}^{-1} H)\hat{x}_i - K_u P H^* R_{He}^{-1} y_i \end{array} \right. . \tag{1.7.62}$$

**Remarks:**

(i) Note that, unlike the $H^2$ measurement feedback control problem of Theorem 1.6.4, the Riccati equations for $P^c$ and $P$ are coupled. Indeed, the DARE for $P$ depends on the solution of the DARE for $P^c$ (but not vice versa). Therefore the separation in measurement feedback $H^\infty$ control is not complete, *i.e.*, the estimation problem depends on the full information controller.

Through a suitable change of variables (essentially a bilinear transformation involving $P$ and $P^c$) it is possible to come up with an auxiliary variable, say $P^d$, that satisfies a DARE which is independent of $P^c$. However, the price we have to pay is a certain coupling condition (on the spectral radius of $P^c P^d$) that has to be added to the existence conditions (i)-(iv). In fact, the first solutions to the measurement feedback $H^\infty$ and risk-sensitive control problems (see [DGKF89] and [Whi90]) presented such Riccati equations and a corresponding separation principle. However, we believe that the separation principle given here is more natural,[29] and shall therefore not go into the details of defining $P^d$ and deriving its DARE (which requires lengthy algebraic manipulations, anyway).

(ii) The solutions of the various $H^\infty$ control and estimation problems presented so far require finding the *stabilizing* solution of a DARE and checking whether the solution has certain inertia properties. It turns out that there are many efficient

[29] In fact, the separation principle given here does not need to appeal to state-space models.

methods for both checking the existence of, and finding stabilizing solutions to, DARE's (see *e.g.*, [BLW91], [Meh91], [LR95] and the references therein). We shall say more about this in Chapter 8.

(iii) Proceeding in a similar vein it is also possible to derive measurement feedback $H^\infty$ controllers that are strictly causal, however, for lack of space we shall not do so here.

**State-Space Models**

Let us finally, consider the finite horizon case where we have (possibly time-varying) state-space models which, for the full information problem, we may write as

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \end{array}\right. , x_0, \quad 0 \leq i \leq N \tag{1.7.63}$$

and, for the measurement feedback problem,

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,1} u_i \\ s_i & = & L_i x_i \\ y_i & = & H_i x_i + v_i \end{array}\right. , x_0, \quad 0 \leq i \leq N. \tag{1.7.64}$$

Moreover, assume that the cost associated with the disturbances, $x_0$, $\{u_i\}$ and $\{v_i\}$ is given by

$$x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} w_i^* Q_i^{-1} w_i + \sum_{i=0}^{N} v_i^* R_i^{-1} u_i, \tag{1.7.65}$$

and that the objective cost function is

$$\sum_{i=0}^{N} s_i^* R_i^c s_i + \sum_{i=0}^{N} u_i^* Q_i^c u_i + x_{N+1}^* P_{N+1}^c x_{N+1}, \tag{1.7.66}$$

where $\Pi_0 > 0$, $Q_i > 0$, $R_i > 0$, $R_i^c \geq 0$, $Q_i^c \geq 0$ and $P_{N+1}^c \geq 0$ are given weighting matrices.

We shall not give the finite horizon full information and measurement feedback controllers here since they are very similar to the solutions given by Theorems 1.7.3

and 1.7.4. Essentially, the only difference is that the DARE's for $P$ and $P^c$ are replaced by Riccati recursions for the time-varying matrices, $P_i$ and $P_i^c$, *i.e.*,

$$\left\{\begin{array}{rcl} P_i^c & = & F_i^* P_{i+1}^c F_i + L_i^* L_i - K_{c,i}^* R_{c,i} K_{c,i} \\ P_{i+1} & = & (F_i - G_{1,i}K_{w,i})P_i(F_i - G_{1,i}K_{w,i})^* + G_{1,i}\Delta_i^{-1}G_{1,i}^* - K_{e,i}R_{e,i}K_{p,i}^* \end{array}\right. , \tag{1.7.67}$$

(initialized with $P_{N+1}^c$ and $P_0 = (\Pi_0^{-1} - P_0^c)^{-1}$) where

$$\left\{\begin{array}{rcl} K_{c,i} & = & R_{c,i}^{-1}\begin{bmatrix} G_{2,i}^* \\ G_{1,i}^* \end{bmatrix} P_{i+1}^c F_i \\ R_{c,i} & = & \begin{bmatrix} I_{m_2} & 0 \\ 0 & -I_{m_1} \end{bmatrix} + \begin{bmatrix} G_{2,i}^* \\ G_{1,i}^* \end{bmatrix} P_{i+1}^c \begin{bmatrix} G_{2,i} & G_{1,i} \end{bmatrix} \\ K_{w,i} & = & \Delta_i^{-1} G_{1,i}^* P_{i+1}^c (I + G_{2,i}G_{2,i}^* P_{i+1}^c)^{-1} F_i \\ \Delta_i & = & I_{m_1} - G_{1,i}^* P_{i+1}^c (I + G_{2,i}G_{2,i}^* P_{i+1}^c)^{-1} G_{1,i} \\ K_{p,i} & = & \left\{(F_i - G_{1,i}K_{w_i})P_i \begin{bmatrix} H_i^* & K_{u,i}^* R_{Gc,i}^{1/2} \end{bmatrix} + \begin{bmatrix} G_{1,i}\Delta_i^{-1}G_{1,i}^* P_{i+1}^c G_{2,i} R_{Gc,i}^{-*/2} & 0 \end{bmatrix}\right\} R_{e,i}^{-1} \\ R_{e,i} & = & \begin{bmatrix} I_p & 0 \\ 0 & -I_{m_2} \end{bmatrix} + \begin{bmatrix} H_i \\ R_{Gc,i}^{*/2} K_{u,i} \end{bmatrix} P_i \begin{bmatrix} H_i^* & K_{u,i}^* R_{Gc,i}^{1/2} \end{bmatrix} \\ \begin{bmatrix} K_{w,i} \\ K_{u,i} \end{bmatrix} & = & K_c \\ R_{Gc,i} & = & I_{m_2} + G_{2,i}^* P_{i+1}^c G_{2,i} \end{array}\right. , \tag{1.7.68}$$

and that the stability requirements of Theorems 1.7.3 and 1.7.4 are dropped.

We end this section by noting that we will return to control problems, albeit through a slightly different approach, in Chapter 6. The relationships between the above Riccati recursions and the problem of recursively performing canonical and dual canonical factorizations will be taken up starting in Chapter 2. Finally, the asymptotic behaviour of the above controllers (in the time-invariant case) will be studied in Chapter 8.

## 1.8 Other Approaches to Estimation and Control

In Secs. 1.3 to 1.7 we gave a rather detailed overview of the $H^2$ and $H^\infty$ approaches to estimation and control. It should be mentioned that, although it is fair to say that these are the two methodologies that have received the most attention in the literature, they are by no means the only approaches to these problems. We have already mentioned, in passing, the $l_1$ approach to robust estimation and control [DP87, DDB95], and should mention the various "mixed" strategy approaches, such as mixed $H^2/H^\infty$ (which we shall explain in Chapter 12) [BH89, YBC92, KR92, LA94, ZGBD94a, ZGBD94b, Meg94, FFT94, FFL95, HHK96], mixed $l_1/H^\infty$ [Rot95], mixed $H^2/l_1$ [Vou95], and other variations, as well as such methods as "unknown but bounded" estimation and control [Sch73b]. Of course, there are also other general approaches to estimation and control, such as adaptive control [AW89, SB89, KKK95, WW96], nonlinear control [Lef65, BMS88, Kha92, Vid93], and convex-optimization-based control [BB91, BGFB94], which are far beyond the scope of this thesis.

However, there are two other recent approaches to estimation and control that immediately lend themselves to the methodology presented in this thesis. These two are the so-called risk-sensitive (or LEQG) and game-theoretic approaches to filtering and control.

### Risk-sensitive Control and Quadratic Game Theory

In the risk-sensitive framework, instead of choosing estimators and controllers that minimize the expected value of a certain quadratic cost function, as the $H^2$ framework suggests, attempt is made to choose estimators and controllers that minimize the expected value of an exponential-quadratic cost function [Jac73, SDJ74, Whi90, SFB92]. The ensuing theory is sometimes called LEQG (linear-exponential-quadratic-Gaussian) theory to reflect the facts that the resulting optimal estimators (and controllers) are linear, the cost function is the exponential of a quadratic, and the disturbances are assumed to be Gaussian random variables. The reason why it is also (following Whittle) called risk-sensitive estimation and control is that the criterion is risk-sensitive, in the sense that it depends on a real parameter that determines whether more or less weight should be given to higher or smaller errors. [Roughly speaking when more weight is given to smaller errors the criterion is risk-seeking, and when more weight is given to large errors the criterion is risk-averse.]

Following some pioneering work in game theory (motivated primarily by the field of economics [NM44, KT50]), since the mid 1960's there has been considerable interest in applying game-theoretic ideas and methods to estimation and control. In this framework, the problem of estimation (or control) is treated as a noncooperative two-player game, with one player (the opponent) being the exogenous signals, and the other player being the estimator (or controller). This approach, which treats the exogenous signals as malignant disturbances that compete against the estimator (or controller), is, of course, fundamentally different from the $H^2$ (or risk-sensitive) approach where the exogenous signals are simply taken to be random variables with known probability distributions. The class of games most often applied to estimation and control is the class of differential games with quadratic payoff [Isa65, Ber64, BH69], which, in the discrete-time case of interest to us, shall henceforth be referred to as a quadratic dynamic game.

An interesting aspect of the risk-sensitive and quadratic dynamic game approaches is that the resulting solutions bear many similarities to the solutions obtained via the $H^2$ and $H^\infty$ approaches — in fact, estimators have an observer structure, full information controllers have state-feedback structure, the various observer and state-feedback gains are found from the solution of certain Riccati equations, etc.

The main reason for the above similarities, and indeed a major claim of this thesis, is that all the above problems can be related to stationarizing certain indefinite quadratic forms. In other words, these (seemingly different) problems are all unified (or can be treated in a unified fashion) since they can be related to certain *indefinite* quadratic forms from which a Krein space optimization problem (as developed in Chapter 2) can be inferred. We shall give the details of why this is the case in Chapters 3 and 4, but, for the time being, we can outline how this comes about in the following (somewhat pictorial) fashion.

(i) $H^\infty$ problems:

$$\frac{\|\cdot\|^2}{\|\cdot\|^2} < \gamma^2$$

$$\Rightarrow \quad \|\cdot\|^2 - \gamma^{-2}\|\cdot\|^2 > 0$$

(ii) Risk-sensitive problems:

$$\min \ E e^{\|\cdot\|^2}$$

$$\Rightarrow \quad \min \int e^{\|\cdot\|^2} e^{-x^* R_x^{-1} x} dx = \min e^{\min_x \left[\|\cdot\|^2 - x^* R_x^{-1} x\right]}$$

(iii) Quadratic game problems:

$$\min_u \max_w J(u, w, y), \quad J \text{ quadratic and indefinite}$$

Therefore, starting from the next chapter, we shall develop a general theory for linear estimation in Krein spaces. This theory will then be applied to $H^\infty$, risk-sensitive and quadratic dynamic game, estimation and control in Chapters 3, 4 and 6. This approach also has the major bonus of allowing for new results in these fields, as demonstrated throughout the thesis, and, in particular, as shown in Chapters 5, 7, 8 and 9.

### Adaptive Filtering

Finally, a few remarks are in order for the fields of adaptive filtering, adaptive signal processing and adaptive neural networks. Adaptive filtering has been widely used (since the late 1950's) to cope with time-variations of system parameters and lack of a priori knowledge of the statistical properties of the input data [WS85, Hay96]. This puts it in contrast to Wiener and Kalman filter theory which require a priori statistical information. Indeed it was claimed that adaptive filtering algorithms "learn" the signal statistics as the signals are observed, and hence "adapt" to the inputs that they are presented with.

Although this may very well be the case, it was also claimed that adaptive filtering is therefore a field different from estimation theory, since estimation theory relies on a priori statistical information. This, of course, is based on too narrow a definition of estimation theory, since recent approaches in this field, such as the $H^\infty$, $l_1$ and game-theoretic approaches, do not require any a priori statistical knowledge and are therefore fully compatible with the objectives of adaptive filtering.

Therefore it is fair to say that adaptive filtering is a subfield of estimation theory. Moreover, insofar as adaptive filtering algorithms are concerned, once these algorithms satisfy a certain optimality criterion (as does, say, the RLS algorithm, which is least-squares or $H^2$-optimal) then they are certainly a part of estimation theory, and once they do not satisfy such a criterion, they can be regarded, to some extent or the other, as ad hoc. In fact, in Chapter 9 we shall show that the celebrated LMS adaptive algorithm [WH60] (along with related instantaneous-gradient-based methods ), which had long been regarded as an intuitively appealing method with many successful practical applications, but little theoretical justification, is indeed $H^\infty$ optimal. This surprising fact demonstrates the connections between adaptive filtering and $H^\infty$ estimation, which are further pursued in Chapter 11.

## 1.9 Scope and Contributions of Thesis

In this last section (of this introductory chapter) we shall present a brief overview of the scope and contributions of this thesis. For ease of reference, and to illustrate some of the pedagogical continuity between successive chapters, we shall survey the scope and contributions on a chapter by chapter basis. Browsing through the opening paragraphs of each chapter should also serve as a useful survey.

### Chapter 2

Chapter 2 develops a self-contained theory for linear estimation in Krein spaces. A first motivation for the study of indefinite metric spaces was noted in Secs. 1.3 to 1.7 where we saw that the solution of various estimation and control problems could be given in terms of the canonical factorizations of certain indefinite transfer operators. Chapter 2 begins with a further motivation of indefinite metric spaces, via the celebrated Kalman-Yakubovich-Popov (KYP) lemma, and then proceeds to study Krein spaces and to compare their geometrical properties (such as the existence of projections onto linear subspaces) with the corresponding geometrical properties of Hilbert (or definite metric) spaces. It turns out that while Hilbert spaces and

Krein spaces share many characteristics, they differ in special ways that mark out the differences between the conventional $H^2$ theories and the more recent $H^\infty$ theories.

We then relate Krein space projections to the computation of stationary points of certain indefinite quadratic forms, and give conditions as to when these stationary points correspond to (stochastic or deterministic) minima. The results are then specialized to state-space models, more specifically, to linear state-space models driven by inputs that lie in a Krein space. This then leads to the centerpiece of the ensuing theory, viz., a Krein space generalization of the classical Kalman filter. This (so-called) Krein space Kalman filter allows one to recursively compute the stationary point of certain indefinite quadratic forms, or to recursively compute the canonical factorization of certain indefinite Gramians (or transfer operators), and therefore plays a crucial role in the solution of problems in several areas such as $H^2$ and $H^\infty$ estimation and control, quadratic game theory, risk-sensitive optimization and adaptive filtering.

The results presented in this chapter serve as a backbone for the remainder of the thesis and will be repeatedly used throughout the remaining chapters. Indeed it will be shown that the Krein space formalism developed in Chapter 2 serves as a unified approach to treating problems in $H^2$, $H^\infty$, risk-sensitive and game-theoretic estimation, control and adaptive filtering, and allows one to extend to these new settings many of the results developed and obtained in $H^2$ theory over the last three to four decades.

The appendix of Chapter 2 presents a new stochastic interpretation of the KYP lemma and constructs a simple proof based on Krein space geometry.

## Chapter 3

Chapter 3 deals with the study and solution of $H^\infty$ filtering problems, using the Krein space estimation theory developed in Chapter 2. The basic approach is to associate an indefinite quadratic form with the $H^\infty$ filtering problem, and to use the Krein space Kalman filter to compute the stationary point of the quadratic form and to check its conditions for a minimum. The major result is that $H^\infty$ filters are nothing more than certain Krein space Kalman filters — thus explaining the surprising similarities between $H^\infty$ filters and conventional Kalman filters. The chapter solves the so-called $H^\infty$ a posteriori, a priori and smoothed estimation problems and parametrizes all possible solutions. Although most of the results on $H^\infty$ filtering, apart from, say, certain equivalent conditions for the existence of solutions, are not new, the approach and the derivations are. [Incidentally, we believe that the approach presented here is the most natural.] The chapter also solves the $l$-step ahead $H^\infty$ prediction problem which appears to be new — we are not aware of solutions to this problem in the literature.

## Chapter 4

Chapter 3 describes some further (filtering) applications of the Krein space estimation theory of Chapter 2. Specifically, these applications are risk-sensitive filtering, quadratic game-theoretic filtering, and finite-memory adaptive filtering. The major point is that all these problems can be cast into the problem of calculating the stationary point of certain indefinite quadratic forms, and that by considering the appropriate state space models and error Gramians, we can use the Krein space estimation theory to calculate these stationary points and study their properties. Although many of the connections between $H^\infty$, risk-sensitive and quadratic game-theoretic estimation and control are known in the literature, the material of this chapter sheds further light on these connections and provides for a new perspective. Moreover, while the solution to finite-memory (or so-called sliding window) adaptive filtering problems are wellknown in the literature, the development presented here, and the connections with Kalman filtering, are new. Finally, some connections to new work on suboptimal recursive total least-squares algorithms are mentioned.

## Chapter 5

We have repeatedly claimed that the major bonus of the Krein space approach to $H^\infty$, game-theoretic, and risk-sensitive, estimation and control is that, apart from rather more transparent derivations of existing results (as done in Chapters 3 and 4), it shows a way to apply to the $H^\infty$ (and these other) settings many of the results developed for Kalman filtering and LQG control over the last three decades. Chapter 5 is the first place where we truly deliver on this claim by developing square-root array algorithms and Chandrasekhar recursions for $H^\infty$ filtering problems. These are the generalizations of the (numerically superior) conventional square-root arrays and (fast) Chandrasekhar recursions to the Krein space setting. The $H^\infty$ square-root algorithms involve propagating the indefinite square-root of the quantities of interest and have the property that the appropriate inertia of these quantities is preserved. For systems that are constant, or whose time-variation is structured in a certain way, the Chandrasekhar recursions allow a reduction in the computational effort per iteration from $O(n^3)$ to $O(n^2)$, where $n$ is the number of states. The $H^\infty$ square-root and Chandrasekhar recursions both have the interesting feature that one does not need to explicitly check for the inertia conditions required for the existence of $H^\infty$ filters. These conditions are built into the algorithms themselves so that an $H^\infty$ estimator of the desired level exists if, and only if, the algorithms can be executed. [All the results of this chapter are new.]

## Chapter 6

Chapter 6 deals with duality in (definite and indefinite metric) linear spaces. Although not the conventional way for introducing duality in systems and control theory, duality here is introduced through the geometrical notion of dual bases for linear spaces spanned by a set of nonorthogonal basis vectors. Apart from having conceptual value, the resulting duality is a useful tool in the study of various problems of interest, and, in particular, in allowing certain a dual approach to (definite and indefinite) quadratic problems. A large part of this chapter is devoted to the use of duality in ($H^2$, $H^\infty$, game-theoretic and risk-sensitive) control. Although the control results given in this chapter are already available in the literature, the derivations are new, and effectively combine the use of duality (here duality with estimation problems) and the Krein space theory of Chapter 2. All results are based on a certain indefinite LQR (linear-quadratic-regulator) problem from which the $H^2$, $H^\infty$, game-theoretic and risk-sensitive solutions follow as special cases.

## Chapter 7

Chapter 7 studies the celebrated discrete-time algebraic Riccati equation (DARE) which arises in an impressive range of applications in systems and control theory. Although a great deal is known about the Riccati equation when the coefficient matrices are positive semi-definite, much less is known when these coefficients are indefinite matrices. In this chapter the DARE is considered in the full generality of this, so-called, indefinite case and the results are then particularized to some important special cases (essentially the special cases that arise in $H^2$ and $H^\infty$ estimation and control). The approach taken in this chapter is through the introduction of a certain (so-called) Popov function whose factorizations are intimately related to solutions of the DARE.[30] The main result is that solutions to the DARE, or more more precisely a system of discrete-time algebraic Riccati equations (SDARE), exists if, and only if, a certain proper factorization of the Popov function exists. Additional conditions are then given under which the solution to the DARE becomes stabilizing, Hermitian, positive semi-definite, etc. The DARE is also related to a so-called Hamiltonian matrix, from which the famous invariant subspace method can be obtained that actually computes solutions to the DARE. Some examples are also included to illustrate the significance of the results.

## Chapter 8

Chapter 8 focuses on the behaviour of the Riccati recursion with time-invariant coefficient matrices, as time progresses to infinity. The main objective is to find conditions under which, for a given initial condition, the solution to the Riccati recursion converges to a solution of the associated DARE. The main result states that if, at each time instant, a certain inertia condition is met, then the Riccati recursion (exponentially) converges to the unique stabilizing solution (assuming such a solution exists) of the associated DARE. In the general case, the aforementioned inertia conditions need to be recursively checked, however, in some special cases they may be reduced

[30] The Popov function can be regarded as the generalization of the usual power spectral density function to indefinite metric spaces.

to more simple and more explicit requirements on the initial condition. In particular, when the coefficient matrices of the Riccati recursion are positive semi-definite, convergence of the Riccati recursion can be guaranteed for some indefinite, and even negative semi-definite, initial conditions (provided they are bounded below by a certain negative semi-definite matrix). Moreover, in the case frequently encountered in $H^\infty$ filtering and control, convergence is guaranteed for all positive semi-definite initial conditions that are less than or equal to the unique positive semi-definite solution of a related Lyapunov equation. [We believes that Chapter 8 furnishes a very direct approach to establishing the convergence of the Riccati recursion. Moreover, to the best of our knowledge, the results obtained here are more general than those to have appeared in the literature and subsume, as special cases, all of the earlier given results.]

## Chapter 9

Chapter 9 uses the connection between adaptive filtering and state-space estimation to study adaptive filtering with an $H^\infty$ criterion. In particular, it is shown that the celebrated LMS (least-mean-squares) adaptive algorithm is $H^\infty$ optimal. The LMS algorithm has been long regarded as an approximate solution to either a stochastic or a deterministic least-squares problem, and it essentially amounts to updating the weight vector estimates along the direction of the instantaneous gradient of a quadratic cost function. In this chapter it is shown that LMS can be regarded as the exact solution to a minimization problem in its own right. Namely, it is established that it is a minimax filter: it minimizes the maximum energy gain from the disturbances to the prediction errors, while the closely related so-called normalized LMS algorithm minimizes the maximum energy gain from the disturbances to the filtered errors. Moreover, since these algorithms are central $H^\infty$ filters, they minimize a certain exponential cost function and are thus also risk-sensitive optimal. The various implications of these results are also discussed, and it is shown how they provide theoretical justification for the widely observed excellent robustness properties of the LMS filter.

## Chapter 10

In order to compare the robustness of other adaptive filtering algorithms with the ($H^\infty$-optimal) LMS and normalized LMS algorithms, Chapter 10 studies the robustness of least-squares-based adaptive filters, such as the RLS algorithm,from the $H^\infty$ point of view. The basic result is the derivation of certain upper and lower bounds for the $H^\infty$ norm of the RLS algorithm (in fact, more generally, of the Kalman filter) with respect to prediction and filtered errors. The main conclusion is that, unlike LMS and normalized LMS which do not allow for any amplification of the disturbances, the RLS algorithm does allow for such amplification. This fact can be especially pronounced in the prediction error case. Moreover, it is also shown that the $H^\infty$ norm for RLS is data-dependent, whereas for LMS and normalized LMS it was not so. [The $H^\infty$ norm was simply unity.] The significance of the results are also discussed.

## Chapter 11

The results of Chapters 9 and 10 indicate that there may be great promise in the interplay of adaptive filtering and $H^\infty$ estimation theory. To continue with this approach and line of reasoning, Chapter 11 presents a preliminary study of the design of adaptive filters using the $H^\infty$ criterion. The strength of $H^\infty$-optimal adaptive filters lies in the fact that they guarantee the smallest possible estimation error energy over all possible disturbances of fixed energy, and are therefore robust with respect to model uncertainties and lack of statistical information on the exogenous signals. Specifically, this chapter studies the problem of prediction of the weight vector itself, and for the purpose of coping with time-variations, exponentially weighted, finite-memory and time-varying adaptive filtering. This results in some new adaptive filtering algorithms that may be useful in uncertain and non-stationary environments. The presentation of the chapter is brief and the major goal is to only demonstrate some of the possibilities.

## Chapter 12

The final chapter concludes with some brief remarks on various directions for future research that are suggested by the methods and results presented in this thesis. In particular, it introduces and motivates the mixed $H^2/H^\infty$ approach to estimation and control.

# Chapter 2

# Linear Estimation in Krein Spaces

In this chapter we develop a self-contained theory for linear estimation in Krein spaces. The presentation is based on simple concepts such as projections and matrix factorizations, and leads to an interesting connection between Krein space projections and the recursive computation of the stationary points of certain second order (or quadratic) forms. The innovations process is then used to obtain a general recursive linear estimation algorithm which, when specialized to state space structure, yields a Krein space generalization of the celebrated Kalman filter with applications in several areas such as $H^\infty$-filtering and control, quadratic dynamic game theory, risk sensitive control, and adaptive filtering.

## 2.1 Introduction

In some recent explorations,[1] we have found that $H^\infty$ estimation and control problems and several related problems (risk-sensitive estimation and control, finite memory adaptive filtering, stochastic interpretation of the KYP lemma, and others) can be studied in a simple and unified way by relating them to Kalman filtering problems, not in the usual (stochastic) Hilbert space but in a special kind of indefinite metric space known as a Krein space (see *e.g.* [Bog74, IKL82, Ist87]). Although the two types of spaces share many characteristics, they differ in special ways that turn out

[1] See *e.g.*, [HSK96c, HSK96b, HSK93c, HSK93b, KHS93, SHK96b, SHK96a, SHK95, HSK94b].

to mark the differences between the LQG or $H^2$ theories and the more recent $H^\infty$ theories.[2] The connections with the conventional Kalman filter theory will allow a lot of the newer numerical algorithms, developed over the last three decades, to be applied to the $H^\infty$ theories (see [HSK94c] and Chapter 5.[3]

In this chapter we develop a self-contained theory for linear estimation in Krein spaces. The ensuing theory is richer than that of the conventional Hilbert space case, which is why it yields a unified approach to the aforementioned problems. Applications will follow in subsequent chapters.

The remainder of the chapter is organized as follows. In Sec. 2.2 we motivate the introduction of Krein spaces by first reviewing the known results for the $H^\infty$ filtering and control problems. We note that the $H^\infty$ filtering and control problems are related to the factorization of indefinite transfer operators and that the solutions resemble the conventional Kalman filter and LQG controller with the exception of the appearance of certain indefinite Gramians and extra conditions; this vaguely suggests the possibility of using indefinite metric spaces, of which Krein spaces are a special case. Further motivation for the introduction of Krein spaces comes from examining the celebrated KYP lemma that characterizes the family of all state-space models that give rise to a given output covariance (or power spectrum in the infinite time case). It turns out that a certain free Hermitian matrix arising in this lemma has a nice interpretation when random variables are allowed to take values in a Krein

[2]Incidentally, indefinite metric were first introduced into the solution of physical problems via the so-called Minkowski spaces of relativity theory [LEMW23, Ein31, Sch73a]. There Minkowski was apparently the first to notice that various *physical* phenomena in special relativity, such as the Lorentz transformations, were best explained by considering the geometrical properties of a four dimensional space-time, $(x, y, z, t)$, with *indefinite* metric, $x^2 + y^2 + z^2 - t^2$. In this, and subsequent, chapters we shall make an analogous observation — namely, that $H^\infty$ estimation and control problems (and their solutions) are best understood, and explained, by considering the geometry of indefinite metric spaces.

[3]We should also remark that indefinite metric spaces were already somewhat implicit in the early (pre-state-space) solutions of $H^\infty$ control (see [Fra87]) and in some of the solutions based on indefinite factorization [BC87, FT88]. However, this fact was neither fully appreciated or exploited in the above works, nor were connections with state-space theory made. On the other hand, it has recently been brought to our attention, that A. Halanay, V. Ionescu, C. Oara and M. Weiss have, during 1990-94, developed an interesting "generalized Popov-Yakubovich theory" for dealing with problems involving indefinite scalar products that has nice connections with the material presented in this chapter [HI94, HI93, IW93].

space.

Krein spaces are introduced in Sec. 2.3, and projections in Krein spaces in Sec. 2.4. Contrary to the Hilbert space case where projections always exist and are unique, the Krein space projection exists and is unique if, and only if, a certain Gramian matrix is nonsingular. In Sec. 2.5, we first remark that while quadratic forms in Hilbert space always have minima (or maxima), in Krein spaces one can only assert that they will always have stationary points. Further conditions will have to be met for these to be minima or maxima. We explore this by first considering the problem of finding a vector $k$ to *stationarize* the quadratic form $\langle \mathbf{z} - k^*\mathbf{y}, \mathbf{z} - k^*\mathbf{y}\rangle$, where $\langle \cdot, \cdot \rangle$ is an indefinite inner product, $*$ denotes conjugate transpose, $\mathbf{y}$ is a collection of vectors in a Krein space (which we can regard as *generalized* random variables) and $\mathbf{z}$ is a vector outside the linear space spanned by the $\mathbf{y}$. If the Gramian matrix $R_y = \langle \mathbf{y}, \mathbf{y}\rangle$ is nonsingular, then there is a unique stationary point $k_o^*\mathbf{y}$, given by the projection of $\mathbf{z}$ onto the linear space spanned by the $\mathbf{y}$; the stationary point will be a minimum if, and only if, $R_y$ is strictly positive definite as well. In a Hilbert space, the nonsingularity of $R_y$ and its strict positive definiteness are equivalent properties, but this is not true with $\mathbf{y}$ in a Krein space.

Now in the Hilbert space theory it is well known (motivated by a Bayesian approach to the problem) that a certain deterministic quadratic form $J(z, y)$, where now $z$ and $y$ are elements of the usual Euclidean vector space, is also minimized by $k_o^* y$, with exactly the same $k$ as before. In the Krein space case, $k_o^* y$ also yields a *stationary* point of the corresponding deterministic quadratic form, but now this point will be a minimum if, and only if, a *different* condition, not $R_y > 0$, but $R_z - R_{zy}R_y^{-1}R_{yz} > 0$, is satisfied. In Hilbert space, unlike Krein space, the two conditions for a minimum hold simultaneously (see Corollary 2.5.3 in Sec. 2.5). This simple distinction turns out to be crucial in understanding the difference between $H^2$ and $H^\infty$ estimation, as we shall show in detail in Chapter 3 of this thesis.

In this chapter, however, we continue with the general theory, by exploring the consequences of assuming that $\{\mathbf{z}, \mathbf{y}\}$ are based on some underlying state-space model. The major ones are a reduction in computational effort, $O(Nn^3)$ vs. $O(N^3)$ where $N$ is the number of observations and $n$ is the number of states, and the possibility of recursive solutions. In fact, it will be seen that the innovations-based derivation[4] of the Hilbert space Kalman filter extends to Krein spaces, except that now the *Riccati* variable $P_i$, and the innovations Gramian $R_{e,i}$ are not necessarily positive (semi)definite. The Krein space Kalman filter continues to have the interpretation of performing the triangular factorization of the Gramian matrix of the observations, $R_y$; this reduces the test for $R_y > 0$ to recursively checking that the $R_{e,i} > 0$.

Similar results are expected for the corresponding indefinite quadratic form. While global expressions for the stationary point of such quadratic forms and of the minimization condition were readily obtained, as previously mentioned, recursive versions are not easy to obtain. Dynamic programming arguments are the ones usually invoked [BH69, BB95], and turn out to be algebraically more complex than the simple innovations (Gram-Schmidt orthogonalization) ideas available in the stochastic (Krein space) case.

Briefly, given a possibly indefinite quadratic form, our approach is to associate with it (by inspection) a Krein space model whose stationary point will have the same gain $k_o^*$ as for the deterministic problem. The Kalman filter recursions can now be invoked and give a recursive algorithm for the stationary point of the deterministic quadratic form; moreover, the condition for a minimum can also be expressed in terms of quantities easily related to the basic Riccati equations of the Kalman filter. These results are developed in Secs. 2.6 and 2.7, with Theorems 2.7.3 and 2.7.4 being the major results.

Finally, in the appendix, we give a stochastic proof of a time-variant version of the Kalman-Yakubovich-Popov lemma, whose infinite horizon time-invariant counterpart was used as a motivation for the introduction of Krein spaces at the beginning of this chapter. The proof is based on introducing state-space models driven by inputs that lie in an indefinite (Krein) space which can be considered as generalizations of standard stochastic state-space models driven by stationary stochastic processes (that lie in a definite (Hilbert) space). We also provide a simple geometric interpretation of the KYP lemma in terms of a certain decomposition of positive vectors in Krein space.

[4] See *e.g.*, [Kai68, Kai81].

While it is possible to pursue many of the results of this chapter in greater depth, the development here is sufficient to solve several problems of interest in estimation theory. In chapter 3 we shall apply these results to $H^\infty$ filtering and in chapter 4 to risk-sensitive and game-theoretic estimation, and to finite memory adaptive filtering. Chapter 6 studies various dualities, and applies them to obtain dual (or so-called complementary) state-space models, and to solve the $H^2$, $H^\infty$, and risk-sensitive *control* problems. We may mention that using these results we have also been able to develop the (possibly) numerically more attractive square root arrays and Chandrasekhar recursions for $H^\infty$ problems (see Chapter 5), to study robust adaptive filtering (see Chapters 9 11 and 10) , and to study convergence issues and obtain steady state results (see Chapter 7 and 8. The point is that the many years of experience and intuition gained from the LQG or $H^2$ theory can be used as a guide to the corresponding $H^\infty$ results.

### 2.1.1 Notation

A remark on the notation used in the paper. Elements in a Krein space are denoted by bold face letters, and elements in the Euclidean space of complex numbers are denoted by normal letters. Whenever the Krein space elements and the Euclidean space elements satisfy the same set of constraints we shall denote them by the same letters, with the former ones being bold and the latter ones being normal. [This convention is similar to the one used in probability theory, where random variables are denoted by bold face letters, and their assumed values are denoted by normal letters.]

## 2.2 Motivation for Indefinite Metric Spaces

In this section we shall motivate the study of Krein spaces by considering the $H^\infty$ filtering and control problems and the Kalman-Yakubovich-Popov (KYP) Lemma [Kal63a, Yak62, Pop64].

### 2.2.1 $H^\infty$ Filtering and Control

The problems of $H^\infty$ filtering and control were introduced and briefly considered in Secs. 1.4 and 1.7 of Chapter 1.[5] There we saw that, despite the fundamental differences in their approaches, the $H^2$ and $H^\infty$ solutions had striking formal similarities. Indeed the only differences essentially arose from the fact that the canonical factorization of *positive definite* operators (or Gramians), that appear in the $H^2$ solutions, were replaced by the canonical factorization of certain *indefinite* operators in the $H^\infty$ solution. The structure of the solutions, however, especially as seen in the state-space case, remained quite similar:

- the $H^2$ and $H^\infty$ filters both had a (so-called Kalman-Luenberger) observer structure [Lue63], with the observer gain being determined by Riccati equations (or recursions, in finite time);
- the $H^2$ and $H^\infty$ full information controllers were both given by a state feedback law, with the state feedback gain being determined by a dual Riccati equation;
- the $H^2$ and $H^\infty$ measurement feedback controllers both possessed a certain separation structure, in which the desired control signals were obtained by estimating, in an $H^2$ and $H^\infty$ sense, respectively, the (unobservable) full information control signals.

The differences, were also due the replacement of (canonical factorizations of) positive definite Gramians (operators) with indefinite ones. Indeed:

- certain indefinite covariance matrices appear in the Riccati equations of $H^\infty$ filtering and control, whereas in the $H^2$ problems they are positive (semi)definite;
- the linear combination of the state we intend to estimate affects the structure of the $H^\infty$ estimators, whereas in $H^2$ problems the best estimate of any linear combinations of the state is just that linear combination of the best state estimate;

[5]The reader at this stage may want to review those sections to refresh his/her memory.

- the full information control and estimation problems obtained from the separation principle of measurement feedback $H^\infty$ control are coupled, whereas in the $H^2$ case they are decoupled.

- there are certain additional conditions that need to be satisfied for the solutions of $H^\infty$ problems to exist; $H^2$ solutions, on the other hand, always exist.

For example, the last of the above differences is readily explained by the fact that, whereas the canonical factorization of bounded positive definite operators always exists, the canonical factorization of bounded indefinite operators need not always exist [Yak70a, KS94].

We will not go any further into the specifics of $H^\infty$ problems here, since we have already done so in Secs. 1.4 and 1.7 and since we will treat them in great detail in Chapter 3.[6] We only remark that the above observations suggest that to further study $H^\infty$ problems it seems reasonable to introduce the concept of indefinite metric spaces. Indeed, we shall shortly see that, such an approach allows for a unified mathematical treatment of the $H^2$ and $H^\infty$ problems (and a host of other problems as well). The differences between the two theories will be explained by the differences in the geometrical properties of the underlying (Hilbert and Krein) spaces.

At this point, we shall further motivate the introduction of Krein space models for state-space processes by considering the celebrated Kalman-Yakubovich-Popov (KYP) lemma on a state-space characterization of power spectral density matrices.

### 2.2.2 The KYP Lemma

The Kalman-Yakubovich-Popov (KYP) Lemma was first introduced and proven in [Kal63a, Yak62, Pop64] in the context of control theory. It is also closely related to passive network synthesis [Kal63c, AV73] and dissipative dynamical systems [Wil72].

[6] Indeed in Chapter 3 we shall study such problems, not from the viewpoint of factorization as presented in Chapter 1, but from the very closely related point of view of finding the stationary points of certain indefinite quadratic forms. In this framework, the essential difference between the $H^2$ and $H^\infty$ solutions will arise from the fact that, whereas positive definite quadratic forms always have minima, indefinite quadratic forms will, in general, only have stationary points. [Further conditions must be met for a minimum to exist.]

Here, however, we shall consider the KYP Lemma from a stochastic viewpoint. We shall see that the KYP Lemma allows for a great deal of freedom in representing stationary stochastic processes with rational power spectral density functions. These degrees of freedom may then be exploited to perform spectral factorization (a key ingredient in linear least-mean-squares estimation theory), or to solve the stochastic realization problem [Pic76].

Consider the time-invariant state-space model

$$\begin{cases} \mathbf{x}_{i+1} = F\mathbf{x}_i + \mathbf{u}_i \\ \mathbf{y}_i = H\mathbf{x}_i + \mathbf{v}_i \end{cases} \tag{2.2.1}$$

where $F$ is stable, $\{F, H\}$ is observable[7] and the disturbances are zero-mean *stationary* random processes with

$$E \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \begin{bmatrix} \mathbf{u}_j^* & \mathbf{v}_j^* & 1 \end{bmatrix} = \begin{bmatrix} Q\delta_{ij} & S\delta_{ij} & 0 \\ S^*\delta_{ij} & R\delta_{ij} & 0 \end{bmatrix}.$$

Taking $z$-transforms, we can rewrite (2.2.1) as

$$\mathbf{y}(z) = H(zI-F)^{-1}\mathbf{u}(z) + \mathbf{v}(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} \mathbf{u}(z) \\ \mathbf{v}(z) \end{bmatrix}.$$

Recall that if a $m \times 1$ stationary process $\{\mathbf{r}_i\}$ with $z$-spectral density function $S_r(z)$ is applied to a $p \times m$ linear system with transfer matrix $H(z)$ to yield an output $\{\mathbf{s}_i\}$, the so-called output $z$-spectrum defined as

$$S_s(z) = \mathcal{Z}\left\{E\mathbf{s}_j\mathbf{s}_{j-i}^*\right\},$$

is given by

$$S_s(z) = H(z)S_r(z)H^*(z^{-*}).$$

Thus, in our case, the output $z$-spectrum of $\{\mathbf{y}_i\}$ is given by

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{2.2.2}$$

[7]In Chapter 7 we shall see that both these conditions can be replaced with the less restrictive condition that $\{F, H\}$ is detectable.

Note that the matrix appearing in the center of (2.2.2) is the covariance of the disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$, so that we have

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \geq 0. \tag{2.2.3}$$

This implies that $S_y(e^{j\omega}) \geq 0$, which is the defining property of a power spectral density matrix generated by a true stochastic process.

However, let us calculate the output spectrum in an alternative fashion. The steady-state covariance of the state $\mathbf{x}_i$, defined by $\bar{\Pi} = \lim_{i\to\infty} E\mathbf{x}_i\mathbf{x}_i^*$, satisfies the (discrete-time) Lyapunov equation

$$\bar{\Pi} = F\bar{\Pi}F^* + Q. \tag{2.2.4}$$

Thus, in the steady-state, the autocorrelation function of the output is given by

$$R_{y,i} = E\mathbf{y}_j\mathbf{y}_{j-i}^* = \begin{cases} HF^i\bar{\Pi}H^* + HF^{i-1}S & i > 0 \\ R + H\bar{\Pi}H^* & i = 0 \\ H\bar{\Pi}F^{*i}H^* + S^*F^{*(i-1)}H^* & i < 0 \end{cases}$$

Taking the $z$-transform of $R_{y,i}$ in the above expression, the output $z$-spectrum can be written as

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & F\bar{\Pi}H^* + S \\ H\bar{\Pi}F^* + S^* & R + H\bar{\Pi}H^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{2.2.5}$$

Comparing (2.2.2) with (2.2.5) we see that the only difference between these two *representations* of the output $z$-spectrum is the matrix appearing in the center of these equations. In the case of (2.2.2) we saw that this matrix was the covariance of the disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$. Now in the case of (2.2.5) the center matrix

$$\begin{bmatrix} 0 & F\bar{\Pi}H^* + S \\ H\bar{\Pi}F^* + S^* & R + H\bar{\Pi}H^* \end{bmatrix}, \tag{2.2.6}$$

is indefinite. Note that $S_y(e^{j\omega}) \geq 0$, of course, even though the center matrix (2.2.6) is not non-negative definite and cannot be thought of as the covariance of some random variables, say $\{\mathbf{u}_i^{(1)}, \mathbf{v}_i^{(1)}\}$. (Indeed $\mathbf{u}_i^{(1)}$ would need to have zero variance but nonzero cross-variance with $\mathbf{v}_i^{(1)}$!) However, if we broaden our domain of discourse, and instead of random variables, consider disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$ that belong to an abstract *indefinite* (so-called Krein) space, then the matrix (2.2.6) can be considered as the covariance of such an abstract disturbance $\{\mathbf{u}_i^{(1)}, \mathbf{v}_i^{(1)}\}$.[8]

The above discussion shows that even when considering state-space models driven by random variable disturbances (that lie in a Hilbert space) it is natural to consider indefinite metric spaces. Indeed there is much more to be gained from this generalization. Thus we shall gain an understanding of the fact that several different *center matrices* (*e.g.* those in (2.2.3) and (2.2.6)) can give rise to the same output $z$-spectrum.

### An Equivalence Class for Input Covariances

To this end, consider the state-space model (2.2.1) but now suppose that the inputs $\{\mathbf{u}_i, \mathbf{v}_i\}$ are such that

$$\left\langle \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \right\rangle = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \delta_{ij}. \tag{2.2.7}$$

Note that we have replaced the notation $E\mathbf{u}_i\mathbf{v}_j^*$ with $\langle \mathbf{u}_i, \mathbf{v}_j \rangle$ since we are now considering the $\{\mathbf{u}_i, \mathbf{v}_i\}$ to live in an indefinite space so that the matrix appearing in (2.2.7) may be indefinite. Now associated with the state-space model (2.2.1) and the inputs (2.2.7), we may define the *Popov function*

$$\Sigma(z) = S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{2.2.8}$$

We can readily see that the Popov function is the generalization of the $z$-power spectral density function since

$$S_y(z) = \mathcal{Z}\left\{\langle \mathbf{y}_j, \mathbf{y}_{j-i} \rangle\right\}.$$

[8] We shall state what is exactly meant by a Krein space in Sec. 2.3. For the time being, it suffices to know that in a Krein space the variables $\{\mathbf{u}_i, \mathbf{v}_i\}$ may have indefinite covariance matrices.

Now suppose that we intend to add white and stationary disturbances $\{\bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i\}$ (orthogonal to the original $\{\mathbf{u}_i, \mathbf{v}_i\}$) to the state-space model (2.2.1) such that the output $z$-spectrum $S_y(z)$ remains unchanged. In other words the output of the state-space model

$$\begin{cases} \mathbf{x}_{i+1} + \bar{\mathbf{x}}_{i+1} & = & F(\mathbf{x}_i + \bar{\mathbf{x}}_i) + \mathbf{u}_i + \bar{\mathbf{u}}_i \\ \mathbf{y}_i + \bar{\mathbf{y}}_i & = & H(\mathbf{x}_i + \bar{\mathbf{x}}_i) + \mathbf{v}_i + \bar{\mathbf{v}}_i \end{cases} \tag{2.2.9}$$

should still have Popov function equal to $S_y(z)$, given in (2.2.8).

The covariance matrix of the new disturbances $\mathbf{u}_i + \bar{\mathbf{u}}_i, \mathbf{v}_i + \bar{\mathbf{v}}_i$ is given by

$$\begin{bmatrix} Q + \bar{Q} & S + \bar{S} \\ S^* + \bar{S}^* & R + \bar{R} \end{bmatrix},$$

and the output $z$-spectrum by

$$S_{y+\bar{y}}(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q + \bar{Q} & S + \bar{S} \\ S^* + \bar{S}^* & R + \bar{R} \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}.$$

Now by linearity, $S_{y+\bar{y}}(z) = S_y(z) + S_{\bar{y}}(z)$. Therefore if $S_y(z)$ is to be unchanged, this implies that $S_{\bar{y}}(z)$, the $z$-spectrum of the process $\{\bar{\mathbf{y}}_i\}$ defined by

$$\begin{cases} \bar{\mathbf{x}}_{i+1} & = & F\bar{\mathbf{x}}_i + \bar{\mathbf{u}}_i \\ \bar{\mathbf{y}}_i & = & H\bar{\mathbf{x}}_i + \bar{\mathbf{v}}_i \end{cases}, \tag{2.2.10}$$

must be *zero*. Now a simple calculation shows that

$$\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_i \rangle = \bar{R} + H\langle \bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i \rangle H^*, \tag{2.2.11}$$

so that if we define $Z = -\langle \bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i \rangle$, (note that since the variables in (2.2.10) belong to an indefinite metric space, $Z$ is in general indefinite) we may write

$$\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_i \rangle = \bar{R} - HZH^* = 0, \tag{2.2.12}$$

or $\bar{R} = HZH^*$. Likewise, a similar computation for $i > j$, shows that

$$\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_j \rangle = HF^{i-j-1}(F\langle \bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i \rangle H^* + \bar{S}) = HF^{i-j-1}(-FZH^* + \bar{S}). \tag{2.2.13}$$

Thus choosing

$$\bar{S} = FZH^* \tag{2.2.14}$$

we see that

$$\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_j \rangle = 0. \tag{2.2.15}$$

Finally, using the state equation in (2.2.10) we may write

$$-Z = -FZF^* + \bar{Q}. \tag{2.2.16}$$

Combining (2.2.12), (2.2.14) and (2.2.16) shows that the indefinite variables $\{\bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i\}$ must have as covariance matrix

$$\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} = \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix}, \tag{2.2.17}$$

for some Hermitian $Z$ (which is the negative of the steady state covariance matrix of the process $\bar{\mathbf{x}}_i$).

We can thus show the following result.

**Lemma 2.2.1 (Equivalence Class for Input Covariances)** *(a) For any Hermitian $Z$, the output $z$-spectrum of the state-space model (2.2.1)*

$$S_y(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

*is invariant under the input covariance transformation*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \to \begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix}. \tag{2.2.18}$$

*(b) If for an observable system $\{F, H\}$, there exist input covariances*

$$\begin{bmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_2 & S_2 \\ S_2^* & R_2 \end{bmatrix}$$

*that yield the same output spectrum,* i.e.,

$$\begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}$$
$$= \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_2 & S_2 \\ S_2^* & R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}$$

*then there exists a unique Hermitian $Z$ such that*

$$\begin{bmatrix} Q_1 & S_1 \\ S_1^* & R_1 \end{bmatrix} = \begin{bmatrix} Q_2 - Z + FZF^* & S_2 + FZH^* \\ S_2^* + HZF^* & R_2 + HZH^* \end{bmatrix}.$$

**Remark:** When $\{F, H\}$ is not observable, part (b) of the above Lemma becomes slightly more complicated (see Chapter 7). Although the results presented below extend to the case where $\{F, H\}$ is detectable instead of observable, to simplify the arguments we shall retain the observability assumption.

**Proof of Lemma 2.2.1:** We have already proven part (a) in the arguments preceding the statement of the Lemma. Another approach is to directly show via a calculation that (2.2.18) is true for any Hermitian matrix $Z$ — just check that

$$0 = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

is true for any $Z = Z^*$.

For part (b), note that since $\{Q_1, R_1, S_1\}$ and $\{Q_2, R_2, S_2\}$ generate the same output $z$-spectrum we can write

$$\begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} = 0.$$

Thus if we define the indefinite variables $\{\bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i\}$ such that

$$\left\langle \begin{bmatrix} \bar{\mathbf{u}}_i \\ \bar{\mathbf{v}}_i \end{bmatrix}, \begin{bmatrix} \bar{\mathbf{u}}_j \\ \bar{\mathbf{v}}_j \end{bmatrix} \right\rangle = \begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} \delta_{ij} = \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} \delta_{ij},$$

then the state-space model

$$\begin{cases} \bar{\mathbf{x}}_{i+1} &= F\bar{\mathbf{x}}_i + \bar{\mathbf{u}}_i \\ \bar{\mathbf{y}}_i &= H\bar{\mathbf{x}}_i + \bar{\mathbf{v}}_i \end{cases},$$

must generate zero output $z$-spectrum. Using the arguments presented before the statement of the Lemma, this implies that

$$\begin{aligned} \bar{Q} &= -Z + FZF^* \\ \bar{R} &= HZH^* \\ 0 &= F^{i-j-1}(-FZH^* + \bar{S}) \quad \text{for } i > j \end{aligned}$$

where as before we have defined $\langle \bar{\mathbf{x}}_i, \bar{\mathbf{x}}_i \rangle = -Z$. Note that since $F$ is stable the first of the above equations shows that $Z$ is unique[9]

Moreover, the last equation shows that

$$\mathcal{O}(-FZH^* + \bar{S}) = 0,$$

where

$$\mathcal{O} = \begin{bmatrix} H^* & F^*H^* & F^{2*}H^* & \dots \end{bmatrix}^*$$

is the observability map. When $\{F, H\}$ is observable, $\mathcal{O}$ is full rank and we conclude that

$$-FZH^* + \bar{S} = 0.$$

We have thus shown that there exists a unique Hermitian $Z$ such that

$$\begin{bmatrix} \bar{Q} & \bar{S} \\ \bar{S}^* & \bar{R} \end{bmatrix} = \begin{bmatrix} Q_1 - Q_2 & S_1 - S_2 \\ S_1^* - S_2^* & R_1 - R_2 \end{bmatrix} = \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix}$$

from which the statement of part (b) follows.

■

Lemma 2.2.1 shows the great freedom that is obtained by allowing the disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$ to have an indefinite covariance matrix. We were thus able to parametrize all input covariance matrices that gave rise to the same Popov function in terms of a Hermitian matrix $Z$. This matrix had the interpretation of being the steady state covariance of the state vector in a state-space model that generates zero output spectrum. [The reader at this point may want to verify that the choice $Z = \bar{\Pi}$, where $\bar{\Pi}$ is as in (2.2.4), relates the input covariances in (2.2.2) and (2.2.5).]

Another application of the degree of freedom available via the matrix $Z$, is to choose $Z$ such that the center matrix in the Popov function drops rank, *i.e.*,

$$\begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix} = \begin{bmatrix} K_p \\ I \end{bmatrix} R_e \begin{bmatrix} K_p^* & I \end{bmatrix}. \tag{2.2.19}$$

[9]In fact, we only require that $F$ have no two eigenvalues such that $\lambda_i = \lambda_j^{-*}$ for the solution to the Lyapunov equation $Z = FZF^* + \bar{Q}$ to be unique.

This is of significance, since it leads to the following factorization of the Popov function

$$S_y(z) = \left[H(zI - F)^{-1}K_p + I\right] R_e \left[H(z^{-*}I - F)^{-1}K_p + I\right]^*. \tag{2.2.20}$$

In particular, when the transfer matrix $H(zI - F)^{-1}K_p + I$ has a stable inverse, the above factorization is known as the canonical, or spectral, factorization of the Popov function. As we have seen, the canonical factorization is a key element in $H^2$ and $H^\infty$ estimation and control. We should remark that the above approach can be used to study solutions of the discrete-time algebraic Riccati equation (DARE) in terms of factorizations of the Popov function.[10] Its major benefit is that it allows one to treat the positive (semi)definite and indefinite cases in a unified fashion. This approach will be taken up in Chapter 7 to obtain general existence results for Riccati equations in the (possibly) indefinite case.

The results of Lemma 2.2.1 are not concerned with the case where the process $\{\mathbf{y}_i\}$ is a true stochastic process, *i.e.*, that its $z$-spectrum, $S_y(z)$, is nonnegative on the unit circle. When that is true, we have a further characterization of the Hermitian matrices, $Z$. The result is the KYP Lemma.

**Theorem 2.2.1 (KYP Lemma)** *Consider the observable pair $\{F, H\}$. Then the following two statements are equivalent:*

*(i) $S_y(z) \geq 0$ for all $z = e^{j\omega} \notin \lambda(F)$, where*

$$S_y(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}.$$

*(ii) There exists a Hermitian $Z$ such that*

$$\begin{bmatrix} Q - Z + FZF^* & S + FZH^* \\ S^* + HZF^* & R + HZH^* \end{bmatrix} \geq 0. \tag{2.2.21}$$

[10]See [McM52, You61, Yak70a] for the factorization of rational matrix functions.

**Remark:** The above Theorem has a remarkable interpretation. Note that in view of Lemma 2.2.1, we may write

$$S_y(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q-Z+FZF^* & S+FZH^* \\ S^*+HZF^* & R+HZH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix}.$$

Thus, Theorem 2.2.1 states that $S_y(z)$ is a true $z$-spectral density function (*i.e.*, it is nonnegative definite on the unit circle) if, and only if, there exists true stochastic inputs with nonnegative definite covariance

$$\begin{bmatrix} Q-Z+FZF^* & S+FZH^* \\ S^*+HZF^* & R+HZH^* \end{bmatrix} \geq 0,$$

that generate it! This has special significance to the problem of stochastic realization since it states that any nonnegative definite rational $z$-spectral density function can be realized by a finite-dimensional state-space model driven by true stochastic processes. The Theorem also gives a recipe as to how to find this realization in terms of the *linear matrix inequality* (LMI), (2.2.21).

In the Appendix we shall show how it is possible to use simple Krein space geometry to prove the KYP Lemma, and in fact a slightly more general time-variant counterpart. For the time being, however, let us proceed with main objective of this chapter — the development of a theory for linear estimation in Krein spaces.

## 2.3 On Krein Spaces

We briefly introduce the definitions and basic properties of Krein spaces, focusing on those results that we shall need later. Detailed expositions can be found in the books [Bog74, Ist87, IKL82]. Most readers will be familiar with finite-dimensional (often called Euclidean) and infinite-dimensional Hilbert spaces. Finite-dimensional (often called *Minkowski*) and infinite-dimensional Krein spaces share many of the properties Hilbert spaces, but differ in some important ways that we shall emphasize in the following.

**Definition 2.3.1 (Krein Spaces)** *An abstract vector space* $\{\mathcal{K}, \langle\cdot,\cdot\rangle\}$ *that satisfies the following requirements is called a Krein Space:*

*(i)* $\mathcal{K}$ *is a linear space over* $\mathcal{C}$*, the complex numbers.*

*(ii) There exists a bilinear form* $\langle \cdot, \cdot \rangle \in \mathcal{C}$ *on* $\mathcal{K}$ *such that*

*(a)* $\langle \mathbf{y}, \mathbf{x} \rangle \ = \ \langle \mathbf{x}, \mathbf{y} \rangle^*$

*(b)* $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle \ = \ a\langle \mathbf{x}, \mathbf{z} \rangle \ + \ b\langle \mathbf{y}, \mathbf{z} \rangle$

*for any* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{K}$*,* $a, b \in \mathcal{C}$*, and where* $*$ *denotes complex conjugation.*

*(iii) The vector space* $\mathcal{K}$ *admits a direct orthogonal sum decomposition*

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$$

*such that* $\{\mathcal{K}_+, \langle \cdot, \cdot \rangle\}$ *and* $\{\mathcal{K}_-, -\langle \cdot, \cdot \rangle\}$ *are Hilbert spaces, and*

$$\langle \mathbf{x}, \mathbf{y} \rangle \ = \ 0$$

*for any* $\mathbf{x} \in \mathcal{K}_+$ *and* $\mathbf{y} \in \mathcal{K}_-$*.*

**Remarks:**

1. Recall that Hilbert spaces satisfy not only (i) and (ii)-(a), (ii)-(b) above, but also the requirement that

$$\langle \mathbf{x}, \mathbf{x} \rangle > 0 \quad \text{when} \quad \mathbf{x} \neq 0.$$

2. The fundamental decomposition of $\mathcal{K}$ defines two projection operators $\mathcal{P}_+$ and $\mathcal{P}_-$ such that

$$\mathcal{P}_+\mathcal{K} = \mathcal{K}_+ \quad \text{and} \quad \mathcal{P}_-\mathcal{K} = \mathcal{K}_-.$$

Therefore for every $\mathbf{x} \in \mathcal{K}$ we can write

$$\mathbf{x} = P_+\mathbf{x} \ + \ P_-\mathbf{x} \ = \ \mathbf{x}_+ \ + \ \mathbf{x}_- \ \ , \ \ \mathbf{x}_\pm \in \mathcal{K}_\pm.$$

Note that for every $\mathbf{x} \in \mathcal{K}_+$, we have $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, but the converse is not true: $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ does not necessarily imply that $\mathbf{x} \in \mathcal{K}_+$.

3. A vector $\mathbf{x} \in \mathcal{K}$ will be said to be *positive* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, *neutral* if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$, or *negative* if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$. Correspondingly, a subspace $\mathcal{M} \subset \mathcal{K}$ can be positive, neutral, or negative, if all its elements are so, respectively.

We now focus on linear subspaces of $\mathcal{K}$. We shall define $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ as the linear subspace of $\mathcal{K}$ spanned by the elements $\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_N$ in $\mathcal{K}$. The *Gramian* of the collection of elements $\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ is defined as the $(N+1) \times (N+1)$ matrix

$$R_y \triangleq [\langle \mathbf{y}_i, \mathbf{y}_j \rangle]_{i,j=0:N} \,. \tag{2.3.1}$$

The reflexivity property, $\langle \mathbf{y}_i, \mathbf{y}_j \rangle = \langle \mathbf{y}_j, \mathbf{y}_i \rangle^*$, shows that the Gramian is a Hermitian matrix.

It is useful to introduce some matrix notation here. We shall write the column vector of the $\{\mathbf{y}_i\}$ as

$$\mathbf{y} = \text{col}\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_N\},$$

and denote the above Gramian of the $\{\mathbf{y}_i\}$ as

$$R_y \triangleq \langle \mathbf{y}, \mathbf{y} \rangle.$$

[ A useful mnemonic device for recalling this is to think of the $\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ as "random variables" and their Gramian as the "covariance matrix"

$$R_y = \left[ E\mathbf{y}_i\mathbf{y}_j^* \right] = E\mathbf{y}\mathbf{y}^*,$$

where $E(\cdot)$ denotes "expectation". We use the quotation marks because in our context, the covariance matrix will generally be indefinite, so we are dealing with some kind of generalized "random variables". We do not pursue this interpretation here since our aim is only to provide readers with a convenient device for interpreting the shorthand notation.]

So also if we have two sets of elements $\{\mathbf{z}_0, \ldots, \mathbf{z}_M\}$ and $\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ we shall write

$$\mathbf{z} = \text{col}\{\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_M\} \quad \text{and} \quad \mathbf{y} = \text{col}\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_N\},$$

and introduce the $(M+1) \times (N+1)$ cross-Gramian matrix

$$R_{zy} = [\langle \mathbf{z}_i, \mathbf{y}_j \rangle]_{\substack{i=0:M \\ j=0:N}} \triangleq \langle \mathbf{z}, \mathbf{y} \rangle.$$

Note the property

$$R_{zy} = R_{yz}^*.$$

We now proceed with a simple result.

**Lemma 2.3.1 (Positive and Negative Linear Subspaces)** *Suppose* $\mathbf{y}_0, \ldots, \mathbf{y}_N$ *are linearly independent elements of* $\mathcal{K}$. *Then* $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ *is a "positive" (negative) subspace of* $\mathcal{K}$ *if, and only if,*

$$R_y > 0 \quad (R_y < 0).$$

**Proof:** Since the $\mathbf{y}_i$ are linearly independent, for any $\mathbf{z} \neq 0 \in \mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ there exists a unique $k \in \mathcal{C}^{N+1}$ such that $\mathbf{z} = k^*\mathbf{y}$. Now

$$\langle \mathbf{z}, \mathbf{z} \rangle = k^* \langle \mathbf{y}, \mathbf{y} \rangle k = k^* R_y k,$$

so that $\langle \mathbf{z}, \mathbf{z} \rangle > 0$ for all $\mathbf{z} \in \mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$, if, and only if, $R_y > 0$. The proof for $R_y < 0$ is similar.

∎

Note that any linear subspace whose Gramian has mixed inertia (both positive and negative eigenvalues) will have elements in both the positive and negative subspaces.

### 2.3.1 A Geometric Interpretation

Indefinite metric spaces were perhaps first introduced into the solution of physical problems via the finite-dimensional Minkowski spaces of special relativity [LEMW23, Ein31, Sch73a], and some geometric insight may be gained by considering the special 3-dimensional Minkowski space of Figure 2.1, defined by the inner product

$$\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = x_1 x_2 + y_1 y_2 - t_1 t_2,$$

where

$$\mathbf{v}_1 = (x_1, y_1, t_1), \quad \mathbf{v}_2 = (x_2, y_2, t_2) \quad \text{and} \quad x_i, y_i, t_i \in \mathcal{C}.$$

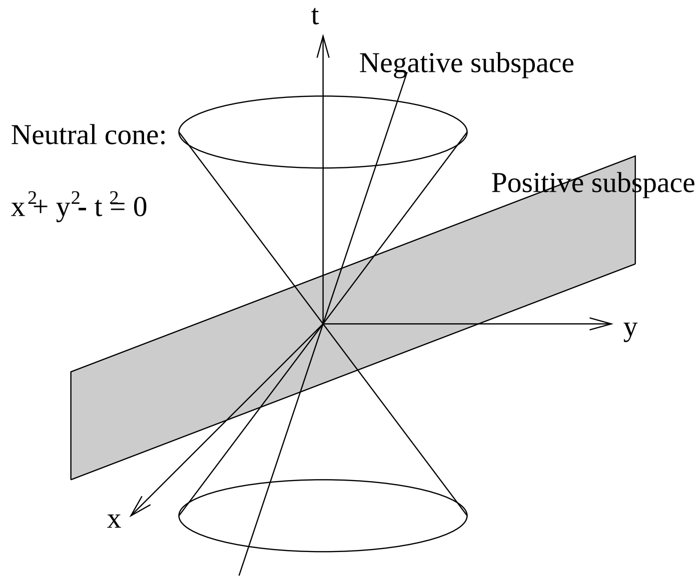


Figure 2.1: 3-dimensional Minkowski space

The (indefinite) squared norm of each vector $\mathbf{v} = (x, y, t)$ is equal to

$$\langle \mathbf{v}, \mathbf{v} \rangle = x^2 + y^2 - t^2.$$

In this case, we can take $\mathcal{K}_+$ to be the $x-y$ plane and $\mathcal{K}_-$ as the $t$ axis. The neutral subspace is given by the cone, $x^2 + y^2 - t^2 = 0$, with points inside the cone belonging to the negative subspace, $x^2 + y^2 - t^2 < 0$, and points outside the cone corresponding to the positive subspace, $x^2 + y^2 - t^2 > 0$.

Moreover, any plane passing through the origin but lying outside the neutral cone will have positive definite Gramian, and any line passing through the origin and inside the neutral cone will have negative definite Gramian. So also, any plane passing through the origin that intersects the neutral cone will have Gramian with mixed inertia, and any plane tangent to the cone will have singular Gramian.

Two key differences between Krein spaces and Hilbert spaces are the existence of *neutral* and *isotropic* vectors. As mentioned earlier, a neutral vector is a nonzero vector that has zero length; an isotropic vector is a nonzero vector lying in a linear subspace of $\mathcal{K}$ that is orthogonal to every element in that linear subspace. There are obviously no such vectors in Euclidean or Hilbert spaces. In the Minkowski space described above, $\begin{bmatrix} 1 & 1 & \sqrt{2} \end{bmatrix}$ is a neutral vector, and if one considers the linear subspace $\mathcal{L}\left\{\left[\begin{array}{ccc} 1 & 1 & \sqrt{2} \end{array}\right], \left[\begin{array}{ccc} \sqrt{2} & 0 & 1 \end{array}\right]\right\}$ then $\left[\begin{array}{ccc} 1 & 1 & \sqrt{2} \end{array}\right]$ is also an isotropic vector in this linear subspace.[11]

## 2.4 Projections in Krein Spaces

An important notion in both Hilbert and Krein spaces is that of the projection onto a subspace.

**Definition 2.4.1 (Projections in Krein Space)** *Given the element* $\mathbf{z}$ *in* $\mathcal{K}$, *and the elements* $\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_N\}$ *also in* $\mathcal{K}$, *we define* $\hat{\mathbf{z}}$ *to be the projection of* $\mathbf{z}$ *onto* $\mathcal{L}\{\mathbf{y}_0, \mathbf{y}_1, \ldots, \mathbf{y}_N\}$ *if,*

$$\mathbf{z} = \hat{\mathbf{z}} + \tilde{\mathbf{z}} \tag{2.4.1}$$

*where* $\hat{\mathbf{z}} \in \mathcal{L}\{\mathbf{y}_0, ..., \mathbf{y}_N\}$ *and* $\tilde{\mathbf{z}}$ *satisfies the orthogonality condition*

$$\tilde{\mathbf{z}} \ \perp \ \mathcal{L}\{\mathbf{y}_0, ..., \mathbf{y}_N\},$$

*or equivalently,* $\langle \tilde{\mathbf{z}}, \mathbf{y}_i \rangle = 0$ *for* $i = 0, 1, \ldots, N$.

In Hilbert space projections always exist and are unique. However, in Krein space this is not always the case. Indeed we have the following result, where for simplicity we have written $\mathcal{L}\{\mathbf{y}\} \stackrel{\Delta}{=} \mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$.

**Lemma 2.4.1 (Existence and Uniqueness of Projections)** *In the Hilbert space setting, projections always exist and are unique. However, in the Krein space setting:*

*(a) If the Gramian matrix* $R_y = \langle \mathbf{y}, \mathbf{y} \rangle$ *is nonsingular, then the projection of* $\mathbf{z}$ *onto* $\mathcal{L}\{\mathbf{y}\}$ *exists, is unique, and is given by*

$$\hat{\mathbf{z}} = \langle \mathbf{z}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{y} \rangle^{-1} \mathbf{y} = R_{zy} R_y^{-1} \mathbf{y}. \tag{2.4.2}$$

*(b) If the Gramian matrix* $R_y = \langle \mathbf{y}, \mathbf{y} \rangle$ *is singular, then*

[11] We should note that Minkowski (and Krein) spaces are closely related to the hyperbolic spaces first introduced by N.I. Lobachevskii in 18?? (see [Lob14]) in his researches on non-Euclidean geometry. In what follows we shall make use of only some rudimentary facts from hyperbolic geometry. For more on this subject consult [Fen89] and [Ive92].

*(i) If $\mathcal{R}(R_{yz}) \subseteq \mathcal{R}(R_y)$ (where $\mathcal{R}(A)$ denotes the column range space of the matrix $A$), the projection $\hat{\mathbf{z}}$ exists but is nonunique. In fact, $\hat{\mathbf{z}} = k_o^* \mathbf{y}$, where $k_o$ is "any" solution to the linear matrix equation*

$$R_y k_o = R_{yz}. \tag{2.4.3}$$

*(ii) If $\mathcal{R}(R_{yz}) \not\subseteq \mathcal{R}(R_y)$, the projection $\hat{\mathbf{z}}$ does not exist.*

**Proof:** Suppose $\hat{\mathbf{z}}$ is a projection of $\mathbf{z}$ onto the desired space. By (2.4.1), we can write

$$\mathbf{z} = k_o^* \mathbf{y} + \tilde{\mathbf{z}},$$

for some $k_o \in \mathcal{C}^{(N+1)}$. Since $\langle \tilde{\mathbf{z}}, \mathbf{y} \rangle = 0$,

$$R_{zy} = \langle \mathbf{z}, \mathbf{y} \rangle = k_o^* \langle \mathbf{y}, \mathbf{y} \rangle + 0 = k_o^* R_y. \tag{2.4.4}$$

If $R_y$ is nonsingular then the solution for $k$ in (2.4.4) is unique and the projection is given by (2.4.2). If $R_y$ is singular, two things may happen: either $\mathcal{R}(R_{yz}) \subseteq \mathcal{R}(R_y)$ in which case (2.4.4) will have a nonunique solution (since any $k_1^*$ in the left null space of $R_y$ can be added to $k_o^*$), or $\mathcal{R}(R_{yz}) \not\subseteq \mathcal{R}(R_y)$ in which case the projection does not exist since a solution to (2.4.4) does not exist.

In Hilbert spaces the projection always exists because it is always true that $\mathcal{R}(R_{yz}) \subseteq \mathcal{R}(R_y)$, or equivalently, that $\mathcal{N}(R_y) \subseteq \mathcal{N}(R_{zy})$, where $\mathcal{N}(A)$ is the right nullspace of the matrix $A$. To show this, suppose that $l \in \mathcal{N}(R_y)$. Then

$$\begin{aligned} R_y l = 0 \;\; &\Rightarrow \;\; l^* R_y l = 0 \\ &\Rightarrow \;\; l^* \langle \mathbf{y}, \mathbf{y} \rangle l \; = \; \langle l^* \mathbf{y}, l^* \mathbf{y} \rangle \; = \; 0 \\ &\Rightarrow \;\; l^* \mathbf{y} = 0, \end{aligned}$$

where the last equality follows from the fact that in Hilbert spaces $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = 0$. We now readily conclude that $\langle \mathbf{z}, l^* \mathbf{y} \rangle = R_{zy} l = 0$, *i.e.* , $l \in \mathcal{N}(R_{zy})$ and hence $\mathcal{N}(R_y) \subseteq \mathcal{N}(R_{zy})$. Therefore a solution to (2.4.4) (and hence a projection) always exists in Hilbert spaces.

In Hilbert spaces the projection is also unique because if $k_1$ and $k_2$ are two different solutions to (2.4.4), then $(k_1 - k_2)^* R_y = 0$. But the above argument shows that we must then have $(k_1 - k_2)\mathbf{y} = 0$. Hence the projection

$$\hat{\mathbf{z}} = k_1^* \mathbf{y} = k_2^* \mathbf{y}$$

is unique.

■

The proof of the above lemma shows that in Hilbert spaces the singularity of $R_y$ implies that the $\{\mathbf{y}_i\}$ are linearly dependent, *i.e.* ,

$$\det(R_y) = 0 \;\Leftrightarrow\; k^* \mathbf{y} = 0 \text{ for some vector } k \in \mathcal{C}^{N+1}.$$

In the Krein space setting, all we can deduce from the singularity of $R_y$ is that there exists a linear combination of the $\{\mathbf{y}_i\}$ that is orthogonal to every vector in $\mathcal{L}\{\mathbf{y}_0, ..., \mathbf{y}_N\}$, *i.e.* , that $\mathcal{L}\{\mathbf{y}_0, ..., \mathbf{y}_N\}$ contains an isotropic vector. This follows by noting that for any complex matrix $k_1$, and for any $k$ in the null space of $R_y$, we have

$$k_1^* R_y k \;=\; \langle k_1^* \mathbf{y}, k^* \mathbf{y} \rangle \;=\; 0 \;\;,$$

which shows that the linear combination $k^* \mathbf{y}$ is orthogonal to $k_1^* \mathbf{y}$, for every $k_1$, *i.e.* , $k^* \mathbf{y}$ is an isotropic vector in $\mathcal{L}\{\mathbf{y}\}$.

**Standing Assumption:** Since existence and uniqueness will be important for all our future results, we shall make the standing assumption that the Gramian

$$R_y \quad \text{is } \textit{nonsingular.}$$

### 2.4.1 Vector-Valued Projections

Consider the $n$-vector $\mathbf{z} = \text{col}\{\mathbf{z}_1, \ldots, \mathbf{z}_n\}$ composed of elements $\mathbf{z}_i \in \mathcal{K}$, and the set $\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ where $\mathbf{y}_j \in \mathcal{K}$; project *each* element $\mathbf{z}_i$ onto $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ to obtain $\hat{\mathbf{z}}_i$. We define $\hat{\mathbf{z}} = \text{col}\{\hat{\mathbf{z}}_1, \ldots, \hat{\mathbf{z}}_n\}$ as the projection of $\mathbf{z}$ onto $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$. [Strictly speaking, we should call $\hat{\mathbf{z}} \in \mathcal{K}^n$ the projection of $\mathbf{z} \in \mathcal{K}^n$ onto $\mathcal{L}^n\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$, since it is an element of $\mathcal{L}^n\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$, and not $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$. However, for simplicity, we shall generally use the looser terminology.]

It is easy to see that the results on the existence and uniqueness of projections in Lemma 2.4.1 continue to hold in the vector case as well.

In this connection, it will be useful to introduce a slight generalization of the definition of Krein spaces that was given in Sec. 2.3. There, in Definition 2.3.1, we mentioned that $\mathcal{K}$ should be linear over the field of complex numbers, $\mathcal{C}$. However, it turns out that we can replace $\mathcal{C}$ with any ring $\mathcal{S}$. In other words, the first two axioms for Krein spaces can be replaced by

(i) $\mathcal{K}$ is a linear space over the ring $\mathcal{S}$.

(ii) There exists a bilinear form $\langle \cdot, \cdot \rangle \in \mathcal{S}$ on $\mathcal{K}$ such that

  (a) $\langle \mathbf{y}, \mathbf{x} \rangle \;=\; \langle \mathbf{x}, \mathbf{y} \rangle^*$

  (b) $\langle a\mathbf{x} + b\mathbf{y}, \mathbf{z} \rangle \;=\; a\langle \mathbf{x}, \mathbf{z} \rangle \;+\; b\langle \mathbf{y}, \mathbf{z} \rangle$

  for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{K}$ and $a, b \in \mathcal{S}$, and where the operation $*$ depends on the ring $\mathcal{S}$.

When the inner product $\langle \cdot, \cdot \rangle \in \mathcal{S}$ is positive, $\{\mathcal{K}, \langle \cdot, \cdot \rangle\}$ is referred to as a *module*. Thus the third axiom for Krein spaces can be replaced by

(iii) The vector space $\mathcal{K}$ admits a direct orthogonal sum decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$$

  such that $\{\mathcal{K}_+, \langle \cdot, \cdot \rangle\}$ and $\{\mathcal{K}_-, -\langle \cdot, \cdot \rangle\}$ are modules, and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for any $\mathbf{x} \in \mathcal{K}_+$ and $\mathbf{y} \in \mathcal{K}_-$.

The most important case for us is when $\mathcal{S}$ is a ring of complex matrices, and the operation $*$ denotes Hermitian transpose.

The point of this generalization is that we can now directly define the projection of a vector $\mathbf{z} \in \mathcal{K}^n$ onto $\mathcal{L}^n\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ as an element $\hat{\mathbf{z}} \in \mathcal{L}^n\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$, such that

$$\hat{\mathbf{z}} = k_o^* \mathbf{y} \quad , \quad k_o^* \in \mathcal{C}^{n \times N}$$

where $k$ is such that

$$0 = \langle \mathbf{z} - k_o^* \mathbf{y}, \mathbf{y} \rangle \stackrel{\Delta}{=} R_{zy} - k_o^* R_y$$

or

$$k_o^* R_y = R_{zy}.$$

Finally, let us remark that to avoid additional notational burden, we shall often refrain from writing $\mathcal{K}^n$ and shall simply use the notation $\mathcal{K}$ for any Krein space. The ring $\mathcal{S}$ over which the Krein space is defined will be obvious from the context.

## 2.5 Projections and Quadratic Forms

In Hilbert space, projections *extremize* (minimize) certain quadratic forms, as we shall briefly first describe. In Krein spaces, we can in general only assert that projections *stationarize* such quadratic forms; further conditions need to be met for the stationary points to be extrema (minima). This will be elaborated in Sec. 2.5.1, in the context of (what we shall call) a stochastic minimization problem. In Sec. 2.5.2, we shall study a closely related quadratic form arising in what we shall call a partially *equivalent* deterministic minimization problem.

### 2.5.1 Stochastic Minimization Problems in Hilbert and Krein Spaces

Consider a collection of elements $\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ in a Krein space $\mathcal{K}$ with indefinite inner product $\langle \cdot, \cdot \rangle$. Let $\mathbf{z} = \text{col}\{\mathbf{z}_0, \ldots, \mathbf{z}_M\}$ be some column vector of elements in $\mathcal{K}$, and consider an arbitrary linear combination of $\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$, say $k^*\mathbf{y}$, where $k^* \in \mathcal{C}^{(M+1)\times(N+1)}$ and $\mathbf{y} = \text{col}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$. A natural object to study is the *error Gramian*

$$P(k) = \langle \mathbf{z} - k^*\mathbf{y}, \mathbf{z} - k^*\mathbf{y} \rangle. \tag{2.5.1}$$

To motivate the subsequent discussion, let us first assume that the $\{\mathbf{y}_i\}$ and $\{\mathbf{z}_j\}$ belong to a Hilbert space of zero-mean random variables, and that their variance and cross-variances are known. In this case the inner product is $\langle \mathbf{z}_i, \mathbf{y}_j \rangle_{\mathcal{H}} = E\mathbf{z}_i\mathbf{y}_j^*$ (where $E(\cdot)$ denotes expectation), and $P(k)$ is simply the mean-square-error (or error variance) matrix in estimating $\mathbf{z}$ using $k^*\mathbf{y}$, viz.,

$$P(k) = E(\mathbf{z} - k^*\mathbf{y})(\mathbf{z} - k^*\mathbf{y})^* = \|\mathbf{z} - k^*\mathbf{y}\|^2_{\mathcal{H}}, \quad \text{say.}$$

It is well-known that the linear least-mean-square estimate, which minimizes $P(k)$, is given by the projection of $\mathbf{z}$ on $\mathcal{L}\{\mathbf{y}\}$:

$$\hat{\mathbf{z}} = k_o^*\mathbf{y},$$

where

$$k_o^* = E\mathbf{z}\mathbf{y}^* \left[E\mathbf{y}\mathbf{y}^*\right]^{-1} = R_{zy}R_y^{-1}.$$

The simple proof will be instructive. Thus note that

$$P(k) = \|\mathbf{z} - k^*\mathbf{y}\|^2_{\mathcal{H}} = \|\mathbf{z} - \hat{\mathbf{z}} + \hat{\mathbf{z}} - k^*\mathbf{y}\|^2_{\mathcal{H}} = \|\mathbf{z} - \hat{\mathbf{z}}\|^2_{\mathcal{H}} + \|\hat{\mathbf{z}} - k^*\mathbf{y}\|^2_{\mathcal{H}},$$

since by the definition of $\hat{\mathbf{z}}$, it holds that

$$\langle \mathbf{z} - \hat{\mathbf{z}}, \hat{\mathbf{z}} - k^*\mathbf{y}\rangle_{\mathcal{H}} = 0.$$

Clearly, since $\hat{\mathbf{z}} = k_o^*\mathbf{y}$,

$$P(k) \geq P(k_o),$$

with equality achieved only when $k = k_o$.

However, this argument breaks down when the elements are in a Krein space, since then we could have

$$\|\hat{\mathbf{z}} - k^*\mathbf{y}\|^2 = \|k_o^*\mathbf{y} - k^*\mathbf{y}\|^2 = 0, \quad \text{even if} \quad k_o \neq k.$$

All we can assert is that

$$k_o^*\mathbf{y} - k^*\mathbf{y} = \text{ an isotropic vector in the linear subspace spanned by } \{\mathbf{y}_0, \ldots, \mathbf{y}_N\}.$$

Moreover, since $\|k_o^*\mathbf{y} - k^*\mathbf{y}\|^2$ could be negative, it is not true that $P(k)$ will be minimized by choosing $k = k_o$. So a closer study is necessary.

We shall start with a definition.

**Definition 2.5.1 (Stationary Point)** *The matrix $k_o \in \mathcal{C}^{(N+1)\times(M+1)}$ is said to be a stationary point of an $(M+1)\times(M+1)$ matrix quadratic form in $k$, say*

$$P(k) = A + Bk + k^*B^* + k^*Ck,$$

*if, and only if, $k_o a$ is a stationary point of the "scalar" quadratic form $a^*P(k)a$ for all complex column vectors $a \in \mathcal{C}^{M+1}$,* i.e. , *if, and only if,*

$$\left.\frac{\partial a^*P(k)a}{\partial ka}\right|_{k=k_o} = 0.$$

Now we can prove the following.

**Lemma 2.5.1 (Condition for Minimum)** *A stationary point of $P(k)$ is a minimum if, and only if, for all $a \in \mathcal{C}^{M+1}$*

$$\left.\frac{\partial^2 a^*P(k)a}{\partial (ka)^2}\right|_{k=k_o} \geq 0. \tag{2.5.2}$$

*Moreover, it is a unique minimum if, and only if,*

$$\left.\frac{\partial^2 a^*P(k)a}{\partial (ka)^2}\right|_{k=k_o} > 0. \tag{2.5.3}$$

**Proof:** Writing the Taylor series expansion of $a^*P(k)a$ around the stationary point $k_o$ yields (since $a^*P(k)a$ is quadratic in $ka$)

$$a^*P(k)a = a^*P(k_o)a + \underbrace{\left.\frac{\partial a^*P(k)a}{\partial ka}\right|_{k=k_o}}_{=0} \cdot (k-k_o)a + a^*(k-k_o)^* \left.\frac{\partial^2 a^*P(k)a}{\partial (ka)^2}\right|_{k=k_o} \cdot (k-k_o)a,$$

or equivalently,

$$a^*P(k)a - a^*P(k_o)a = a^*(k-k_o)^* \left.\frac{\partial^2 a^*P(k)a}{\partial (ka)^2}\right|_{k=k_o} \cdot (k-k_o)a.$$

Using the above expression, we see that $k_o$ is a minimum, *i.e.* , $a^*P(k)a - a^*P(k_o)a \geq 0$ for all $k \neq k_o$ if, and only if, (2.5.2) is satisfied. Moreover, $k_o$ will be a unique minimum, *i.e.* , $a^*P(k)a - a^*P(k_o)a > 0$ for all $k \neq k_o$ if, and only if, (2.5.3) is satisfied.

■

Let us now return to the error Gramian $P(k)$ in (2.5.1), and expand it as

$$P(k) = \langle \mathbf{z}, \mathbf{z} \rangle_{\mathcal{K}} - \langle \mathbf{z}, \mathbf{y} \rangle_{\mathcal{K}} k - k^* \langle \mathbf{y}, \mathbf{z} \rangle_{\mathcal{K}} + k^* \langle \mathbf{y}, \mathbf{y} \rangle_{\mathcal{K}} k, \tag{2.5.4}$$

or more compactly,

$$P(k) = \begin{bmatrix} I & -k^* \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} \begin{bmatrix} I \\ -k \end{bmatrix}. \tag{2.5.5}$$

Note that the center matrix appearing in (2.5.5) is the Gramian of the vector $\mathrm{col}\{\mathbf{z}, \mathbf{y}\}$.

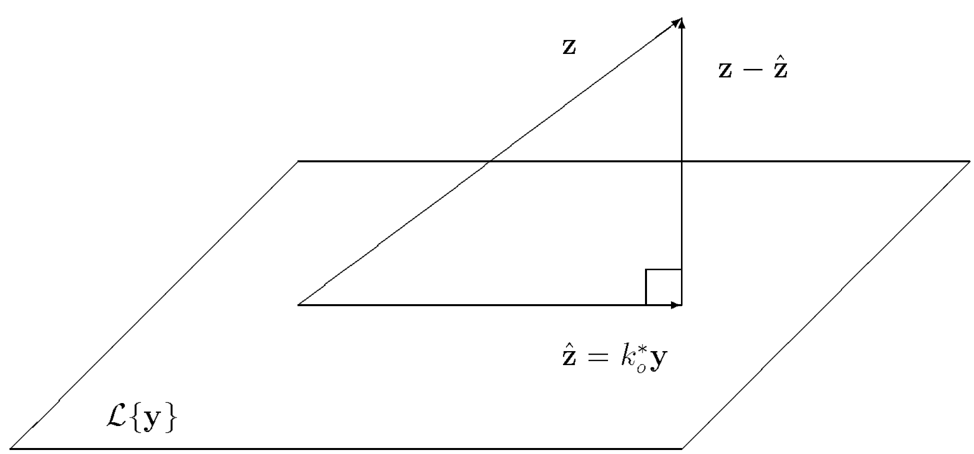


Figure 2.2: The projection $\hat{\mathbf{z}} = k_o^* \mathbf{y}$ stationarizes the error Gramian $P(k) = \langle \mathbf{z} - k^*\mathbf{y}, \mathbf{z} - k^*\mathbf{y} \rangle$ over all $k^*\mathbf{y} \in \mathcal{L}\{\mathbf{y}\}$.

For this particular quadratic form, we can use the easily verified triangular factorization (recall our standing assumption that $R_y$ is nonsingular),

$$\begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} =$$

$$\begin{bmatrix} I & R_{zy} R_y^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} R_z - R_{zy} R_y^{-1} R_{yz} & 0 \\ 0 & R_y \end{bmatrix} \begin{bmatrix} I & 0 \\ R_y^{-1} R_{yz} & I \end{bmatrix} \tag{2.5.6}$$

to write

$$a^*P(k)a =$$

$$\begin{bmatrix} a^* & a^*k^* - a^*R_{zy}R_y^{-1} \end{bmatrix} \begin{bmatrix} R_z - R_{zy}R_y^{-1}R_{yz} & 0 \\ 0 & R_y \end{bmatrix} \begin{bmatrix} a \\ ka - R_y^{-1}R_{yz}a \end{bmatrix} \tag{2.5.7}$$

Calculating the stationary point of $P(k)$, and the corresponding condition for a minimum, is now straightforward. Note, moreover, that $R_y$ nonsingular implies that the stationary point is unique.

**Theorem 2.5.1 (Stationary Point of the Error Gramian)** *When $R_y$ is nonsingular, $k_o$, the unique coefficient matrix in the projection of $\mathbf{z}$ onto $\mathcal{L}\{\mathbf{y}\}$,*

$$\hat{\mathbf{z}} = k_o^*\mathbf{y} \quad , \quad k_o = R_y^{-1}R_{yz}$$

*yields the unique stationary point of the error Gramian*

$$P(k) \triangleq \langle \mathbf{z} - k^*\mathbf{y}, \mathbf{z} - k^*\mathbf{y} \rangle = \begin{bmatrix} I & -k^* \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} \begin{bmatrix} I \\ -k \end{bmatrix} \tag{2.5.8}$$

*over all $k \in \mathcal{C}^{(N+1)\times(M+1)}$. Moreover, the value of $P(k)$ at the stationary point is given by*

$$P(k_o) = R_z - R_{zy}R_y^{-1}R_{yz}.$$

**Proof:** The claims follow easily from (2.5.7) by differentiation.

∎

Further differentiation and use of Lemma 2.5.1 yields the following result.

**Corollary 2.5.1 (Condition for a Minimum)** *In Theorem 2.5.1, $k_o$ is a unique minimum if, and only if,*

$$R_y > 0.$$

*i.e., $R_y$ is not only nonsingular but also positive definite.*

### 2.5.2 A Partially Equivalent Deterministic Problem

We shall now consider what we call a partially equivalent deterministic problem. We refer to it as *deterministic*, because it involves computing the stationary point of a certain *scalar* quadratic form over ordinary complex variables (not Krein space ones). Moreover, it is called partially equivalent since its solution, *i.e.* , the stationary point, is given by the *same* expression as the projection of one suitably defined Krein space vector onto another, while the condition for a minimum is *different* than that for the Krein space projection.

To this end, consider the scalar second order form

$$J(z,y) \triangleq \begin{bmatrix} z^* & y^* \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \begin{bmatrix} z \\ y \end{bmatrix} \tag{2.5.9}$$

where the central matrix is the inverse of the Gramian matrix in the stochastic problem of Theorem 2.5.1 - see (2.5.5). Suppose we seek the stationarizing element $z_o$ for a given $y$. [Of course now we assume not only that $R_y$ is nonsingular, but so also the block matrix appearing in (2.5.9).] Note that $z$ and $y$ are no longer bold face, meaning that they are to be regarded as (ordinary) vectors of complex numbers.

Referring to the discussion at the beginning of Sec. 2.5.1 on Hilbert spaces, the motivation for this problem is the fact that for jointly Gaussian random vectors $\{\mathbf{z},\mathbf{y}\}$, the linear least-mean-squares estimate can be found as the conditional mean of the conditional density $p_{\mathbf{zy}}(z,y)/p_{\mathbf{y}}(y)$. When $\{\mathbf{z},\mathbf{y}\}$ are zero-mean with covariance matrix $\begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}$, taking logarithms of the conditional density results in the quadratic form (2.5.9), which is the negative of the so-called *log-likelihood function.* In this case, the relation between (2.5.9) and the projection follows from the fact that the linear least-mean-squares estimate is the same as the maximum likelihood estimate (obtained by minimizing (2.5.9)). With this motivation, we now introduce and study the quadratic form $J(z,y)$, without any reference to $\{\mathbf{z},\mathbf{y}\}$ being Gaussian.

**Theorem 2.5.2 (Deterministic Stationary Point)** *Suppose both $R_y$ and the block matrix in (2.5.9) are nonsingular. Then* *(a) The stationary point $z_o$ of $J(z,y)$ over $z$ is given by*

$$z_o = R_{zy}R_y^{-1}y.$$

*(b) The value of $J(z,y)$ at the stationary point is*

$$J(z_o, y) = y^* R_y^{-1} y.$$

**Corollary 2.5.2 (Condition for a Minimum)** *In Theorem 2.5.2, $z_o$ is a minimum if, and only if,*

$$R_z - R_{zy}R_y^{-1}R_{yz} > 0.$$

**Proof:** We note that (see (2.5.6))

$$\begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} =$$

$$\begin{bmatrix} I & 0 \\ -R_y^{-1}R_{yz} & I \end{bmatrix} \begin{bmatrix} R_z - R_{zy}R_y^{-1}R_{yz} & 0 \\ 0 & R_y \end{bmatrix}^{-1} \begin{bmatrix} I & -R_{zy}R_y^{-1} \\ 0 & I \end{bmatrix}$$

so that we can write

$$J(z,y) =$$

$$\begin{bmatrix} (z^* - y^* R_y^{-1} R_{yz}) & y^* \end{bmatrix} \begin{bmatrix} R_z - R_{zy}R_y^{-1}R_{yz} & 0 \\ 0 & R_y \end{bmatrix}^{-1} \begin{bmatrix} (z - R_{zy}R_y^{-1}y) \\ y \end{bmatrix}.$$

It now follows by differentiation that the stationary point of $J(z,y)$ is equal to $z_o = R_{zy}R_y^{-1}y$, and that $J(z_o, y) = y^* R_y^{-1} y$. To prove the Corollary, we differentiate once again, and use Lemma 2.5.1.

∎

**Remark 1.** Comparing the results of Theorems 2.5.1 and 2.5.2 shows that the stationary point $z_o$, of the scalar quadratic form (2.5.9), is given by a formula that is exactly the same as that in Theorem 2.5.1 for the Krein space projection of a vector $\mathbf{z}$

onto the linear span $\mathcal{L}\{\mathbf{y}\}$. However, in Theorem 2.5.2 there is no Krein space: $z$ and $y$ are just vectors (in general of different dimensions) in Euclidean space and $z_o$ is *not* the projection of $z$ onto the vector $y$. What we have shown in Theorem 2.5.2 is that by properly defining the scalar quadratic form as in (2.5.9) using coefficient matrices $R_z$, $R_y$, $R_{zy}$, and $R_{yz}$ that are arbitrary, but can be regarded as being obtained from Gramians and cross Gramians of some Krein space vectors $\{\mathbf{z}, \mathbf{y}\}$, we can calculate the stationary point using the same recipe as in Theorem 2.5.1.

**Remark 2.** However, although the stationary points of the matrix quadratic form $P(k)$ and the scalar quadratic form $J(z, y)$ are found by the same computations, the two forms do not necessarily simultaneously have a minimum, since one requires the condition $R_y > 0$ (Cor. 2.5.1), and the other requires the condition $R_z - R_{zy}R_y^{-1}R_{yz} > 0$ (Cor. 2.5.2). This is the major difference from the classical Hilbert space context where we have

$$\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \rangle_{\mathcal{H}} = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} > 0. \tag{2.5.10}$$

When (2.5.10) holds, the approaches of Theorems 2.5.1 and 2.5.2 give equivalent results.

**Corollary 2.5.3 (Simultaneous Minima)** *For vectors* $\mathbf{z}$ *and* $\mathbf{y}$ *of linear independent elements in a Hilbert space* $\mathcal{H}$*, the conditions* $R_z - R_{zy}R_y^{-1}R_{yz} > 0$ *and* $R_y > 0$ *occur simultaneously.*

**Proof:** Immediate from the factorization (2.5.5).

∎

We shall see in more detail in Chapters 3 and 4, and to some extent in Sec. 2.7.2, that this difference is what makes $H^\infty$ (and risk-sensitive and finite memory adaptive filtering) results different from $H^2$ results. Briefly, $H^\infty$ problems will lead directly to certain indefinite quadratic forms: to stationarize them we shall find it useful to set up the corresponding Krein space problem and appeal to Theorem 2.5.1. While this will give an algorithm, further work will be necessary to check for the minimum condition of Theorem 2.5.2 in the $H^\infty$ problem.

It is this difference that leads us to say that the deterministic problem is only *partially equivalent* to the stochastic problem of Sec. 2.5.1. [We may remark that we are making a distinction between equivalence and "duality": one can in fact define duals to both the above problems, but we defer this topic to Chapter 6.]

**Remark 3.** Finally, recall that Lemma 2.4.1 on the existence and uniqueness of the projection implies that the stochastic problem of Theorem 2.5.1 has a unique solution if, and only if, $R_y$ is nonsingular, thus explaining our standing assumption. The following result is the analog for the deterministic problem.

**Lemma 2.5.2 (Existence of Stationarizing Solutions)** *The deterministic problem of Theorem 2.5.2 has a unique stationarizing solution for all $y$ if, and only if, $R_y$ is nonsingular.*

**Proof:** Let us denote

$$\begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} = \begin{bmatrix} A & B \\ B & C \end{bmatrix},$$

so that

$$J(z,y) = \begin{bmatrix} z^* & y^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}.$$

If $J(z,y)$ has a unique stationarizing solution for all $y$ then $A$ must be nonsingular (since by differentiation the stationary point must satisfy the equation $Az_o = By$). But the invertibility of $A$ and the whole center matrix appearing in $J(z,y)$ imply the invertibility of the Schur complement $C - B^*A^{-1}B$. But it is easy to check that this Schur complement must be the inverse of $R_y$. Thus $R_y$ must be invertible.

On the other hand if $R_y$ is invertible then the deterministic problem has a unique stationarizing solution as given by Theorem 2.5.2.

■

### 2.5.3 Alternative Inertia Conditions for Minima

In many cases it can be complicated to directly check for the positivity condition of the deterministic problem, namely $R_z - R_{zy}R_y^{-1}R_{yz} > 0$. On the other hand, it is often easier to compute the inertia (the number of positive, negative and zero eigenvalues) of $R_y$ itself. This often suffices [SHK96b, SHK96a, SHK95].

**Lemma 2.5.3 (Inertia Conditions for Deterministic Minimization)** *(a) If $R_y$ and $R_z$ are nonsingular, then the deterministic problem of Theorem 2.5.2 will have a minimizing solution (*i.e. *$R_z - R_{zy}R_y^{-1}R_{yz}$ will be $> 0$) if, and only if,*

$$I_-[R_y] = I_-[R_z] + I_-[(R_y - R_{yz}R_z^{-1}R_{zy})], \tag{2.5.11}$$

*where $I_-[A]$ denotes the negative inertia (number of negative eigenvalues) of $A$.*

*(b) When $R_z > 0$ (rather than just being nonsingular) then we will have a minimizing solution if, and only if,*

$$I_-[R_y] = I_-[R_y - R_{yz}R_z^{-1}R_{zy}], \tag{2.5.12}$$

i.e. *if, and only if, $R_y$ and $R_y - R_{yz}R_z^{-1}R_{zy}$ have the same inertia.*

**Proof:** If $R_y$ and $R_z$ are both nonsingular, then equating the lower-upper and upper-lower block triangular factorizations of the Gramian matrix in (2.5.6) will yield the result that

$$\begin{bmatrix} R_z - R_{zy}R_y^{-1}R_{yz} & 0 \\ 0 & R_y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} R_z & 0 \\ 0 & R_y R_{yz}R_z^{-1}R_{zy} \end{bmatrix}$$

are congruent. By Sylvester's Law that congruent matrices have the same inertia [GL89], we have

$$I_-[R_z - R_{zy}R_y^{-1}R_{yz}] + I_-[R_y] = I_-[R_z] + I_-[(R_y - R_{yz}R_z^{-1}R_{zy})].$$

Now if (2.5.11) holds, then $I_-[R_z - R_{zy}R_y^{-1}R_{yz}] = 0$, so that $R_z - R_{zy}R_y^{-1}R_{yz} > 0$.

Conversely if $I_-[R_z - R_{zy}R_y^{-1}R_{yz}] = 0$, then (2.5.11) holds.

When $R_z > 0$, we have $I_-[R_z] = 0$, and (2.5.12) follows immediately.

■

The general results presented so far can be made even more explicit when there is more structure in the problems. In particular, we shall see that when we have state-space structure both $R_z$ and $R_y - R_{yz}R_z^{-1}R_{zy}$ are block-diagonal. Moreover, a "Krein space Kalman filter" will yield a direct method for computing the inertia of $R_y$. Thus, when we have state-space structure, it will be much easier to use the results of Lemma 2.5.3 than to directly check for the positivity of $R_z - R_{zy}R_y^{-1}R_{yz}$ [HSK94c, SHK96b, SHK96a, SHK95].

## 2.6 State-Space Structure

One approach at this point is to begin by assuming that the components $\{\mathbf{y}_j\}$ of $\mathbf{y}$ arise from an underlying Krein space state-space model. However, to better motivate the introduction of such state-space models, we shall start with the following (indefinite) quadratic minimization problem.

Consider a system described by the state-space equations

$$\left\{ \begin{array}{rcl} x_{j+1} & = & F_j x_j + G_j u_j, \quad 0 \le j \le N \\ y_j & = & H_j x_j + v_j \end{array} \right. \tag{2.6.1}$$

where $F_j \in \mathcal{C}^{n\times n}$, $G_j \in \mathcal{C}^{n\times m}$ and $H_j \in \mathcal{C}^{p\times n}$ are given matrices and the initial state $x_0 \in \mathcal{C}^n$, the driving disturbance $u_j \in \mathcal{C}^m$, and the measurement disturbance $v_j \in \mathcal{C}^p$, are unknown complex vectors. The output $y_j \in \mathcal{C}^p$ is assumed known for all $j$.

In many applications one is confronted with the following deterministic minimization problem: Given $\{y_j\}_{j=0}^N$, minimize over $x_0$ and $\{u_j\}_{j=0}^N$ the quadratic form

$$J(x_0, u, y) = x_0^*\Pi_0^{-1}x_0 + \sum_{j=0}^N \left[ \begin{array}{cc} u_j^* & v_j^* \end{array} \right] \left[ \begin{array}{cc} Q_j & S_j \\ S_j^* & R_j \end{array} \right]^{-1} \left[ \begin{array}{c} u_j \\ v_j \end{array} \right], \tag{2.6.2}$$

subject to the state-space constraints (2.6.1), and where $Q_j \in \mathcal{C}^{m\times m}$, $S_j \in \mathcal{C}^{m\times p}$, $R_j \in \mathcal{C}^{p\times p}$, $\Pi_0 \in \mathcal{C}^{n\times n}$ are (possibly indefinite) given Hermitian matrices.

The above deterministic quadratic form is usually encountered in filtering problems; a special case that we shall see in the nect chapter is the $H^\infty$ filtering problem where the weighting matrices are $\Pi_0$, $Q_j = I$ and $R_j = \left[ \begin{array}{cc} I & 0 \\ 0 & -\gamma_f^2 I \end{array} \right]$, and where $H_i$ is now replaced by $\text{col}\{H_i, L_i\}$. Another application arises in adaptive filtering, in which case we usually have $u_j \equiv 0$ and $F_j \equiv I$ [SK94b, HSK96a]. In the general case, however, $\Pi_0$ represents the penalty on the initial state, and $\{Q_j, R_j, S_j\}$ represents the penalty on the driving and measurement disturbances $\{u_j, v_j\}$. (There is also a "dual" quadratic form that arises in control applications, which we shall study elsewhere.)

Such deterministic problems can be solved via a variety of methods, such as dynamic programming or Lagrange multipliers (see *e.g.* , [BB95]), but we shall find it easier to use the equivalence discussed in Sec. 2.5: construct a (partially) equivalent Krein space (or stochastic) problem. In order to do so we first need to express the $J(x_0, u, y)$ of (2.6.2) in the form of (2.5.9) of Sec. 2.5.2.

For this, we first introduce some vector notation. Note that the states $\{x_j\}$ and the outputs $\{y_j\}$ are linear combinations of the fundamental quantities $\{x_0, \{u_j, v_j\}_{j=0}^N\}$. We introduce (the state transition matrix)

$$\Phi(j,k) = F_{j-1} \dots F_{k+1}, \quad j > k \quad , \quad \Phi(j,j) = I.$$

and define

$$h_{jk} \triangleq H_j F_{j-1} \dots F_{k+1} G_k = H_j \Phi(j,k) G_k$$

as the response at time $j$ to an impulse at time $k < j$ (assuming both $x_0 = 0$ and $v_k \equiv 0$).

Then with

$$y \triangleq \text{col}\{y_0, \dots, y_N\} \quad , \quad u \triangleq \text{col}\{u_0, \dots, u_N\} \quad , \quad v \triangleq \text{col}\{v_0, \dots, v_N\}$$

the state-space equations (2.6.1) allow us to write

$$y = \mathcal{O} x_0 + \Gamma u + v = \begin{bmatrix} \mathcal{O} & \Gamma \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} + v \tag{2.6.3}$$

where $\mathcal{O}$ and $\Gamma$ are the observability map and the impulse response matrix, respectively,

$$\mathcal{O} = \begin{bmatrix} H_0 \\ H_1\Phi(1,0) \\ H_2\Phi(2,0) \\ \vdots \\ H_N\Phi(N,0) \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} 0 & & & & \\ h_{10} & 0 & & & \\ h_{20} & h_{21} & 0 & & \\ h_{30} & h_{31} & h_{32} & . & \\ . & . & . & . & . \end{bmatrix}.$$

With these definitions we can rewrite $J(x_0,u,y)$ as

$$J(x_0,u,y) = \begin{bmatrix} x_0^* & u^* & v^* \end{bmatrix} \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q & S \\ 0 & S^* & R \end{bmatrix}^{-1} \begin{bmatrix} x_0 \\ u \\ v \end{bmatrix} \tag{2.6.4}$$

where we have defined

$$Q \triangleq Q_0 \oplus \ldots \oplus Q_N \quad , \quad R \triangleq R_0 \oplus \ldots \oplus R_N \quad , \quad S \triangleq S_0 \oplus \ldots \oplus S_N.$$

Finally we make the change of coordinates

$$\begin{bmatrix} x_0 \\ u \\ y \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mathcal{O} & \Gamma & I \end{bmatrix} \begin{bmatrix} x_0 \\ y \\ v \end{bmatrix}$$

to obtain

$$\begin{aligned} J(x_0,u,y) &= \begin{bmatrix} x_0 \\ u \\ y \end{bmatrix}^* \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\mathcal{O} & -\Gamma & I \end{bmatrix}^* \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q & S \\ 0 & S^* & R \end{bmatrix}^{-1} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\mathcal{O} & -\Gamma & I \end{bmatrix} \begin{bmatrix} x_0 \\ u \\ y \end{bmatrix} \\ &= \begin{bmatrix} x_0 \\ u \\ y \end{bmatrix}^* \left\{ \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mathcal{O} & \Gamma & I \end{bmatrix} \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q & S \\ 0 & S^* & R \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mathcal{O} & \Gamma & I \end{bmatrix}^* \right\}^{-1} \begin{bmatrix} x_0 \\ u \\ y \end{bmatrix} \end{aligned} \tag{2.6.5}$$

This is now of the desired form (2.5.9) (with $z \triangleq \text{col}\{x_0, u\}$). Therefore comparing with (2.5.8) in Theorem 2.5.1, we introduce a Krein space state-space model

$$\left\{ \begin{array}{rcl} \mathbf{x}_{j+1} &=& F_j\mathbf{x}_j + G_j\mathbf{u}_j, \quad 0 \le j \le N \\ \mathbf{y}_j &=& H_j\mathbf{x}_j + \mathbf{v}_j \end{array} \right. \tag{2.6.6}$$

where the initial state, $\mathbf{x}_0$, and the driving and measurement disturbances, $\{\mathbf{u}_j\}$ and $\{\mathbf{v}_j\}$ are such that

$$\left\langle \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \\ \mathbf{x}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{u}_k \\ \mathbf{v}_k \\ \mathbf{x}_0 \end{bmatrix} \right\rangle = \begin{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \delta_{jk} & 0 \\ 0 & \Pi_0 \end{bmatrix}. \tag{2.6.7}$$

The condition (2.6.7) is the Krein space version of the usual assumption made in the stochastic (Hilbert space) state-space models, viz., that the initial condition $\mathbf{x}_0$ and the driving and measurement disturbances $\{\mathbf{u}_i, \mathbf{v}_i\}$ are zero-mean uncorrelated random variables with variance matrices $\Pi_0$ and $\begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix}$, respectively, and that the $\{\mathbf{u}_i, \mathbf{v}_i\}$ form a white (uncorrelated) sequence. As mentioned before, the Krein space elements can be thought of as some kind of generalized random variables.

Now if, as was done earlier, we define

$$\mathbf{y} = \text{col}\{\mathbf{y}_0, \ldots \mathbf{y}_\mathbf{N}\} \quad , \quad \mathbf{u} = \text{col}\{\mathbf{u}_0, \ldots \mathbf{u}_\mathbf{N}\} \quad , \quad \mathbf{v} = \text{col}\{\mathbf{v}_0, \ldots \mathbf{v}_\mathbf{N}\}$$

then we can use the state-space model (2.6.6) to write

$$\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mathcal{O} & \Gamma & I \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} \tag{2.6.8}$$

and to see that

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{y} \end{bmatrix} \right\rangle = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mathcal{O} & \Gamma & I \end{bmatrix} \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q & S \\ 0 & S^* & R \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \mathcal{O} & \Gamma & I \end{bmatrix}^*, \tag{2.6.9}$$

which is exactly the inverse of the central matrix appearing in expression (2.6.5) for $J(x_0, u, v)$. Therefore, referring to Theorems 2.5.1 and 2.5.2, the main point is that to find the stationary point of $J(x_0, u, y)$ over $\{x_0, u\}$, we can alternatively find the projection of $\{\mathbf{x}_0, \mathbf{u}\}$ onto $\mathcal{L}\{\mathbf{y}\}$ in the Krein space model (2.6.6).

Now that we have identified the stochastic and deterministic problems when a state-space structure is assumed, we can give the analogs of Theorems 2.5.1 and 2.5.2.

**Lemma 2.6.1 (Stochastic Interpretation)** *Suppose* $\mathbf{z} = col\{\mathbf{x}_0, \mathbf{u}\}$ *and* $\mathbf{y}$ *are related through the state-space model (2.6.6-2.6.7), and that* $R_y$ *given by (2.6.12) is nonsingular. Then the stationary point of the error Gramian*

$$\langle \mathbf{z} - k^*\mathbf{y}, \mathbf{z} - k^*\mathbf{y} \rangle \tag{2.6.10}$$

*over all* $k^*\mathbf{y}$ *is given by the projection*

$$\begin{bmatrix} \hat{\mathbf{x}}_{0|N} \\ \hat{\mathbf{u}}_{|N} \end{bmatrix} = \begin{bmatrix} \Pi_0 \mathcal{O}^* \\ Q\Gamma^* + S \end{bmatrix} R_y^{-1} \mathbf{y} \tag{2.6.11}$$

*where*

$$R_y = \mathcal{O}\Pi_0\mathcal{O}^* + \begin{bmatrix} \Gamma & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \Gamma^* \\ I \end{bmatrix}. \tag{2.6.12}$$

*Moreover this stationary point is a minimum if, and only if,* $R_y > 0$.

We can now also give the analog result to Theorem 2.5.2.

**Lemma 2.6.2 (Deterministic Quadratic Form)** *The expression*

$$\begin{bmatrix} \hat{x}_{0|N} \\ \hat{u}_{|N} \end{bmatrix} = \begin{bmatrix} \Pi_0 \mathcal{O}^* \\ Q\Gamma^* + S \end{bmatrix} R_y^{-1} \mathbf{y} \tag{2.6.13}$$

*yields the stationary point of the quadratic order form:*

$$J(x_0, u, y) = x_0^*\Pi_0^{-1}x_0 + \sum_{j=0}^N \begin{bmatrix} u_j^* & (y_j - H_j x_j)^* \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix}^{-1} \begin{bmatrix} u_j \\ y_j - H_j x_j \end{bmatrix}, \tag{2.6.14}$$

*over* $x_0$ *and* $u = col\{u_0, \ldots, u_N\}$, *and subject to the state-space constraints*

$$\begin{cases} x_{j+1} = F_j x_j + G_j u_j, & 0 \le j \le N \\ y_j = H_j x_j + v_j \end{cases}$$

*In particular, when* $S_j \equiv 0$, *the quadratic form is*

$$J(x_0, u, y) = x_0^*\Pi_0^{-1}x_0 + \sum_{j=0}^N u_j^* Q_j^{-1} u_j + \sum_{j=0}^N (y_j - H_j x_j)^* R_j^{-1} (y_j - H_j x_j). \tag{2.6.15}$$

*The value of* $J(x_0, u, y)$ *(with either* $S_j \equiv 0$ *or* $S_j \neq 0$*) at the stationary point is*

$$J(\hat{x}_{0|N}, \hat{u}_{|N}, y) = y^* R_y^{-1} y.$$

### 2.6.1 The Conditions for a Minimum

As mentioned earlier, the important point is that the conditions for minima in these two problems are different: $R_y > 0$ in the stochastic problem, and

$$M \triangleq R_z - R_{zy}R_y^{-1}R_{yz} > 0 \quad \text{where} \quad \mathbf{z} = \text{col}\{\mathbf{x}_0, \mathbf{u}\}$$

in the deterministic problem. In the state-space case $R_y$ is given by (2.6.12). In this section we shall explore the condition for a deterministic minimum under the state-space assumption. First note that for $M$ we have

$$M = \begin{bmatrix} \Pi_0 & 0 \\ 0 & Q \end{bmatrix} - \begin{bmatrix} \Pi_0 & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \mathcal{O}^* \\ \Gamma^* + Q^{-1}S \end{bmatrix} R_y^{-1} \begin{bmatrix} \mathcal{O} & \Gamma + S^*Q^{-1} \end{bmatrix} \begin{bmatrix} \Pi_0 & 0 \\ 0 & Q \end{bmatrix} =$$

$$\begin{bmatrix} \Pi_0 - \Pi_0\mathcal{O}^*R_y^{-1}\mathcal{O}\Pi_0 & -\Pi_0\mathcal{O}^*R_y^{-1}(\Gamma Q + S^*) \\ -(Q\Gamma^* + S)R_y^{-1}\mathcal{O}\Pi_0 & Q - (Q\Gamma^* + S)R_y^{-1}(\Gamma Q + S^*) \end{bmatrix}. \tag{2.6.16}$$

Now we know that $M > 0$ if, and only if, both the $(1,1)$ block entry in (2.6.16) and its Schur complement are positive definite. The $(1,1)$ block entry may be identified as the Gramian of the error $\mathbf{x_0} - \hat{\mathbf{x}}_{0|N}$, *i.e.*,

$$\Pi_0 - \Pi_0\mathcal{O}^*R_y^{-1}\mathcal{O}\Pi_0 = \langle \mathbf{x_0} - \hat{\mathbf{x}}_{\mathbf{0|N}}, \mathbf{x_0} - \hat{\mathbf{x}}_{\mathbf{0|N}} \rangle \triangleq P_{0|N}. \tag{2.6.17}$$

To obtain a nice form for the Schur complement of the $(1,1)$ block entry, say $\Delta$, we have to use a little matrix algebra. Recall that

$$\begin{aligned} R_y &= \mathcal{O}\Pi_0\mathcal{O}^* + \begin{bmatrix} \Gamma & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \Gamma^* \\ I \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{O} & \Gamma + S^*Q^{-1} \end{bmatrix} \begin{bmatrix} \Pi_0 & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \mathcal{O}^* \\ \Gamma^* + Q^{-1}S \end{bmatrix} + R - S^*Q^{-1}S. \end{aligned}$$

Using the second expression for $R_y$, and a well-known matrix inversion formula, leads to the expression

$$M^{-1} = \begin{bmatrix} \Pi_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O}^* \\ \Gamma^* + Q^{-1}S \end{bmatrix} (R - S^*Q^{-1}S)^{-1} \begin{bmatrix} \mathcal{O} & \Gamma + S^*Q^{-1} \end{bmatrix}. \tag{2.6.18}$$

Now we use another well known fact: the (2,2) block element of $M^{-1}$ is just $\Delta^{-1}$ (where $\Delta^{-1}$ exists since $M$ is positive-definite). Therefore the condition now becomes

$$Q^{-1} + (\Gamma^* + Q^{-1}S)(R - S^*Q^{-1}S)^{-1}(\Gamma + S^*Q^{-1}) > 0,$$

so that we have the following result.

**Lemma 2.6.3 (A Condition for a Minimum)** *If $Q$ and $R - S^*Q^{-1}S$ are invertible, a necessary and sufficient condition for the stationary point of Lemma 2.6.2 to be a minimum is that*

*(i)* $P_{0|N} > 0$, *and*

*(ii)* $Q^{-1} + (\Gamma^* + Q^{-1}S)(R - S^*Q^{-1}S)^{-1}(\Gamma + S^*Q^{-1}) > 0$.

*When $S \equiv 0$, the second condition becomes $Q^{-1} + \Gamma^* R^{-1}\Gamma > 0$.*

The conditions of Lemma 2.6.3 need to be reduced further in order to provide useful computational tests. This can be done in several ways, leading to more specific tests. One interesting way is by showing that $Q^{-1} + (\Gamma^* + Q^{-1}S)(R - S^*Q^{-1}S)^{-1}(\Gamma + S^*Q^{-1})$ may be regarded as the Gramian matrix of the output of a so-called backwards *dual* state-space model. This identification will be useful in studying the $H^\infty$ control problem (and in other ways), but we shall not pursue it here.

Instead we shall use the alternative inertia conditions of Lemma 2.5.3 to circumvent the need for direct analysis of the matrix $R_z - R_{zy}R_y^{-1}R_{yz}$. Recall from Lemma 2.5.3 that if $R_z > 0$, a unique minimizing solution to the deterministic problem of Theorem 2.5.2 exists if, and only if $R_y$ and $R_y - R_{yz}R_z^{-1}R_{zy}$ have the same inertia. However, for the state-space structure that we are considering,

$$R_z = \begin{bmatrix} \Pi_0 & 0 \\ 0 & Q \end{bmatrix}$$

so that after some simple algebra we have

$$R_y - R_{yz}R_z^{-1}R_{zy} = R - S^*Q^{-1}S = (R_0 - S_0^*Q_0^{-1}S_0) \oplus \ldots \oplus (R_N - S_N^*Q_N^{-1}S_N). \quad (2.6.19)$$

Thus $R_y - R_{yz}R_z^{-1}R_{zy}$ is block-diagonal, and we have the following result.

**Lemma 2.6.4 (Inertia Condition for Minimum)** *If* $\Pi_0 > 0$ *and* $Q > 0$*, then a necessary and sufficient condition for the stationary point of Lemma 2.6.2 to be a minimum is that the matrices* $R_y$ *and* $R - S^* Q^{-1} S$ *have the same inertia.*

*In particular, if* $S \equiv 0$*, then* $R_y$ *and* $R$ *must have the same inertia.*

As we shall see in the next section, the Krein space Kalman filter provides the block triangular factorization of $R_y$, and thereby allows one to easily compare the inertia of $R_y$ and $R - S^* Q^{-1} S$.

## 2.7 Recursive Formulas

So far we have obtained *global* expressions for computing projections and for checking the conditions for deterministic and stochastic minimization. Computing the projection requires inverting the Gramian matrix $R_y$, and checking for the minimization conditions requires checking the inertia of $R_y$, both of which require $O(N^3)$ (where $N$ is the dimension of $R_y$) computations.

The key consequence of state-space structure in Hilbert space is that the computational burden of finding projections can be significantly reduced, to $O(Nn^3)$ (where $n$ is the dimension of the state-space model), by using the Kalman filter recursions. Moreover, the Kalman filter also recursively factors the positive definite Gramian matrix $R_y$ as $LDL^*$, $L$ lower triangular with unit diagonal and $D$ diagonal.

We shall presently see that similar recursions hold in Krein space as well, provided

$$R_y \text{ is strongly nonsingular (or strongly regular)} \tag{2.7.1}$$

in the sense that all its (block) leading minors are nonzero. Recall that in Hilbert space if the $\{\mathbf{y}_i\}$ are linearly independent, then $R_y$ is strictly positive definite; so that (2.7.1) holds automatically. In the Krein space theory, we have so far only assumed that $R_y$ is invertible, which does not necessarily imply (2.7.1). However, recursive projection, *i.e.* , projection onto $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_i\}$ for *all* $i$, requires that all the (block) leading submatrices of $R_y$ are nonsingular; recall also that (2.7.1) implies that $R_y$ has a unique triangular decomposition

$$R_y = LDL^*, \tag{2.7.2}$$

Therefore, $\text{In}(R_y) = \text{In}(D)$, and in particular, $R_y > 0$ if, and only if, $D > 0$. This is the standard way of recursively computing the inertia of $R_y$.

The standard method of recursive estimation, which also gives a very useful geometric insight into the triangular factorization of $R_y$, is to introduce the innovations

$$\mathbf{e}_j = \mathbf{y}_j - \hat{\mathbf{y}}_j, \quad 0 \le j \le N \tag{2.7.3}$$

where $\hat{\mathbf{y}}_j \triangleq \hat{\mathbf{y}}_{j|j-1}$ = the projection of $\mathbf{y}_j$ onto $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_{j-1}\}$.

Note that due to the construction (2.7.3), the innovations form an orthogonal basis for $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ (with respect to the Krein space inner product), which simplifies the calculation of projections. For example, we can express the projection of the fundamental quantities $\mathbf{x}_0$ and $\mathbf{u}_j$ onto $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$ as

$$\hat{\mathbf{x}}_{0|N} = \sum_{i=0}^{N} \langle \mathbf{x}_0, \mathbf{e}_i \rangle \langle \mathbf{e}_i, \mathbf{e}_i \rangle^{-1} \mathbf{e}_i \tag{2.7.4}$$

and

$$\hat{\mathbf{u}}_{j|N} = \sum_{i=0}^{N} \langle \mathbf{u}_j, \mathbf{e}_i \rangle \langle \mathbf{e}_i, \mathbf{e}_i \rangle^{-1} \mathbf{e}_i \tag{2.7.5}$$

where the state-space structure may be used to calculate the above inner products recursively.

However, before proceeding to show this, let us note that any method for computing the innovations yields the triangular factorization of the Gramian $R_y$. To this end, let us write

$$\mathbf{y}_i = \hat{\mathbf{y}}_i + \mathbf{e}_i = \langle \mathbf{y}_i, \mathbf{e}_0 \rangle R_{e,0}^{-1} \mathbf{e}_0 + \ldots + \langle \mathbf{y}_i, \mathbf{e}_{i-1} \rangle R_{e,i-1}^{-1} \mathbf{e}_{i-1} + \mathbf{e}_i$$

and collect such expressions in matrix form,

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{bmatrix} = \begin{bmatrix} I & & & \\ \langle \mathbf{y}_1, \mathbf{e}_0 \rangle R_{e,0}^{-1} & I & & \\ \vdots & & \ddots & \\ \langle \mathbf{y}_N, \mathbf{e}_0 \rangle R_{e,0}^{-1} & \langle \mathbf{y}_N, \mathbf{e}_1 \rangle R_{e,1}^{-1} & \ldots & I \end{bmatrix} \begin{bmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_N \end{bmatrix} = L\mathbf{e},$$

where $L$ is lower triangular with unit diagonal. Therefore, since the $\mathbf{e}_i$ are orthogonal, the Gramian of $\mathbf{y}$ is

$$R_y = L R_e L^*, \quad \text{where} \quad R_e = R_{e,0} \oplus R_{e,1} \oplus \ldots \oplus R_{e,N}.$$

We thus have the following result.

**Lemma 2.7.1 (Inertia of $R_y$)** *The Gramian $R_y$ of $\mathbf{y}$ has the same inertia as the Gramian of the innovations, $R_e$. The strong regularity of $R_y$ implies the nonsingularity of $R_{e,i}$, $0 \le i \le N$. In particular, $R_y > 0$, if and only if*

$$R_{e,i} > 0, \quad \text{for all} \quad i = 0, 1, \ldots, N.$$

We should also point out that the value at the stationary point of the quadratic form in Theorem 2.5.2 can also be expressed in terms of the innovations,

$$J(z_o, y) = y^* R_y^{-1} y = y^* L^{-*} R_e^{-1} L^{-1} y = e^* R_e^{-1} e = \sum_{j=0}^{N} e_j^* R_{e,j}^{-1} e_j. \tag{2.7.6}$$

### 2.7.1 The Krein Space Kalman Filter

Now we shall show that the state-space structure allows us to efficiently compute the innovations by an immediate extension of the Kalman filter.

**Theorem 2.7.1 (Kalman Filter in Krein Space)** *Consider the Krein-space state equations*

$$\begin{cases} \mathbf{x}_{i+1} &= F_i \mathbf{x}_i + G_i \mathbf{u}_i, \qquad 0 \le i \le N \\ \mathbf{y}_i &= H_i \mathbf{x}_i + \mathbf{v}_i \end{cases} \tag{2.7.7}$$

*with*

$$\left\langle \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \\ \mathbf{x}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{u}_k \\ \mathbf{v}_k \\ \mathbf{x}_0 \end{bmatrix} \right\rangle = \begin{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix} \delta_{jk} & 0 \\ 0 & \Pi_0 \end{bmatrix}$$

*Assume that $R_y = [\langle \mathbf{y}_i, \mathbf{y}_j \rangle]$ is strongly regular. Then the innovations can be computed via the formulas*

$$\mathbf{e}_i = \mathbf{y}_i - H_i \hat{\mathbf{x}}_i, \qquad 0 \le i \le N \tag{2.7.8}$$

$$\hat{\mathbf{x}}_{i+1} = F_i \hat{\mathbf{x}}_i + K_{p,i}(\mathbf{y}_i - H_i \hat{\mathbf{x}}_i), \quad \hat{\mathbf{x}}_0 = 0 \tag{2.7.9}$$

$$K_{p,i} = (F_i P_i H_i^* + G_i S_i) R_{e,i}^{-1} \tag{2.7.10}$$

*where*

$$R_{e,i} = \langle \mathbf{e}_i, \mathbf{e}_i \rangle = R_i + H_i P_i H_i^*$$

*and the* $\{P_i\}$ *can be recursively computed via the Riccati recursion*

$$P_{i+1} = F_i P_i F_i^* - K_{p,i} R_{e,i} K_{p,i}^* + G_i Q_i G_i^*, \qquad P_0 = \Pi_0. \tag{2.7.11}$$

*The number of computations is dominated by those in (2.7.11) and is readily seen to be* $O(n^3)$ *per iteration.*

**Remark:** The only difference from the conventional Kalman filter expressions is that the matrices $P_i$ and $R_{e,i}$ (and, by assumption, $\Pi_0$, $Q_i$ and $R_i$) may now be indefinite.
**Proof:** The same as in the usual Kalman filter theory (see, *e.g.*, [Kai81]). However, for completeness and to show the power of the geometric viewpoint, we present a simple derivation. There is absolutely no formal difference between the steps in the (usual) Hilbert space case and in the Krein space case.

Begin by noting that

$$\mathbf{e}_i = \mathbf{y}_i - \hat{\mathbf{y}}_i = \mathbf{y}_i - (H_i \hat{\mathbf{x}}_i + \hat{\mathbf{v}}_i) = \mathbf{y}_i - H_i \hat{\mathbf{x}}_i = H_i \tilde{\mathbf{x}}_i + \mathbf{v}_i \tag{2.7.12}$$

where $\hat{\mathbf{x}}_i$ is the projection of $\mathbf{x}_i$ on $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_{i-1}\}$ and where we have defined $\tilde{\mathbf{x}}_i = \mathbf{x}_i - \hat{\mathbf{x}}_i$. It follows readily that

$$R_{e,i} = \langle \mathbf{e}_i, \mathbf{e}_i \rangle = R_i + H_i P_i H_i^*, \qquad P_i \triangleq \langle \tilde{\mathbf{x}}_i, \tilde{\mathbf{x}}_i \rangle. \tag{2.7.13}$$

Recall (see Lemma 2.7.1) that the strong nonsingularity (all leading minors non-zero) of $R_y$ implies that the $\{R_{e,i}\}$ are nonsingular (rather than positive-definite, as in the Hilbert space case). The Kalman filter can now be readily derived by using the orthogonality of the innovations and the state-space structure. Thus we first write

$$\hat{\mathbf{x}}_{i+1|i} = \hat{\mathbf{x}}_{i+1} = \sum_{j=0}^{i} \langle \mathbf{x}_{i+1}, \mathbf{e}_j \rangle \langle \mathbf{e}_j, \mathbf{e}_j \rangle_{\mathcal{K}}^{-1} \mathbf{e}_j,$$

and to seek a recursion we decompose the above as

$$\hat{\mathbf{x}}_{i+1} = \sum_{j=0}^{i-1} \langle \mathbf{x}_{i+1}, \mathbf{e}_j \rangle R_{e,j}^{-1} \mathbf{e}_j + K_{p,i} \mathbf{e}_i, \qquad K_{p,i} \triangleq \langle \mathbf{x}_{i+1}, \mathbf{e}_i \rangle R_{e,i}^{-1}.$$

Now

$$\begin{aligned}\langle \mathbf{x}_{i+1}, \mathbf{e}_i\rangle &= F_i\langle \mathbf{x}_i, \mathbf{e}_i\rangle + G_i\langle \mathbf{u}_i, \mathbf{e}_i\rangle \\ &= F_i\langle \mathbf{x}_i, H_i\tilde{\mathbf{x}}_i + \mathbf{v}_i\rangle + G_i\langle \mathbf{u}_i, H_i\tilde{\mathbf{x}}_i + \mathbf{v}_i\rangle \\ &= F_i\langle \tilde{\mathbf{x}}_i, H_i\tilde{\mathbf{x}}_i\rangle + 0 + 0 + G_i\langle \mathbf{u}_i, \mathbf{v}_i\rangle = F_i P_i H_i^* + G_i S_i.\end{aligned}$$

Note also that the first summation can be rewritten as

$$F_i \sum_{j=0}^{i-1}\langle \mathbf{x}_i, \mathbf{e}_j\rangle R_{e,j}^{-1}\mathbf{e}_j + G_i \sum_{j=0}^{i-1}\langle \mathbf{u}_i, \mathbf{e}_j\rangle R_{e,j}^{-1}\mathbf{e}_j = F_i\hat{\mathbf{x}}_i + 0$$

Combining these facts we find

$$\hat{\mathbf{x}}_{i+1} = F_i\hat{\mathbf{x}}_i + K_{p,i}\mathbf{e}_i \tag{2.7.14}$$

and

$$K_{p,i} = (F_i P_i H_i^* + G_i S_i) R_{e,i}^{-1}. \tag{2.7.15}$$

It now remains to find a recursion for $P_i$. To this end, note that if we define the Gramians $\Pi_i = \langle \mathbf{x_i}, \mathbf{x_i}\rangle$ and $\Sigma_i = \langle \hat{\mathbf{x}}_\mathbf{i}, \hat{\mathbf{x}}_\mathbf{i}\rangle$, then the orthogonality of the $\hat{\mathbf{x}}_i$ and $\tilde{\mathbf{x}}_i$ yields,

$$P_i = \Pi_i - \Sigma_i.$$

The state-space equation (2.6.6) shows that the state variance $\Pi_i$, obeys the recursion

$$\Pi_{i+1} = F_i \Pi_i F_i^* + G_i Q_i^* G_i^*.$$

Likewise, the orthogonality of the innovations implies that (2.7.14) will yield

$$\Sigma_{i+1} = F_i \Sigma_i F_i^* + K_{p,i} R_{e,i} K_{p,i}^*, \quad \Sigma_0 = 0.$$

Subtracting the above two equations yields the desired Riccati recursion for $P_i$,

$$P_{i+1} = F_i P_i F_i^* + G_i Q_i G_i^* - K_{p,i} R_{e,i} K_{p,i}^*, \quad P_0 = \Pi_0. \tag{2.7.16}$$

Equations (2.7.13) to (2.7.16) constitute the Kalman filter of Theorem 2.7.1.

■

In Kalman filter theory there are many variations of the above formulas and we note one here. Let us define the filtered estimate, $\hat{\mathbf{x}}_{i|i}$ as the projection of $\mathbf{x}_i$ onto $\mathcal{L}\{\mathbf{y}_0, \ldots, \mathbf{y}_i\}$.

**Theorem 2.7.2 (Measurement and Time Updates)** *Consider the Krein state-space equations of Theorem 2.7.1 and assume that $R_y$ is strongly regular. Then when $S_i \equiv 0$, the filtered estimates $\hat{\mathbf{x}}_{i|i}$ can be computed via the following (measurement and time update) formulas*

$$\begin{aligned} \hat{\mathbf{x}}_{i+1|i+1} &= \hat{\mathbf{x}}_{i+1} + K_{f,i+1}\mathbf{e}_{i+1}, \quad \hat{\mathbf{x}}_0 = 0, \quad K_{f,i+1} = P_{i+1}H_{i+1}^* R_{e,i+1}^{-1} && (2.7.17) \\ \hat{\mathbf{x}}_{i+1} &= F_i \hat{\mathbf{x}}_{i|i} && (2.7.18) \end{aligned}$$

*where $\mathbf{e}_i$, $R_{e,i}$ and $P_i$ are as in Theorem 2.7.1.*

**Corollary 2.7.1 (Filtered Recursions)** *The two step recursions of Theorem 2.7.2 can be combined into the single recursion*

$$\hat{\mathbf{x}}_{i+1|i+1} = F_i \hat{\mathbf{x}}_{i|i} + K_{f,i+1}(\mathbf{y}_{i+1} - H_{i+1}F_i\hat{\mathbf{x}}_{i|i}), \quad \hat{\mathbf{x}}_{-1|-1} = 0. \qquad (2.7.19)$$

For numerical reasons, certain square-root versions of the KF are now more often used in state-space estimation. Furthermore for constant systems, or in fact for systems where the time-variation is structured in a certain way, the Riccati recursions and the square-root recursions, both of which take $O(n^3)$ elementary computations (flops) per iteration can be replaced by the more efficient Chandrasekhar recursions, which require only $O(n^2)$ flops per iteration [MSK74, SK94a]. The square-root and Chandrasekhar recursions can both be extended to the Krein space setting, as described in [HSK94c], and as we shall see in Chapter 5.

Before closing this section we shall note how the innovations computed in Theorem 2.7.1 can be used to determine the projections $\hat{\mathbf{x}}_{0|N}$ and $\hat{\mathbf{u}}_{|N}$ using the formulas (2.7.4)and (2.7.5).

**Lemma 2.7.2 (Computation of Inner Products)** *We can write*

$$\langle \mathbf{x}_0, \mathbf{e}_i \rangle = \Pi_0 \Phi_{F-KH}^*(i,0) H_i^* \qquad (2.7.20)$$

*and*

$$\langle \mathbf{u}_j, \mathbf{e}_i \rangle = \begin{cases} Q_j G_j^* \Phi_{F-KH}^*(i, j+1) H_i^* + S_i \delta_{ij} & j \leq i \\ 0 & j > i \end{cases} \qquad (2.7.21)$$

*where*

$$\Phi_{F-KH}(i,j) \triangleq \prod_{k=j}^{i-1}(F_k - K_{p,k}H_k).$$

*These lead to the recursions:*

$$\hat{\mathbf{x}}_{0|i} = \hat{\mathbf{x}}_{0|i-1} + \Pi_0 \Phi^*_{F-KH}(i,0) H_i^* R_{e,i}^{-1} \mathbf{e}_i, \quad \hat{\mathbf{x}}_{0|-1} = 0 \tag{2.7.22}$$

*and*

$$\hat{\mathbf{u}}_{j|i} = \begin{cases} \hat{\mathbf{u}}_{j|i-1} + Q_j G_j^* \Phi^*_{F-KH}(i,j+1) H_i^* R_{e,i}^{-1} \mathbf{e}_i, \quad \hat{\mathbf{u}}_{j|j} = S_j R_{e,j}^{-1} \mathbf{e}_i & j \leq i \\ 0 & j > i \end{cases} \tag{2.7.23}$$

*where* $\Phi^*_{F-KH}(i,j)$ $(i \geq j)$ *satisfies the recursion*

$$\Phi^*_{F-KH}(i+1,j) = \Phi^*_{F-KH}(i,j)(F_i - K_{p,i}H_i)^*, \quad \Phi^*_{F-KH}(j,j) = I.$$

**Proof:** Straightforward computation.

∎

### 2.7.2 Recursive State-Space Estimation and Quadratic Forms

Theorems 2.7.3 and 2.7.4 below are essentially restatements of Theorems 2.5.1 and 2.5.2, when a state space model is assumed, and a recursive solution is sought.

The error Gramian associated with the problem of projecting $\{\mathbf{x}_0, \mathbf{u}\}$ onto $\mathcal{L}\{\mathbf{y}\}$ has already been identified in Lemma 2.6.1, and (2.7.22-2.7.23) furnishes a recursive procedure for calculating this projection. The condition for a minimum is $R_y > 0$, where $R_y$ has been shown to be congruent to the diagonal matrix $R_e$. This gives the following theorem.

**Theorem 2.7.3 (Stochastic Problem)** *Suppose* $\mathbf{z} = col\{\mathbf{x}_0, \mathbf{u}\}$ *and* $\mathbf{y}$ *are related through the state-space model (2.6.6) and (2.6.7), and that* $R_y$ *is strongly regular.*

*Then the state-space estimation algorithm (2.7.22-2.7.23) recursively computes the stationary point of the error Gramian*

$$\langle \mathbf{z} - k^*\mathbf{y}, \mathbf{z} - k^*\mathbf{y} \rangle$$

*over all* $k^*\mathbf{y}$. *Moreover this stationary point is a minimum if and only if*

$$R_{e,j} > 0 \quad \textit{for } j = 0, \ldots, i.$$

Similarly, the scalar quadratic form associated with the (partially) equivalent deterministic problem has already been identified in Lemma 2.6.2:

$$J_N(x_0, u, y) = x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{N} \begin{bmatrix} u_j^* & (y_j - H_j x_j)^* \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix}^{-1} \begin{bmatrix} u_j \\ y_j - H_j x_j \end{bmatrix}. \tag{2.7.24}$$

In particular, $\hat{x}_{0|N}$ and $\hat{u}_{j|N}$ are the stationary points of $J_N(x_0, u, y)$ over $x_0$ and $u_j$, and subject to the state-space constraints $x_{j+1} = F_j x_j + G_j u_j$, $j = 0, \ldots, N$. In the recursions, for each time $i$, we find $\hat{x}_{0|i}$ and $\hat{u}_{j|i}$, which are the stationary points of

$$J_i(x_0, u, y) = x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{i} \begin{bmatrix} u_j^* & (y_j - H_j x_j)^* \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix}^{-1} \begin{bmatrix} u_j \\ y_j - H_j x_j \end{bmatrix}. \tag{2.7.25}$$

**Theorem 2.7.4 (Deterministic Problem)** *If* $R_y$ *is strongly regular, the stationary point of the quadratic form*

$$J_i(x_0, u, y) = x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{i} \begin{bmatrix} u_j^* & (y_j - H_j x_j)^* \end{bmatrix} \begin{bmatrix} Q_j & S_j \\ S_j^* & R_j \end{bmatrix}^{-1} \begin{bmatrix} u_j \\ y_j - H_j x_j \end{bmatrix} \tag{2.7.26}$$

*over* $x_0$ *and* $u_j$, *subject to the state-space constraints* $x_{j+1} = F_j x_j + G_j u_j$, $j = 0, 1, \ldots, i$ *can be recursively computed as*

$$\hat{x}_{0|i} = \hat{x}_{0|i-1} + \Pi_0 \Phi_{F-KH}^*(i, 0) H_i^* R_{e,i}^{-1} e_i, \qquad \hat{x}_{0|-1} = 0$$

*and*

$$\hat{u}_{j|i} = \begin{cases} \hat{u}_{j|i-1} + Q_j G_j^* \Phi_{F-KH}^*(i,j+1) H_i^* R_{e,i}^{-1} e_i, \quad \hat{u}_{j|j} = S_j R_{e,j}^{-1} e_i & j \leq i \\ 0 & j > i \end{cases}$$

*where the innovations* $e_j$ *can be computed via the recursions*

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i} e_i, \qquad \hat{x}_0 = 0$$

*with* $K_{p,i} = (F_i P_i H_i^* + G_i S_i) R_{e,i}^{-1}$, $R_{e,i} = R_i + H_i P_i H_i^*$, $e_i = y_i - H_i \hat{x}_i$, *and* $P_i$ *satisfying the Riccati recursion*

$$P_{i+1} = F_i P_i F_i^* + G_i Q_i G_i^* - K_{p,i} R_{e,i}^{-1} K_{p,i}^* \quad , \quad P_0 = \Pi_0.$$

*Moreover, the value of* $J_i(x_0, u, y)$ *at the stationary point is given by*

$$J_i(\hat{x}_{0|i}, \hat{u}_{|i}, y) = \sum_{j=0}^{i} e_j^* R_{e,j}^{-1} e_j.$$

**Proof:** The proof follows from the basic equivalence between the deterministic and stochastic problems. The recursions for $\hat{x}_{0|i}$ and $\hat{u}_{j|i}$ are the same as those in the stochastic problem of Lemma 2.7.2, and the innovations $e_i$ are found via the Krein space Kalman filter of Theorem 2.7.1.

■

As mentioned earlier, the deterministic quadratic form of Theorem 2.7.4 is often encountered in estimation problems. By appeal to Gaussian assumptions on the $v_i$, $u_i$ and $x_0$, and maximum likelihood arguments, it is well-known that state estimates can be obtained via a deterministic quadratic minimization problem. Here we have shown this result using simple projection arguments, and have generalized it to indefinite quadratic forms.

The result of Theorem 2.7.4 is probably the most important result of this paper, and we shall make frequent use of it in the next two chapters to solve the problems of $H^\infty$ and risk-sensitive estimation, and finite-memory adaptive filtering. In those problems we shall also need to recursively check for the condition for a minimum, and therefore we will now study these conditions in more detail.

Recall from Lemma 2.6.4 that the above deterministic problem has a minimum if, and only if, $R_y$ and $R - S^*Q^{-1}S$ have the same inertia. Since $R_y$ is congruent to the block diagonal matrix $R_e$, and since $R - S^*Q^{-1}S$ is also block diagonal, the solution of the recursive stationarization problem will give a minimum *at each step* if and only if all the block diagonal elements of $R_e$ and $R - S^*Q^{-1}S$ have the same inertia. This leads to the following result.

**Lemma 2.7.3 (Inertia Conditions for a Minimum)** *If $\Pi_0 > 0$, $Q > 0$ and $R$ is non-singular, then the (unique) stationary points of the quadratic forms (2.7.26), for $i = 0, 1, \ldots N$, will each be a unique minimum if, and only if, the matrices*

$$R_{e,j} \quad \text{and} \quad R_j - S_j^* Q_j^{-1} S_j$$

*have the same inertia for all $j = 0, 1, \ldots N$. In particular, when $S_j \equiv 0$, the condition becomes that $R_{e,j}$ and $R_j$ should have the same inertia for all $j = 0, 1, \ldots N$.*

The conditions of the above Lemma are easy to check since the Krein space Kalman filter used to compute the stationary point also computes the matrices $R_{e,j}$. There is another condition, more frequently quoted in the $H^\infty$ literature, which we restate here (see *e.g.* , [ST92]).

**Lemma 2.7.4 (Condition for a Minimum)** *If $\Pi_0 > 0$, $Q > 0$, $R$ is invertible, $Q - SR^{-1}S^* > 0$ and $\begin{bmatrix} F_j & G_j \end{bmatrix}$ has full rank for all $j$, then the quadratic forms (2.7.26) will each have a unique minimum if, and only if,*

$$P_{j|j}^{-1} = P_j^{-1} + H_j^* R_j^{-1} H_j > 0 \qquad j = 0, 1, ..., N.$$

*It also follows in the minimum case that $P_{j+1} > 0$ for $j = 0, 1, ..., N$.*

**Remark:** In comparison to our result in Lemma 2.7.3, we here have the additional requirement that the $\begin{bmatrix} F_j & G_j \end{bmatrix}$ must be full rank. Furthermore, we not only have to compute the $P_j$ (which is done via the Riccati recursion of the Kalman filter), but we also have to invert $P_j$ (and $R_j$) at each step and then check for the positivity of $P_j^{-1} + H_j^* R_j^{-1} H_j$. The test of Lemma 2.7.3 uses only quantities already present in the Kalman filter recursion, viz. $R_{e,j}$ and $R_j$. Moreover, these are $p \times p$ matrices (as opposed to $P_{j|j}^{-1}$ which is $n \times n$), with $p$ typically less than $n$, and whose inertia is easily determined via a triangular factorization. Furthermore, it can be shown (see [HSK94c] and Chapter 5) that even this computation can be effectively blended into the filter recursions by going to a square-root-array version of the Riccati recursion. Here, however, for completeness we shall show how Lemma 2.7.4 follows from our Lemma 2.7.3.

**Proof of Lemma 2.7.4:** We shall prove the Lemma by induction. Consider the matrix

$$\begin{bmatrix} -\Pi_0^{-1} & 0 & H_0^* \\ 0 & -Q_0^{-1} & Q_0^{-1}S_0 \\ H_0 & S_0^*Q_0^{-1} & R_0 - S_0^*Q_0^{-1}S_0 \end{bmatrix}.$$

Two different triangular factorizations (lower-upper and upper-lower) of the above matrix show that the matrices,

$$\begin{bmatrix} -\Pi_0^{-1} & 0 & 0 \\ 0 & -Q_0^{-1} & 0 \\ 0 & 0 & R_0 + H_0\Pi_0H_0^* \end{bmatrix},$$

and

$$\begin{bmatrix} -(\Pi_0^{-1} + H_0^*R_0^{-1}H_0) & 0 & 0 \\ 0 & -(Q_0 - S_0R_0^{-1}S_0^*)^{-1} & 0 \\ 0 & 0 & R_0 - S_0^*Q_0^{-1}S_0 \end{bmatrix},$$

have the same inertia. Thus, since $\Pi_0 > 0$, $Q_0 > 0$, and $Q_0 - S_0R_0^{-1}S_0^* > 0$, then the matrices $R_{e,0} = R_0 + H_0\Pi_0H_0^*$ and $R_0 - S_0^*Q_0^{-1}S_0$ will have the same inertia (and we will have a minimum for $J_0$) if, and only if,

$$\Pi_0^{-1} + H_0^*R_0^{-1}H_0 > 0.$$

Now with some effort we may write the first step of the Riccati recursion as

$$P_1 = \begin{bmatrix} F_0 & G_0 \end{bmatrix} \left( \begin{bmatrix} \Pi_0^{-1} & 0 \\ 0 & Q_0^{-1} \end{bmatrix} + \begin{bmatrix} H_0^* \\ Q_0^{-1}S_0 \end{bmatrix} (R_0^{-1} - S_0^*Q_0^{-1}S_0)^{-1} \begin{bmatrix} H_0 & S_0^*Q_0^{-1} \end{bmatrix} \right)^{-1} \begin{bmatrix} F_0^* \\ G_0^* \end{bmatrix}.$$

Moreover, the center matrix appearing in the above expression is congruent to

$$\begin{bmatrix} \Pi_0^{-1} + H_0^* R_0^{-1} H_0 & 0 \\ 0 & (Q_0 - S_0 R_0^{-1} S_0^*)^{-1} \end{bmatrix},$$

and hence is positive definite. Thus if $\begin{bmatrix} F_0 & G_0 \end{bmatrix}$ has full rank, we can conclude that $P_1 > 0$. We can now repeat the argument for the next time instant and so on. ∎

We close this section with yet another condition which will be useful in control problems.

**Lemma 2.7.5 (Condition for a Minimum)** *If in addition to the conditions of Lemma 2.7.4, the matrices $F_j - G_j S_j R_j^{-1} H_j$ are invertible for all $j$, then the deterministic problems of Theorem 2.7.4 will each have a unique minimum if, and only if, $P_{N+1} > 0$ and*

$$(Q_j - S_j R_j^{-1} S_j^*)^{-1} - G_j^* P_{j+1}^{-1} G_j > 0 \qquad j = 0, 1, ..., N.$$

**Proof:** Let us first note that the Riccati recursion can be rewritten as

$$\begin{aligned} P_{i+1} &= F_i P_i F_i^* + G_i Q_i G_i^* - (F_i P_i H_i^* + G_i S_i)(R_i + H_i P_i H_i^*)^{-1}(F_i P_i H_i^* + G_i S_i)^* \\ &= (F_i - G_i S_i R_i^{-1} H_i) P_i (F_i - G_i S_i R_i^{-1} H_i)^* + G_i (Q_i - S_i R_i^{-1} S_i^*) G_i^* \\ &\quad -(F_i - G_i S_i R_i^{-1} H_i) P_i H_i^* (R_i + H_i P_i H_i^*)^{-1} H_i P_i (F_i - G_i S_i R_i^{-1} H_i)^* \\ &= (F_i - G_i S_i R_i^{-1} H_i)(P_i^{-1} + H_i^* R_i^{-1} H_i)^{-1}(F_i - G_i S_i R_i^{-1} H_i)^* + G_i (Q_i - S_i R_i^{-1} S_i^*) G_i^*. \end{aligned}$$

The proof, which uses the last of the above equalities, now follows from the following sequence of congruences and Lemma 2.7.4.

$$\begin{aligned} & \begin{bmatrix} P_{i+1} & 0 \\ 0 & (Q_i - S_i R_i^{-1} S_i^*)^{-1} - G_i^* P_{i+1}^{-1} G_i \end{bmatrix} \\ \sim & \begin{bmatrix} P_{i+1} & G_i \\ G_i^* & (Q_i - S_i R_i^{-1} S_i^*)^{-1} \end{bmatrix} \\ \sim & \begin{bmatrix} P_{i+1} - G_i (Q_i - S_i R_i^{-1} S_i^*) G_i^* & 0 \\ 0 & (Q_i - S_i R_i^{-1} S_i^*)^{-1} \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} (F_i - G_iS_iR_i^{-1}H_i)(P_i^{-1} + H_i^*R_i^{-1}H_i)^{-1}(F_i - G_iS_iR_i^{-1}H_i)^* & 0 \\ 0 & (Q_i - S_iR_i^{-1}S_i^*)^{-1} \end{bmatrix} \\
&\sim \begin{bmatrix} (P_i^{-1} + H_i^*R_i^{-1}H_i)^{-1} & 0 \\ 0 & (Q_i - S_iR_i^{-1}S_i^*)^{-1} \end{bmatrix}.
\end{aligned}
$$

■

## 2.8 Concluding Remarks

We developed a self-contained theory for linear estimation in Krein spaces. We started with the notion of projections and discussed their relation to stationary points of certain quadratic forms encountered in a pair of partially equivalent stochastic deterministic problems. By assuming an additional state-space structure, we showed that projections could be recursively computed by a Krein space Kalman filter, several applications for which are described in the next chapters.

The approach, in all these applications, is that given an indefinite deterministic quadratic form, to which $H^\infty$, risk-sensitive, and finite-memory problems lead almost by inspection, one can relate them to a corresponding Krein space stochastic problem for which the Kalman filter can be written down immediately and used to obtain recursive solutions of the above problems.

# 2.A Proof of the Time-Variant KYP Lemma

In this section we give a stochastic proof and geometric interpretation of the time-variant Kalman-Yakubovich-Popov (KYP) Lemma whose steady-state counterpart was introduced in Sec. 2.2.2. The method is based on considering linear time-variant systems driven by processes with elements in a Krein space rather than the usual Hilbert space setting.

## 2.A.1 Statement of the Lemma

**Theorem 2.A.1 (Time-Variant KYP Lemma)** *Consider the observable linear time-variant state-space model*

$$\begin{cases} \mathbf{x}_{i+1} &= F_i\mathbf{x}_i + \mathbf{u}_i, \quad 0 \leq i \leq N \\ \mathbf{y}_i &= H_i\mathbf{x}_i + \mathbf{v}_i \end{cases}$$

*with* $\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & S_i\delta_{ij} \\ 0 & S_i^*\delta_{ij} & R_i\delta_{ij} \end{bmatrix}$. *Then the following two statements are equivalent:*

*(i) The output spectrum is positive semi-definite,* i.e.

$$R_y \geq 0.$$

*(ii) There exists a sequence of Hermitian matrices* $\{Z_i\}_{i=0}^{N+1}$, *such that*

$$\Pi_0 - Z_0 \geq 0 \tag{2.A.1}$$

*and*

$$\begin{bmatrix} -Z_{i+1} + F_iZ_iF_i^* + Q_i & S_i + F_iZ_iH_i^* \\ S_i^* + H_iZ_iF_i^* & R_i + H_iZ_iH_i^* \end{bmatrix} \geq 0, \quad i = 0, 1, \ldots, N. \tag{2.A.2}$$

### 2.A.2 Computation of the Output Gramian

We begin by computing the entries of the output Gramian matrix of $\mathbf{y}$ in the standard state-space model of Lemma 2.A.1.

**Lemma 2.A.1 (Gramian Expressions)** *Consider the standard state-space model and denote the state Gramian matrix by*

$$\langle \mathbf{x}_i, \mathbf{x}_i \rangle = \|\mathbf{x}_i\|^2 = \Pi_i.$$

*Then $\Pi_i$ satisfies*

$$\Pi_{i+1} = F_i \Pi_i F_i^* + G_i Q_i G_i^*, \quad i \geq 0, \tag{2.A.3}$$

*which is often called the discrete-time Lyapunov recursion. Moreover, if we define the state transition matrix*

$$\Phi(i,j) = F_{i-1} F_{i-2} \dots F_j, \quad i > j, \quad \Phi(i,i) = I, \tag{2.A.4}$$

*then the Gramians of the state variables can be computed via*

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \begin{cases} \Phi(i,j)\Pi_j & i \geq j, \\ \Pi_i \Phi^*(j,i) & i \leq j, \end{cases} \tag{2.A.5}$$

*and the Gramians of the output process $\{\mathbf{y}_i\}$ are given by*

$$\langle \mathbf{y}_i, \mathbf{y}_j \rangle = \begin{cases} H_i \Phi(i, j+1) N_j & i > j, \\ R_i + H_i \Pi_i H_i^* & i = j, \\ N_i^* \Phi^*(j, i+1) H_j^* & i < j, \end{cases} \tag{2.A.6}$$

*where*

$$N_i = F_i \Pi_i H_i^* + G_i S_i. \tag{2.A.7}$$

**Proof:** The Lyapunov recursion follows by computing the Gramian of both sides of the state equation and using the property $\langle \mathbf{x}_i, \mathbf{u}_i \rangle = 0$.

To compute the Gramians of the state variables, note that from the state equation we can write

$$\mathbf{x}_i = \Phi(i,j)\mathbf{x}_j + \text{ some linear combination of } \mathcal{L}\{\mathbf{u}_j, \dots, \mathbf{u}_{i-1}\}, \quad i \geq j. \tag{2.A.8}$$

Therefore,

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \Phi(i,j)\langle \mathbf{x}_j, \mathbf{x}_j \rangle + 0, \quad \text{for} \;\; i \geq j,$$

Next note that

$$\text{for } i < j, \quad \langle \mathbf{x}_i, \mathbf{x}_j \rangle = \langle \mathbf{x}_j, \mathbf{x}_i \rangle^* = \Pi_i \Phi^*(j,i). \tag{2.A.9}$$

Finally, we note that

$$\langle \mathbf{y}_i, \mathbf{y}_j \rangle = H_i \langle \mathbf{x}_i, \mathbf{x}_j \rangle H_i^* + H_i \langle \mathbf{x}_i, \mathbf{v}_j \rangle + \langle \mathbf{v}_i, \mathbf{x}_j \rangle H_j^* + \langle \mathbf{v}_i, \mathbf{v}_j \rangle. \tag{2.A.10}$$

Now for $i > j$, the last two terms on the RHS are zero, while the second term is,

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \Phi(i, j+1) G_j \langle \mathbf{u}_i, \mathbf{v}_j \rangle = \Phi(i, j+1) G_j S_j, \tag{2.A.11}$$

and the first term is

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \Phi(i,j)\Pi_j = \Phi(i, j+1) F_j \Pi_j. \tag{2.A.12}$$

Collecting these results gives the first of the desired expressions for $\langle \mathbf{y}_i, \mathbf{y}_j \rangle$. The remaining two (for $i = j$ and $i < j$) follow in similar fashion.

■

The above Lemma allows us to computes the entries of the output Gramian $R_y$. It is also possible to give the following *global* expression of the output Gramian,

$$R_y = \mathcal{O}\Pi_0\mathcal{O}^* + \begin{bmatrix} \Gamma & I \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} \Gamma^* \\ I \end{bmatrix} \tag{2.A.13}$$

where

$$Q = \text{diag}(Q_0, \ldots, Q_N) \;\; , \quad R = \text{diag}(R_0, \ldots, R_N) \;\; , \quad S = \text{diag}(S_0, \ldots, S_N)$$

and where we have defined the *observability map*

$$\mathcal{O} \triangleq \mathcal{O}(0,N) = \begin{bmatrix} H_0\Phi(0,0) \\ H_1\Phi(1,0) \\ H_2\Phi(2,0) \\ \vdots \\ H_N\Phi(N,0) \end{bmatrix} = \begin{bmatrix} H_0 \\ H_1F_0 \\ H_2F_1F_0 \\ \vdots \\ H_NF_{N-1}\ldots F_0 \end{bmatrix}, \tag{2.A.14}$$

and the *impulse response* matrix

$$\Gamma = \begin{bmatrix} 0 & & & & \\ \Gamma_{10} & 0 & & & \\ \Gamma_{20} & \Gamma_{21} & 0 & & \\ \Gamma_{30} & \Gamma_{31} & \Gamma_{32} & 0 & \\ . & . & . & . & . \end{bmatrix}, \tag{2.A.15}$$

with

$$\Gamma_{ij} \triangleq H_i \Phi(i, j+1) G_j = H_i F_{i-1} \ldots F_{j+1} G_j. \tag{2.A.16}$$

The expression (2.A.13) for the output Gramian follows readily from the global relation

$$\mathbf{y} = \mathcal{O}\mathbf{x}_0 + \Gamma\mathbf{u} + \mathbf{v} = \begin{bmatrix} \mathcal{O} & \Gamma \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \end{bmatrix} + \mathbf{v}, \tag{2.A.17}$$

where we have defined,

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_N \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_N \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{v}_0 \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_N \end{bmatrix}.$$

Alternative expressions can be obtained by exploiting the fact that the entries of $\mathcal{O}$ and $\Gamma$ inherit the assumed state-space structure. Indeed, let us denote by

$$Z = \begin{bmatrix} 0 & & & & \\ I & 0 & & & \\ & I & 0 & & \\ & & \ddots & \ddots & \\ & & & I & 0 \end{bmatrix}, \tag{2.A.18}$$

the lower triangular shift matrix (with identity on the first lower block sub-diagonal and zeros elsewhere); $Z$ has the property that it shifts column vectors one block element downwards: if $a = \text{col}\{a_0, \ldots, a_N\}$, then

$$Za = \text{col}\{0, a_0, \ldots, a_{N-1}\}.$$

Using the shift matrix $Z$, we can rewrite the $i$th (block) column of the impulse response matrix as (refer to Eqs. (2.A.14)-(2.A.15) where the structure of $\Gamma$ is shown)

$$\text{i-th column of } \ \Gamma = Z^i \begin{bmatrix} 0 \\ H_i\Phi(i,i) \\ H_{i+1}\Phi(i+1,i) \\ \vdots \\ H_N\Phi(N,i) \end{bmatrix} G_i. \tag{2.A.19}$$

Consequently, the following expressions for $R_y$ also result.

**Lemma 2.A.2 (Two Expressions for the Output Gramian)** *The output Gramian, $R_y$, corresponding to the standard state-space model can be written in either of the following two forms:*

*(i)*

$$R_y = \sum_{i=0}^{N} Z^i \begin{bmatrix} 0 & I \\ H_i\Phi(i,i) & 0 \\ H_{i+1}\Phi(i+1,i) & 0 \\ \vdots & \vdots \\ H_N\Phi(N,i) & 0 \end{bmatrix} \begin{bmatrix} 0 & N_i \\ N_i^* & R_i + H_i\Pi_i H_i^* \end{bmatrix} \begin{bmatrix} 0 & I \\ H_i\Phi(i,i) & 0 \\ H_{i+1}\Phi(i+1,i) & 0 \\ \vdots & \vdots \\ H_N\Phi(N,i) & 0 \end{bmatrix}^* Z^{*i}, \tag{2.A.20}$$

*(ii)*

$$R_y = \mathcal{O}\Pi_0\mathcal{O}^* + \sum_{i=0}^{N} Z^i \begin{bmatrix} 0 & I \\ H_i\Phi(i,i) & 0 \\ H_{i+1}\Phi(i+1,i) & 0 \\ \vdots & \vdots \\ H_N\Phi(N,i) & 0 \end{bmatrix} \begin{bmatrix} Q_i & S_i \\ S_i^* & R_i \end{bmatrix} \begin{bmatrix} 0 & I \\ H_i\Phi(i,i) & 0 \\ H_{i+1}\Phi(i+1,i) & 0 \\ \vdots & \vdots \\ H_N\Phi(N,i) & 0 \end{bmatrix}^* Z^{*i}. \tag{2.A.21}$$

**Proof:** The proof of (i) follows from inspection of the expressions (2.A.6) for the elements of $R_y$. Likewise (ii) follows by inspecting (2.A.13) and using (2.A.19).

■

We thus have obtained two equivalent representations for the output Gramian, $R_y$. [These are, incidentally, the time-varying counterparts of the equivalent representations (2.2.2) and (2.2.5) of the $z$-spectral density function, $S_y(z)$.] We shall now obtain a full parametrization of all such equivalent representations.

## 2.A.3 An Equivalence Class for the Input Gramians

In this section we construct equivalent classes for the initial condition/input Gramians that lead to the same output Gramian when the standard state-space model is assumed. To this end, suppose that we *add* initial conditions $\bar{\mathbf{x}}_0$ and inputs $\{\bar{\mathbf{u}}_i, \bar{\mathbf{v}}_i\}$, orthogonal to the original $\mathbf{x}_0$ and $\{\mathbf{u}_i, \mathbf{v}_i\}$, to the state-space model, *i.e.*

$$\left\{\begin{array}{rcl} \mathbf{x}_{i+1} + \bar{\mathbf{x}}_{i+1} & = & F_i(\mathbf{x}_i + \bar{\mathbf{x}}_i) + \mathbf{u}_i + \bar{\mathbf{u}}_i, \quad \mathbf{x}_0 + \bar{\mathbf{x}}_0, \quad 0 \leq i \leq N \\ \mathbf{y}_i + \bar{\mathbf{y}}_i & = & H_i\mathbf{x}_i + \mathbf{v}_i + \bar{\mathbf{v}}_i \end{array}\right. \tag{2.A.22}$$

in such a manner that the output Gramian of (2.A.22) does not change. Now using linearity and the orthogonality of the inputs, the output Gramian of (2.A.22) is

$$R_y + \bar{R}_y,$$

where $\bar{R}_y$ is the output Gramian of the system

$$\left\{\begin{array}{rcl} \bar{\mathbf{x}}_{i+1} & = & F_i\bar{\mathbf{x}}_i + \bar{\mathbf{u}}_i, \quad \bar{\mathbf{x}}_0, \quad 0 \leq i \leq N \\ \bar{\mathbf{y}}_i & = & H_i\bar{\mathbf{x}}_i + \bar{\mathbf{v}}_i \end{array}\right. \tag{2.A.23}$$

Now for the output Gramian to remain invariant we require $\bar{R}_y = 0$, *i.e.* (2.A.23) must generate zero output Gramian. Let us assume

$$\langle \begin{bmatrix} \bar{\mathbf{x}}_0 \\ \bar{\mathbf{u}}_i \\ \bar{\mathbf{v}}_i \end{bmatrix}, \begin{bmatrix} \bar{\mathbf{x}}_0 \\ \bar{\mathbf{u}}_i \\ \bar{\mathbf{v}}_i \end{bmatrix} \rangle = \begin{bmatrix} \bar{\Pi}_0 & 0 & 0 \\ 0 & \bar{Q}_i\delta_{ij} & \bar{S}_i\delta_{ij} \\ 0 & \bar{S}_i^*\delta_{ij} & \bar{R}_i\delta_{ij} \end{bmatrix}.$$

From the first of the state equations we may write

$$\langle \mathbf{x}_{i+1}, \mathbf{x}_{i+1} \rangle = F_i \langle \mathbf{x}_i, \mathbf{x}_i \rangle_{\mathcal{K}} F_i^* + \underbrace{\langle \mathbf{u}_i, \mathbf{u}_i \rangle}_{\bar{Q}_i}$$

so that if we define $Z_i = -\langle \mathbf{x}_i, \mathbf{x}_i \rangle$, we have

$$\bar{Q}_i = -Z_{i+1} + F_i Z_i F_i^*. \tag{2.A.24}$$

Now

$$\langle \bar{\mathbf{y}}_i, \bar{\mathbf{y}}_i \rangle = H_i \langle \mathbf{x}_i, \mathbf{x}_i \rangle H_i^* + \bar{R}_i = -H_i Z_i H_i^* + \bar{R}_i = 0 \tag{2.A.25}$$

and for $j > 0$

$$\langle \bar{\mathbf{y}}_{i+j}, \bar{\mathbf{y}}_i \rangle = H_{i+j} F_{i+j-1} \ldots F_{i+1}(-F_i Z_i H_i^* + \bar{S}_i) = 0.$$

If we write the above equation for all $j > 0$:

$$\begin{bmatrix} H_{i+1} \\ H_{i+2}F_{i+1} \\ H_{i+3}F_{i+2}F_{i+1} \\ \vdots \end{bmatrix} (-F_i Z_i H_i^* + \bar{S}_i) = 0$$

and since we have assumed the observability of the system, the matrix on the left-hand side is full rank, so that

$$\bar{S}_i = F_i Z_i H_i^*. \tag{2.A.26}$$

Combining (2.A.24), (2.A.25) and (2.A.26) yields

$$\bar{\Pi}_0 = -Z_0 \quad \text{and} \quad \begin{bmatrix} \bar{Q}_i & \bar{S}_i \\ \bar{S}_i^* & \bar{R}_i \end{bmatrix} = \begin{bmatrix} -Z_{i+1} + F_i Z_i F_i^* & F_i Z_i H_i^* \\ H_i Z_i F_i^* & H_i Z_i H_i^* \end{bmatrix}.$$

It is now straightforward to show the following Lemma.

**Lemma 2.A.3 (Equivalent Class for Input Gramians)** *(a) The output Gramian, $R_y$, of the state-space model*

$$\begin{cases} \mathbf{x}_{i+1} = F_i \mathbf{x}_i + \mathbf{u}_i, & 0 \leq i \leq N \\ \mathbf{y}_i = H_i \mathbf{x}_i + \mathbf{v}_i \end{cases} \tag{2.A.27}$$

*with* $\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i \delta_{ij} & S_i \delta_{ij} \\ 0 & S_i^* \delta_{ij} & R_i \delta_{ij} \end{bmatrix}$ *is invariant under the transformation*

$$\Pi_0 \to \Pi_0 - Z_0 \quad \textit{and} \quad \begin{bmatrix} Q_i & S_i \\ S_i^* & R_i \end{bmatrix} \to \begin{bmatrix} -Z_{i+1} + F_i Z_i F_i^* + Q_i & F_i Z_i H_i^* + S_i \\ H_i Z_i F_i^* + S_i^* & H_i Z_i H_i^* + R_i \end{bmatrix}$$

*for any sequence of Hermitian matrices* $\{Z_i\}_{i=0}^{N+1}$.

*(b) If the system (2.A.27) is observable, and there exist*

$$\left\{\Pi_0^{(1)},\ \begin{bmatrix} Q_i^{(1)} & S_i^{(1)} \\ S_i^{*(1)} & R_i^{(1)} \end{bmatrix}\right\} \quad \textit{and} \quad \left\{\Pi_0^{(2)},\ \begin{bmatrix} Q_i^{(2)} & S_i^{(2)} \\ S_i^{*(2)} & R_i^{(2)} \end{bmatrix}\right\}$$

*that yield identical output Gramians,* i.e.,

$$R_y^{(1)} = R_y^{(2)},$$

*then there exists a* <u>*unique*</u> *sequence of Hermitian matrices* $\{Z_i\}_{i=0}^{N+1}$ *such that*

$$\Pi_0^{(1)} = \Pi_0^{(2)} - Z_0 \quad \textit{and} \quad \begin{bmatrix} Q_i^{(1)} & S_i^{(1)} \\ S_i^{*(1)} & R_i^{(1)} \end{bmatrix} = \begin{bmatrix} -Z_{i+1} + F_i Z_i F_i^* + Q_i^{(2)} & F_i Z_i H_i^* + S_i^{(2)} \\ H_i Z_i F_i^* + S_i^{*(2)} & H_i Z_i H_i^* + R_i^{(2)} \end{bmatrix}.$$

**Proof:** We have already shown part (a) prior to the statement of the lemma. We therefore proceed to prove part (b). To this end, suppose that there exist $\{\Pi_0^{(1)}, \{Q_i^{(1)}, R_i^{(1)}, S_i^{(1)}\}\}$ and $\{\Pi_0^{(2)}, \{Q_i^{(2)}, R_i^{(2)}, S_i^{(2)}\}\}$ that yield the same $R_y$, and define the *unique* sequences of matrices, $\{Z_i^{(1)}, N_i^{(1)}, R_{\pi,i}^{(1)}\}$ and $\{Z_i^{(2)}, N_i^{(2)}, R_{\pi,i}^{(2)}\}$, via the recursions,

$$\left\{\begin{array}{lllllllll} Z_{i+1}^{(1)} &=& F_i Z_i^{(1)} F_i^* + Q_i^{(1)} &,& N_i^{(1)} &=& F_i Z_i^{(1)} H_i^* + S_i^{(1)} &,& R_{\pi,i}^{(1)} = R_i^{(1)} + H_i Z_i^{(1)} H_i^* \\ Z_{i+1}^{(2)} &=& F_i Z_i^{(2)} F_i^* + Q_i^{(2)} &,& N_i^{(2)} &=& F_i Z_i^{(2)} H_i^* + S_i^{(2)} &,& R_{\pi,i}^{(2)} = R_i^{(2)} + H_i Z_i^{(2)} H_i^* \end{array}\right. \tag{2.A.28}$$

initialized with $Z_0^{(1)} = Z_0^{(1)} = \Pi_0$. Using the result of Lemma 2.A.2, part (i), this means that we can write

$$\begin{aligned} R_y &= \sum_{i=0}^{N} Z^i \begin{bmatrix} 0 & I \\ H_i\Phi(i,i) & 0 \\ H_{i+1}\Phi(i+1,i) & 0 \\ \vdots & \vdots \\ H_N\Phi(N,i) & 0 \end{bmatrix} \begin{bmatrix} 0 & N_i^{(j)} \\ N_i^{*(j)} & R_{\pi,i}^{(j)} \end{bmatrix} \begin{bmatrix} 0 & I \\ H_i\Phi(i,i) & 0 \\ H_{i+1}\Phi(i+1,i) & 0 \\ \vdots & \vdots \\ H_N\Phi(N,i) & 0 \end{bmatrix}^* Z^{*i} \\ &= \sum_{i=0}^{N} Z^i \begin{bmatrix} 0 \\ H_i\Phi(i,i)N_i^{(j)} \\ H_{i+1}\Phi(i+1,i)N_i^{(j)} \\ \vdots \\ H_N\Phi(N,i)N_i^{(j)} \end{bmatrix} + \operatorname{diag}(R_{\pi,0}^{(j)}, \ldots, R_{\pi,N}^{(j)}) + \begin{bmatrix} \text{strictly upper} \\ \text{triangular matrix} \end{bmatrix} \end{aligned}$$

for $j = 1, 2$. The above expression allows us to identify the $i$-th block diagonal entry of $R_y$ as $R_{\pi,i}^{(1)} = R_{\pi,i}^{(2)} = R_{y,ii}$, and the strictly lower triangular $(i,j)$-th block entry as,

$$H_i \Phi(i,j) N_j^{(1)} = H_i \Phi(i,j) N_j^{(2)} = R_{y,ij}.$$

The observability of the system allows us to conclude that $N_i^{(1)} = N_i^{(2)}$. Therefore

$$\begin{bmatrix} 0 & N_i^{(1)} \\ N_i^{*(1)} & R_{\pi,i}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & N_i^{(2)} \\ N_i^{*(2)} & R_{\pi,i}^{(2)} \end{bmatrix}$$

or, equivalently,

$$\begin{bmatrix} -Z_{i+1}^{(1)} + F_i Z_i^{(1)} F_i^* + Q_i^{(1)} & F_i Z_i^{(1)} H_i^* + S_i^{(1)} \\ H_i Z_i^{(1)} F_i^* + S_i^{*(1)} & R_i^{(1)} + H_i Z_i^{(1)} H_i^* \end{bmatrix} =$$
$$\begin{bmatrix} -Z_{i+1}^{(2)} + F_i Z_i^{(2)} F_i^* + Q_i^{(2)} & F_i Z_i^{(2)} H_i^* + S_i^{(2)} \\ H_i Z_i^{(2)} F_i^* + S_i^{*(2)} & R_i^{(2)} + H_i Z_i^{(2)} H_i^* \end{bmatrix},$$

from which we conclude that the unique sequence of matrices, $\{Z_i \triangleq Z_i^{(2)} - Z_i^{(1)}\}_{i=0}^{N+1}$ relates the two sets of input Gramians.

■

### 2.A.4 The Proof

We can now proceed with the proof of the KYP Lemma. First note that the in view of Lemma 2.A.2, part (ii), and Lemma 2.A.3, part (a), going from (ii) to (i) in the KYP Lemma 2.A.1 is trivial. Therefore we shall focus on the other direction — from (i) to (ii).

To this end, suppose that $R_y \geq 0$. If this is the case, then $R_y$ will admit the following *unique* block lower-diagonal-upper factorization,[12]

$$R_y = L R_e L^*, \tag{2.A.29}$$

where $L$ is block lower diagonal with block unit diagonal, and $R_e$ is block diagonal with

$$R_e \geq 0. \tag{2.A.30}$$

[12]This factorization is, of course, related to the Cholesky factorization of positive semidefinite matrices (see *e.g.*, [GL89, HJ90, Str93]).

Note that the block diagonal entries of $R_e$ may be singular. Now the block lower triangular matrix $L$ represents a linear time-variant causal mapping.[13] Therefore we can construct a *minimal* state-space realization $\{F_i^a, G_i^a, H_i^a, I\}$ of this mapping (see *e.g.*, [Mar64, HI94]). In other words, we can write,

$$L = \begin{bmatrix} I & & & & \\ H_1^a G_0^a & I & & & \\ H_2^a \Phi^a(2,1) G_0^a & H_2^a G_1^a & I & & \\ & & & & \\ \vdots & & & \ddots & \\ H_N^a \Phi^a(N,1)^a G_0^a & H_N^a \Phi^a(N,2) G_1^a & \dots & H_N^a G_{N-1}^a & I \end{bmatrix}, \tag{2.A.31}$$

where $\Phi^a(i,j) = F_{i-1}^a F_{i-2}^a \dots F_j^a$, for $i > j$. Note that we may also write

$$L = \sum_{i=0}^{N} Z^i \begin{bmatrix} 0 & I \\ H_i^a \Phi^a(i,i) & 0 \\ H_{i+1}^a \Phi^a(i+1,i) & 0 \\ \vdots & \vdots \\ H_N^a \Phi^a(N,i) & 0 \end{bmatrix} \begin{bmatrix} G_i^a \\ I \end{bmatrix}, \tag{2.A.32}$$

from which we can infer that,

$$R_y = \sum_{i=0}^{N} Z^i \begin{bmatrix} 0 & I \\ H_i^a \Phi^a(i,i) & 0 \\ H_{i+1}^a \Phi^a(i+1,i) & 0 \\ \vdots & \vdots \\ H_N^a \Phi^a(N,i) & 0 \end{bmatrix} \begin{bmatrix} G_i^a R_{e,i} G_i^{a*} & G_i^a R_{e,i} \\ R_{e,i} G_i^{a*} & R_{e,i} \end{bmatrix} \begin{bmatrix} 0 & I \\ H_i^a \Phi^a(i,i) & 0 \\ H_{i+1}^a \Phi^a(i+1,i) & 0 \\ \vdots & \vdots \\ H_N^a \Phi^a(N,i) & 0 \end{bmatrix}^* Z^{*i}. \tag{2.A.33}$$

Now if we define the sequence of matrices,

$$\begin{cases} Z_{i+1}^a &= F_i^a Z_i^a F_i^{a*} + G_i^a R_{e,i} G_i^{a*}, \quad Z_0^a = \Pi_0 \\ N_i^a &= F_i^a Z_i^a H_i^{a*} + G_i^a R_{e,i} \\ R_{\pi,i}^a &= R_{e,i} + H_i Z_i^a H_i^* \end{cases} \tag{2.A.34}$$

[13]Note that if $L$ were Toeplitz it would represent a *time-invariant* linear mapping. In general, this is not the case.

then using an argument similar to the one presented in the proof of Lemma 2.A.3 allows us to conclude that the diagonal entries of $R_y$ are

$$R_{y,ii} = R_{e,i} + H_i Z_i^a H_i^*, \tag{2.A.35}$$

and that the strictly lower triangular entries of $R_y$ are

$$R_{y,ij} = H_i^a \Phi^a(i,j) N_j^a, \quad i > j. \tag{2.A.36}$$

Had we, instead of the representation (2.A.33) for $R_y$, used the representation of Lemma 2.A.2, part (ii), then we would have identified the entries of $R_y$ as

$$\left\{ \begin{array}{rcl} R_{y,ii} & = & R_i + H_i Z_i H_i^* \\ R_{y,ij} & = & H_i \Phi(i,j) N_j, \quad i > j \end{array} \right. \tag{2.A.37}$$

where

$$\left\{ \begin{array}{rcl} Z_{i+1} & = & F_i Z_i F_i^{a*} + Q_i, \quad Z_0 = \Pi_0 \\ N_i & = & F_i Z_i H_i^* + S_i \\ R_{\pi,i} & = & R_i + H_i Z_i H_i^* \end{array} \right. . \tag{2.A.38}$$

Equating these two results yields,

$$R_{e,i} + H_i Z_i^a H_i^* = R_i + H_i Z_i H_i^*, \tag{2.A.39}$$

and

$$H_i^a \Phi^a(i,j) N_j^a = H_i \Phi(i,j) N_j, \quad i > j. \tag{2.A.40}$$

This latter expression means that the two systems with state-space representations, $\{F_i^a, N_i^a, H_i^a\}$ and $\{F_i, N_i, H_i\}$ have the same impulse response matrix (or input-output map). Thus there must exist a (possibly time-variant) similarity transformation, $T_i$, such that

$$\left\{ \begin{array}{rcl} \begin{bmatrix} F_a^i & \times \\ 0 & F_i^{a,c} \end{bmatrix} & = & T_i^{-1} F_i T_i \\ \begin{bmatrix} N_i^a \\ 0 \end{bmatrix} & = & T_i^{-1} N_i \\ \begin{bmatrix} H_i^a & H_i^{a,c} \end{bmatrix} & = & H_i T_i \end{array} \right. , \tag{2.A.41}$$

where the superscript '$c$' represents the uncontrollable states of the system $\{F_i, N_i, H_i\}$. [Note that we have no unobservable modes since $\{F, H\}$ is assumed observable.] On the other hand, we can write

$$\begin{aligned} H_i^a \Phi^a(i,j) G_j^a &= \begin{bmatrix} H_i^a & H_i^{a,c} \end{bmatrix} \prod_{k=j}^{i-1} \begin{bmatrix} F_a^i & \times \\ 0 & F_i^{a,c} \end{bmatrix} \begin{bmatrix} G_i^a \\ 0 \end{bmatrix} \\ &= H_i \Phi(i,j) \underbrace{T_j \begin{bmatrix} G_i^a \\ 0 \end{bmatrix}}_{K_{p,j}}. \end{aligned}$$

But using (2.A.33), this implies that

$$R_y = \sum_{i=0}^{N} Z^i \begin{bmatrix} 0 & I \\ H_i\Phi(i,i) & 0 \\ H_{i+1}\Phi(i+1,i) & 0 \\ \vdots & \vdots \\ H_N\Phi(N,i) & 0 \end{bmatrix} \begin{bmatrix} K_{p,i}R_{e,i}K_{p,i}^* & K_{p,i}R_{e,i} \\ R_{e,i}K_{p,i}^* & R_{e,i} \end{bmatrix} \begin{bmatrix} 0 & I \\ H_i\Phi(i,i) & 0 \\ H_{i+1}\Phi(i+1,i) & 0 \\ \vdots & \vdots \\ H_N\Phi(N,i) & 0 \end{bmatrix}^* Z^{*i}. \tag{2.A.42}$$

Comparing the above representation of $R_y$ with that given in Lemma 2.A.2, part (ii), indicates that there must exist a unique sequence of Hermitian matrices, say $\{\bar{Z}_i\}_{i=0}^{N+1}$, such that

$$\Pi_0 - \bar{Z}_0 = 0, \tag{2.A.43}$$

and

$$\begin{bmatrix} -\bar{Z}_{i+1} + F_i\bar{Z}_iF_i^* + Q_i & S_i + F_i\bar{Z}_iH_i^* \\ S_i^* + H_i\bar{Z}_iF_i^* & R_i + H_i\bar{Z}_iH_i^* \end{bmatrix} = \begin{bmatrix} K_{p,i}R_{e,i}K_{p,i}^* & K_{p,i}R_{e,i} \\ R_{e,i}K_{p,i}^* & R_{e,i} \end{bmatrix} \geq 0, \tag{2.A.44}$$

where we have used the fact that $R_{e,i} \geq 0$. But this is the desired result.

## 2.A.5 Geometric Interpretation

Since we motivated and introduced the KYP lemma by considering state-space models whose inputs were elements of a Krein space, it seems natural to suspect that the KYP Lemma should admit further interpretation, or a simpler proof, in the context of Krein spaces. We shall presently see that this is indeed the case, and that the KYP Lemma has a simple geometric interpretation in terms of a certain decomposition in Krein space.

### The Geometric Setup

Since our interpretation of the KYP Lemma will use Krein space geometry, we begin by carefully defining the spaces in which our inputs and outputs lie.

The initial condition/input space, $\mathcal{K}^{(d)}$, consists of all initial conditions, $\mathbf{x}_0$, and all input sequences, $\{\mathbf{u}_i, \mathbf{v}_i\}$, that are orthogonal to one another and that have (possibly indefinite) arbitrary Gramian matrices, $\Pi_0$ and $\begin{bmatrix} Q_i & S_i \\ S_i^* & R \end{bmatrix}$. Thus, we may write

$$\mathcal{K}^{(d)} = \left\{ \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_N \\ \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_N \end{bmatrix} \;\middle|\; \left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & S_i\delta_{ij} \\ 0 & S_i^*\delta_{ij} & R_i\delta_{ij} \end{bmatrix} \right\}. \tag{2.A.45}$$

Note that $\mathcal{K}_+^{(d)}$ and $\mathcal{K}_-^{(d)}$ have the obvious structures,

$$\mathcal{K}_+^{(d)} = \left\{ \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_N \\ \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_N \end{bmatrix} \;\middle|\; \left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & S_i\delta_{ij} \\ 0 & S_i^*\delta_{ij} & R_i\delta_{ij} \end{bmatrix} \geq 0 \right\},$$

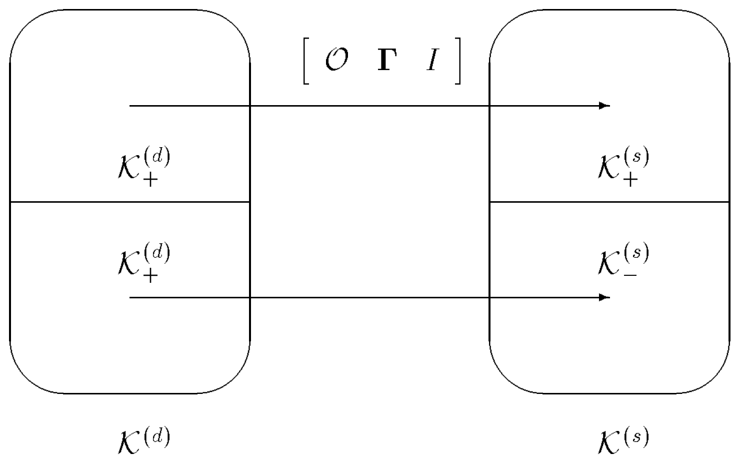


Figure 2.3: Mapping from input space to output space.

and

$$\mathcal{K}_+^{(d)} = \left\{ \left[\begin{array}{c} \mathbf{x}_0 \\ \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_N \\ \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_N \end{array}\right] \;\middle|\; \left\langle \left[\begin{array}{c} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{array}\right], \left[\begin{array}{c} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{array}\right] \right\rangle = \left[\begin{array}{ccc} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & S_i\delta_{ij} \\ 0 & S_i^*\delta_{ij} & R_i\delta_{ij} \end{array}\right] \leq 0 \right\}.$$

To now construct the output space, consider the time-variant state-space model

$$\left\{ \begin{array}{rcl} \mathbf{x}_{i+1} & = & F_i\mathbf{x}_i + \mathbf{u}_i, \quad 0 \leq i \leq N \\ \mathbf{y}_i & = & H_i\mathbf{x}_i + \mathbf{v}_i \end{array} \right. \tag{2.A.46}$$

where $\mathbf{x}_0$ and the $\{\mathbf{u}_i, \mathbf{v}_i\}$ belong to $\mathcal{K}^{(d)}$. Now if, as before, we define

$$\mathbf{y} = \text{col}\{\mathbf{y}_0, \ldots, \mathbf{y}_N\} \ , \ \ \mathbf{u} = \text{col}\{\mathbf{u}_0, \ldots, \mathbf{u}_N\} \ , \ \ \mathbf{v} = \text{col}\{\mathbf{v}_0, \ldots, \mathbf{v}_N\}$$

then we can write

$$\mathbf{y} = \mathcal{O}\mathbf{x}_0 + \Gamma\mathbf{u} + \mathbf{v} = \left[\begin{array}{ccc} \mathcal{O} & \Gamma & I \end{array}\right] \left[\begin{array}{c} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{v} \end{array}\right]. \tag{2.A.47}$$

Thus, the $\mathbf{y}$ also form a Krein space, $\mathcal{K}^{(s)}$. We can formally represent this Krein space as

$$\mathbf{y} \in \mathcal{K}^{(s)} = \begin{bmatrix} \mathcal{O} & \Gamma & I \end{bmatrix} \mathcal{K}^{(d)},$$

and in particular $\mathcal{K}^{(s)} = \mathcal{K}^{(s)}_+ \oplus \mathcal{K}^{(s)}_-$, with

$$\mathcal{K}^{(s)}_+ = \begin{bmatrix} \mathcal{O} & \Gamma & I \end{bmatrix} \mathcal{K}^{(d)}_+ \quad \text{and} \quad \mathcal{K}^{(s)}_- = \begin{bmatrix} \mathcal{O} & \Gamma & I \end{bmatrix} \mathcal{K}^{(d)}_-. \tag{2.A.48}$$

Thus, $\mathcal{K}^{(s)}$ is the Krein space of all possible outputs of (2.A.46) when the inputs are from the Krein space $\mathcal{K}^{(d)}$, and $\mathcal{K}^{(s)}_+$ is the Hilbert space of all possible outputs of (2.A.46) when the inputs are from the Hilbert space $\mathcal{K}^{(d)}_+$. Likewise for $\mathcal{K}^{(s)}_-$. See Fig. 2.3.

### A Simple Decomposition

We now develop a simple geometric interpretation of the KYP Lemma in terms of a certain decomposition in Krein space. Note that the Gramian $\langle \mathbf{y}, \mathbf{y} \rangle = R_y$ can be regarded as the squared norm of $\mathbf{y}$, some element of the Krein space $\mathcal{K}^{(s)}$. Now the premise of the KYP Lemma is such that $R_y \geq 0$, *i.e.*, that $\mathbf{y}$ has positive squared norm and belongs to the positive subspace of $\mathcal{K}^{(s)}$.

Suppose therefore that $R_y \geq 0$ and use the KYP Lemma to choose a sequence of Hermitian matrices, $\{Z_i\}_{i=0}^{N+1}$, such that

$$\Pi_0 - Z_0 \geq 0 \quad \text{and} \quad \begin{bmatrix} -Z_{i+1} + F_i Z_i F_i^* + Q_i & S_i + F_i Z_i H_i^* \\ S_i^* + H_i Z_i F_i^* & R_i + H_i Z_i H_i^* \end{bmatrix} \geq 0$$

for $i = 0, 1, \ldots, N$. Since this corresponds to a nonnegative definite initial state Gramian and nonnegative definite input Gramians, the inputs in this case lie in $\mathcal{K}^{(d)}_+$, and output associated with it, say $\mathbf{y}^+$, belongs to $\mathcal{K}^{(s)}_+$. Moreover, the output process, $\mathbf{y}^0$, corresponding to initial state Gramian $Z_0$ and input Gramian

$$\begin{bmatrix} Z_{i+1} - F_i Z_i F_i^* & -F_i Z_i H_i^* \\ -H_i Z_i F_i^* & -H_i Z_i H_i^* \end{bmatrix},$$

is a neutral element of $\mathcal{K}^{(s)}$ since $\langle \mathbf{y}^0, \mathbf{y}^0 \rangle = 0$. The KYP Lemma obviously states

$$\langle \mathbf{y}, \mathbf{y} \rangle = \langle \mathbf{y}^+, \mathbf{y}^+ \rangle + \langle \mathbf{y}^0, \mathbf{y}^0 \rangle. \tag{2.A.49}$$

The following result is now straightforward.

**Geometric Interpretation of the KYP Lemma** *Consider an element* $\mathbf{y} \in \mathcal{K}^{(s)}$. *Then* $\mathbf{y}$ *has positive squared norm,* i.e., $\langle \mathbf{y}, \mathbf{y} \rangle \geq 0$, *if, and only if, it can be decomposed as follows*

$$\mathbf{y} = \mathbf{y}^{+} + \mathbf{y}^{0}, \tag{2.A.50}$$

*where* $\mathbf{y}^{+} \in \mathcal{K}_{+}^{(s)}$ *is such that* $\langle \mathbf{y}^{+}, \mathbf{y}^{+} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle \geq 0$, *and* $\mathbf{y}^{0}$ *is neutral,* i.e., $\langle \mathbf{y}^{0}, \mathbf{y}^{0} \rangle = 0$. *Moreover, note that* $\langle \mathbf{y}^{+}, \mathbf{y}^{0} \rangle = 0$, i.e., *that* $\mathbf{y}^{+}$ *and* $\mathbf{y}^{0}$ *are orthogonal.*

To gain further insight into the above decomposition of $\mathbf{y}$, let us write the (unique) fundamental decomposition of the elements $\mathbf{y}$ and $\mathbf{y}^{0}$ into their components in $\mathcal{K}_{+}^{(s)}$ and $\mathcal{K}_{-}^{(s)}$

$$\mathbf{y} = \mathbf{y}_{+} + \mathbf{y}_{-} \quad \text{and} \quad \mathbf{y}^{0} = \mathbf{y}_{+}^{0} + \mathbf{y}_{-}^{0}.$$

Therefore, using (2.A.50) we may write

$$\mathbf{y}_{+} + \mathbf{y}_{-} = \mathbf{y}^{+} + (\mathbf{y}_{+}^{0} + \mathbf{y}_{-}^{0}).$$

Equating the components of the above equality that belong to $\mathcal{K}_{+}^{(s)}$ and $\mathcal{K}_{-}^{(s)}$, respectively, yields

$$\mathbf{y}_{+} = \mathbf{y}^{+} + \mathbf{y}_{+}^{0}, \tag{2.A.51}$$

and $\mathbf{y}_{-} = \mathbf{y}_{-}^{0}$. Eq. (2.A.51) has a very interesting interpretation. First note that since $\mathbf{y}^{+}$ is orthogonal to both $\mathbf{y}^{0}$ and $\mathbf{y}_{-}^{0}$ it must be orthogonal to $\mathbf{y}_{+}^{0}$ as well. Thus (2.A.51) is an orthogonal decomposition in $\mathcal{K}_{+}^{(s)}$: it shows that the given element $\mathbf{y}_{+}$ with squared norm larger than the squared norm of $\mathbf{y}$, *i.e.*, $\langle \mathbf{y}_{+}, \mathbf{y}_{+} \rangle \geq \langle \mathbf{y}, \mathbf{y} \rangle \geq 0$ , can be orthogonally decomposed into two elements, $\mathbf{y}^{+}$ and $\mathbf{y}_{+}^{0}$, one of which has squared norm equal to the squared norm of $\mathbf{y}$, *i.e.*, $\langle \mathbf{y}^{+}, \mathbf{y}^{+} \rangle = \langle \mathbf{y}, \mathbf{y} \rangle$. Roughly speaking, if we consider the hypersphere in $\mathcal{K}_{+}^{(s)}$ of radius $\langle \mathbf{y}, \mathbf{y} \rangle \geq 0$, then $\mathbf{y}^{+}$ is obtained from drawing the tangent from $\mathbf{y}_{+}$ to this hypersphere (see Fig. 2.4).

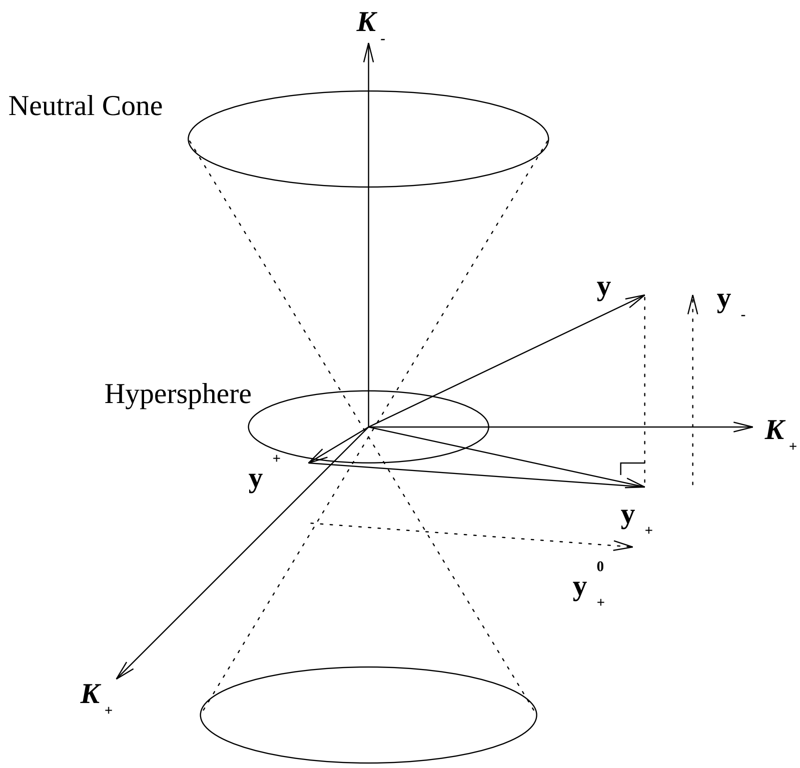


Figure 2.4: Decomposition of positive vectors.

# Chapter 3

# Finite Horizon $H^{\infty}$ Filtering

In this chapter we study the problem of finite horizon $H^{\infty}$ filtering. We consider the two cases of a posteriori a priori filtering, as well as smoothing and $l$-step prediction. We essentially show that all these problems can be cast into the problem of calculating the stationary point of certain indefinite quadratic forms. This allows us to exploit (to great effect) the machinery developed in Chapter 2. In particular, we show that by considering the appropriate state space models and error Gramians, we can use the Krein space estimation theory to calculate the stationary points and study their properties and thereby solve the various $H^{\infty}$ filtering problems. The approach discussed here allows for interesting generalizations and extensions, which we will discuss in Chapters 4 and 5.

## 3.1 Introduction

Classical results in linear least-squares estimation and Kalman filtering are based on an $L^2$ or $H^2$ criterion and require a-priori knowledge of the statistical properties of the noise signals. In some applications however, one is faced with model uncertainties and lack of statistical information on the exogenous signals, which has led to an increasing interest in minimax estimation (see, *e.g.,* [Kwa86, DGKF89, KN91, Bas91, LS91, ST92, YS91, Gri93] and the references therein), with the belief that the resulting so-called $H^{\infty}$ algorithms will be more robust and less sensitive to parameter variations.

$H^\infty$ filtering problems (as well as many other problems in risk-sensitive estimation and control problems, quadratic games and finite memory adaptive filtering) lead almost by inspection to indefinite deterministic quadratic forms. Following Chapter 2, we solve these problems by constructing the corresponding Krein space "stochastic" problems for which the Kalman filter solutions can be written down immediately; moreover, the conditions for a minimum (related to the existence of solutions to the $H^\infty$ problem) can also be expressed in terms of quantities easily related to the basic Riccati equations of the Kalman filter. This approach also explains the many similarities between say, the $H^\infty$ solutions and the classical LQ solutions, and in addition marks out their key differences.

The chapter is organized as follows. In Section 3.2 we introduce the $H^\infty$ estimation problem, state the conventional solution and discuss its similarities with and differences from the conventional Kalman filter. In Section 3.3 we reduce the $H^\infty$ estimation problem to guaranteeing the positivity of a certain indefinite quadratic form. We then relate this quadratic form to a certain Krein state-space model, which allows us to use the results of Chapter 2 to derive conditions for its positivity, and to show that projection in the Krein space allows us to solve the $H^\infty$ estimation problem. In this context we derive the $H^\infty$ a posteriori, a priori, and smoothing filters, as well as $l$-step predictors,[1] and show that $H^\infty$ estimation is essentially Kalman filtering in Krein space; we also obtain a natural parametrization of all $H^\infty$ estimators. One advantage of our approach is that it suggests how well known conventional Kalman filtering algorithms, such as square root arrays and Chandrasekhar recursions, can be extended to the $H^\infty$ setting. Much more will be said about this in Chapter 5.

As was done in Chapter 2, we shall use bold letters for elements in a Krein space, and normal letters for corresponding complex numbers. Also, we shall use $\check{s}$ to denote the estimate of $s$ (according to some criterion), and $\hat{s}$ to denote the Krein space projection, thereby stressing the fact that they need not coincide. Many of the results discussed here were obtained earlier by several other authors, using different methods and arguments. Our approach, we believe, provides a powerful unification,

[1] The solution to the $l$-step prediction problem presented here is, to the best of our knowledge, new. We have not encountered general $l$-step $H^\infty$ filters in the literature.

with immediate insights to various extensions.

## 3.2 $H^\infty$ Estimation

Several $H^\infty$-filtering algorithms have been recently derived by a variety of methods in both the continuous and discrete-time cases (see, *e.g.,* [Kwa86, DGKF89, KN91, Bas91, LS91, ST92, YS91, Gri93] and the references therein).

Many authors have noticed some formal similarities between the $H^\infty$ filters and the conventional Kalman filter, however we shall further clarify this connection by showing that $H^\infty$ filters are nothing more than certain Krein space Kalman filters. In other words, the $H^\infty$ filters can be viewed as recursively performing a (Gram-Schmidt) orthogonalization (or projection) procedure on a convenient set of observation data that obey a state-space model whose state evolves in an indefinite metric space. This is of significance since it yields a geometric derivation of the $H^\infty$ filters, and because it unifies $H^2$ and $H^\infty$ estimation in a simple framework. Moreover, once this connection has been made explicit, many known alternative and more efficient algorithms, such as square-root arrays and Chandrasekhar equations[HSK94c], can be applied to the $H^\infty-$setting as well. Also our results deal directly with the time-varying scenario. Finally, we note that although we restrict ourselves here, to the discrete-time case, however, the continuous time analogs follow the same principles.

### 3.2.1 Formulation of the $H^\infty$ Filtering Problem

Consider a time-variant state-space model of the form

$$\left\{ \begin{array}{rcll} x_{i+1} & = & F_i x_i + G_i u_i, & x_0 \\ y_i & = & H_i x_i + v_i, & i \geq 0 \end{array} \right. \tag{3.2.1}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_i \in \mathcal{C}^{n\times m}$ and $H_i \in \mathcal{C}^{p\times n}$ are known matrices, $x_0$, $\{u_i\}$, and $\{v_i\}$ are *unknown* quantities and $y_i$ is the measured output. We can regard $v_i$ as a measurement noise and $u_i$ as a process noise or driving disturbance. We make no assumption on the nature of the disturbances (*e.g.* , normally distributed, uncorrelated, etc). In general, we would like to estimate some arbitrary linear combination of the states, say

$$s_i = L_i x_i,$$

where $L_i \in \mathcal{C}^{q\times n}$ is given, using the observations $\{y_j\}$. Let $\check{s}_{i|i} = \mathcal{F}_f(y_0, y_1, \ldots, y_i)$ denote the estimate of $s_i$ given observations $\{y_j\}$ from time 0 up to and including time $i$, and $\check{s}_i = \mathcal{F}_p(y_0, y_1, \ldots, y_{i-1})$ denote the estimate of $s_i$ given observations $\{y_j\}$ from time 0 to time $i-1$. We then have the following two estimation errors: the *filtered* error

$$e_{f,i} = \check{s}_{i|i} - L_i x_i, \tag{3.2.2}$$

and the *predicted* error

$$e_{p,i} = \check{s}_i - L_i x_i. \tag{3.2.3}$$

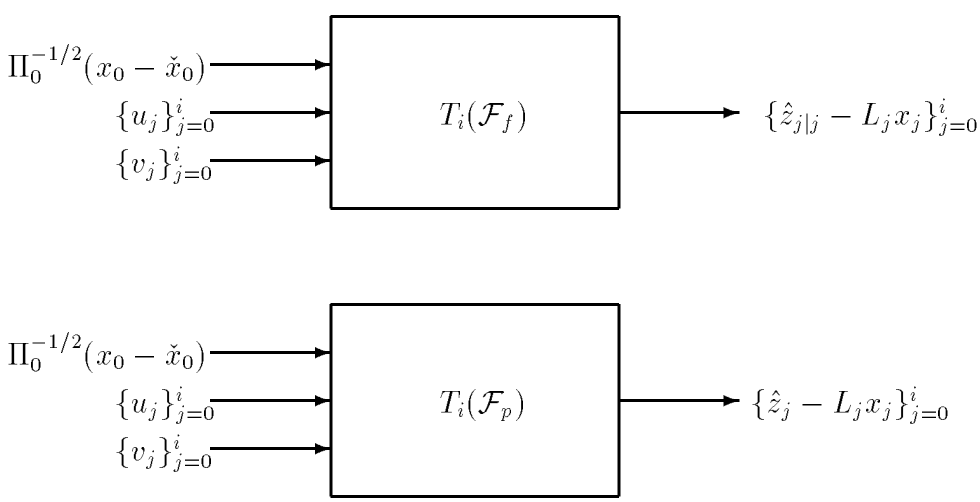


Figure 3.1: Transfer matrix from disturbances to filtered and predicted estimation errors.

As depicted in Fig. 3.1, let $T_i(\mathcal{F}_f)$ and $T_i(\mathcal{F}_p)$ denote the transfer operators that map the unknown disturbances $\{\Pi_0^{-1/2}(x_0 - \check{x}_0), \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ (where $\check{x}_0$ denotes an initial guess for $x_0$, and $\Pi_0$ is a given positive definite matrix) to the filtered and predicted errors $\{e_{f,j}\}_{j=0}^{i}$ and $\{e_{p,j}\}_{j=0}^{i}$, respectively. The problem is to choose the functionals $\mathcal{F}_f(\cdot)$ and $\mathcal{F}_p(\cdot)$ so as to respectively minimize the $H^\infty$ norm of the transfer operators $T_i(\mathcal{F}_f)$ and $T_i(\mathcal{F}_p)$.

**Definition 3.2.1** *The $H^\infty$ norm of a transfer operator $T$ is defined as*

$$\|T\|_\infty = \sup_{u\in h^2, u\neq 0} \frac{\|Tu\|_2}{\|u\|_2}$$

*where $\|u\|_2$ is the $h^2-$norm of the causal sequence $\{u_k\}$,* i.e. $\|u\|_2^2 = \sum_{k=0}^{\infty} u_k^* u_k$.

The $H^\infty$ norm thus has the interpretation of being the maximum energy gain from the input $u$ to the output $y$. Our problem may now be formally stated as follows.

**Problem 3.2.1 (Optimal $H^\infty$ Problem)** *Find filtered and predicted $H^\infty$-optimal estimation strategies $\check{s}_{i|i} = \mathcal{F}_f(y_0, y_1, \ldots, y_i)$ and $\check{s}_i = \mathcal{F}_p(y_0, y_1, \ldots, y_{i-1})$ that respectively minimize $\| T_i(\mathcal{F}_f) \|_\infty$ and $\| T_i(\mathcal{F}_p) \|_\infty$, and obtain the resulting*

$$\begin{aligned} \gamma_{f,o}^2 &= \inf_{\mathcal{F}_f} \| T_i(\mathcal{F}_f) \|_\infty^2 \\ &= \inf_{\mathcal{F}_f} \sup_{x_0, u\in h^2, v\in h^2} \frac{\sum_{j=0}^{i} e_{f,j}^* e_{f,j}}{(x_0-\check{x}_0)^* \Pi_0^{-1}(x_0-\check{x}_0) + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} v_j^* v_j} \end{aligned} \tag{3.2.4}$$

*and*

$$\begin{aligned} \gamma_{p,o}^2 &= \inf_{\mathcal{F}_p} \| T_i(\mathcal{F}_p) \|_\infty^2 \\ &= \inf_{\mathcal{F}_p} \sup_{x_0, u\in h^2, v\in h^2} \frac{\sum_{j=0}^{i} e_{p,j}^* e_{p,j}}{(x_0-\check{x}_0)^* \Pi_0^{-1}(x_0-\check{x}_0) + \sum_{j=0}^{i-1} u_j^* u_j + \sum_{j=0}^{i-1} v_j^* v_j} \end{aligned} \tag{3.2.5}$$

*where $\Pi_0$ is a positive definite matrix that reflects a priori knowledge as to how close $x_0$ is to the initial guess $\check{x}_0$.*

Note that the infimum in (3.2.5) is taken over all *strictly* causal estimators $\mathcal{F}_p$, whereas in (3.2.4) the estimators $\mathcal{F}_f$ are only causal, since they have additional access to $y_i$. This is relevant since the solution to the $H^\infty$ problem, as we shall see, depends on the structure of the information available to the estimator.

The above problem formulation shows that $H^\infty$ optimal estimators guarantee the smallest estimation error energy over all possible disturbances of fixed energy. They are therefore conservative (and at times over-conservative), which results in a better robust behaviour to disturbance variation.

A closed form solution to the optimal $H^\infty$ estimation problem is available only in some special cases (see *e.g.* , [HSK96a] and Chapter 9), and so it is common in the literature to settle for a suboptimal solution.

**Problem 3.2.2 (Sub-optimal $H^\infty$ Problem)** *Given scalars $\gamma_f > 0$ and $\gamma_p > 0$, $H^\infty$ suboptimal estimation strategies $\check{s}_{i|i} = \mathcal{F}_f(y_0, y_1, \ldots, y_i)$ (known as an a posteriori filter) and $\check{s}_i = \mathcal{F}_p(y_0, y_1, \ldots, y_{i-1})$ (known as an a priori filter) that respectively achieve $\| T_i(\mathcal{F}_f) \|_\infty < \gamma_f$ and $\| T_i(\mathcal{F}_p) \|_\infty < \gamma_p$. In other words, find strategies that respectively achieve*

$$\sup_{x_0, u \in h^2, v \in h^2} \frac{\sum_{j=0}^{i} e_{f,j}^* e_{f,j}}{(x_0 - \check{x}_0)^* \Pi_0{}^{-1} (x_0 - \check{x}_0) + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} v_j^* v_j} < \gamma_f^2 \tag{3.2.6}$$

*and*

$$\sup_{x_0, u \in h^2, v \in h^2} \frac{\sum_{j=0}^{i} e_{p,j}^* e_{p,j}}{(x_0 - \check{x}_0)^* \Pi_0{}^{-1} (x_0 - \check{x}_0) + \sum_{j=0}^{i-1} u_j^* u_j + \sum_{j=0}^{i-1} v_j^* v_j} < \gamma_p^2. \tag{3.2.7}$$

*This clearly requires checking whether $\gamma_f \geq \gamma_{f,o}$ and $\gamma_p \geq \gamma_{p,o}$.*

Note that the solutions to Problem 3.2.1 can be obtained to desired accuracy by iterating on the $\gamma_f$ and $\gamma_p$ of Problem 3.2.2. From here on we shall be only dealing with Problem 3.2.2.

Note that since the upper limits in the aforementioned sums are the finite number $i$, the problems defined above are *finite-horizon* problems. So-called *infinite-horizon* problems can be considered if we define $T(\mathcal{F}_f)$ and $T(\mathcal{F}_p)$ as the transfer operators that map $\{x_0 - \check{x}_0, \{u_j\}_{j=0}^\infty, \{v_j\}_{j=0}^\infty\}$ to $\{e_{f,j}\}_{j=0}^\infty$ and $\{e_{p,j}\}_{j=0}^\infty$, respectively. Then by guaranteeing $\| T_i(\mathcal{F}_f) \|_\infty < \gamma_f$ and $\| T_i(\mathcal{F}_p) \|_\infty < \gamma_p$ for all $i$, we can solve the infinite-horizon problems $\| T(\mathcal{F}_f) \|_\infty \leq \gamma_f$ and $\| T(\mathcal{F}_p) \|_\infty \leq \gamma_p$, respectively. However, as we have seen in Sec. 1.4.1, direct solutions are also possible. We shall study the infinite horizon problem in more detail in Chapter 7.

### 3.2.2 Solution of the Suboptimal $H^\infty$ Filtering Problem

We now present the existing solutions (see *e.g.*, [KN91, ST92]) to the suboptimal $H^\infty$ filtering problem and note that they are intriguingly similar, in several ways, to the conventional Kalman filter. It was this similarity in structure that led us to extend Kalman filtering to Krein spaces (see Chapter 2): in effect, $H^\infty$ filters are just Kalman filters in Krein space.

**Theorem 3.2.1 (A Finite Horizon $H^\infty$ A Posteriori Filter)** *[ST92] For a given $\gamma > 0$, if the $\begin{bmatrix} F_j & G_j \end{bmatrix}$ have full rank, then an estimator that achieves $\|T_i(\mathcal{F}_f)\|_\infty < \gamma$ exists if, and only if,*

$$P_j^{-1} + H_j^* H_j - \gamma^{-2} L_j^* L_j > 0, \qquad j = 0, \ldots, i \tag{3.2.8}$$

*where $P_0 = \Pi_0$ and $P_j$ satisfies the Riccati recursion*

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j F_j^*, \tag{3.2.9}$$

*with*

$$R_{e,j} = \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix}. \tag{3.2.10}$$

*If this is the case, then one possible level-$\gamma$ $H^\infty$ filter is given by*

$$\check{s}_{j|j} = L_j \hat{x}_{j|j},$$

*where $\hat{x}_{j|j}$ is recursively computed as*

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{s,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}) \quad , \quad \hat{x}_{-1|-1} = \textit{initial guess} \tag{3.2.11}$$

*and*

$$K_{s,j+1} = P_{j+1} H_{j+1}^* (I + H_{j+1} P_{j+1} H_{j+1}^*)^{-1}. \tag{3.2.12}$$

**Theorem 3.2.2 (A Finite Horizon $H^\infty$ A Priori Filter)** *[ST92] Given $\gamma > 0$, if the $\begin{bmatrix} F_j & G_j \end{bmatrix}$ have full rank, then an estimator that achieves $\|T_i(\mathcal{F}_p)\|_\infty < \gamma$ exists if, and only if,*

$$\tilde{P}_j^{-1} = P_j^{-1} - \gamma^{-2} L_j^* L_j > 0, \qquad j = 0, \ldots, i \tag{3.2.13}$$

*where $P_j$ is the same as in Theorem 3.2.1. If this is the case, then one possible level-$\gamma$ $H^\infty$ filter is given by*

$$\check{s}_j = L_j \hat{x}_j, \tag{3.2.14}$$

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{a,j}(y_j - H_j \hat{x}_j) \quad , \quad \hat{x}_0 = \textit{initial guess} \tag{3.2.15}$$

*where*

$$K_{a,j} = F_j \tilde{P}_j H_j^* (I + H_j \tilde{P}_j H_j^*)^{-1}. \tag{3.2.16}$$

### Comparisons with the Kalman Filter

The Kalman filter algorithm for estimating the states in (3.2.1), assuming that the $\{u_i\}$ and $\{v_i\}$ are uncorrelated unit variance white noise processes is, in the a posteriori case,

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + P_{j+1} H_{j+1}^* (I + H_{j+1} P_{j+1} H_{j+1}^*)^{-1} (y_{j+1} - H_{j+1} \hat{x}_{j+1}), \tag{3.2.17}$$

and, in the a priori case,

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + P_{j+1} H_{j+1}^* (I + H_{j+1} P_{j+1} H_{j+1}^*)^{-1} (y_{j+1} - H_{j+1} \hat{x}_{j+1}), \tag{3.2.18}$$

where

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - F_j P_j (I + H_j P_j H_j^*)^{-1} P_j F_j^*, \quad P_0 = \Pi_0. \tag{3.2.19}$$

As several authors have noted, the $H^\infty$ solutions are very similar to the conventional Kalman filter. The major differences are the following:

- The structure of the $H^\infty$ estimators depends, via the Riccati recursion (3.2.9), on the linear combination of the states that we intend to estimate (*i.e.* the $L_i$). This is as opposed to the Kalman filter, where the estimate of any linear combination of the state is given by that linear combination of the state estimate. Intuitively, this means that the $H^\infty$ filters are specifically tuned towards the linear combination $L_i x_i$.

- We have additional conditions, (3.2.8) or (3.2.13), that must be satisfied for the filter to exist; in the Kalman filter problem the $L_i$ would not appear, and the $P_i$ would be positive definite, so that (3.2.8) and (3.2.13) would be immediate.

- We have indefinite (*covariance*) matrices, *e.g.* , $\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -\gamma^2 I \end{bmatrix}$ vs. just $I$ in the Kalman filter.

- As $\gamma \to \infty$, the Riccati recursion (3.2.9) reduces to the Kalman filter recursion (3.2.19). [This suggests that the $H^\infty$ norm of the conventional Kalman filter may be quite large, and that it may have poor robustness properties.[2] Note also that condition (3.2.13) is more stringent than condition (3.2.8), indicating that the existence of an a priori filter of level $\gamma$ implies the existence of an a posteriori filter of the same level, but not necessarily vice versa.]

Despite these differences, we shall show by applying the results of Chapter 2, that the filters of Theorems 3.2.1 and 3.2.2 can in fact be obtained as certain Kalman filters, not in an $H^2$ (Hilbert) space, but in a certain indefinite vector space, called a Krein space. The indefinite covariances and the appearance of $L_i$ in the Riccati equation will be easily explained in this framework. The additional condition (3.2.8) will be seen to arise from the fact that in Krein space, unlike as in the usual Hilbert space context, quadratic forms need not always have minima or maxima, unless certain additional conditions are met. Moreover, our approach will provide a simpler and more general alternative to the tests (3.2.8) and (3.2.13).

[2] We shall have more to say about this in Chapter 10.

## 3.3 Derivation of the $H^\infty$ Filters

In Sec. 1.4.1 we derived the $H^\infty$ suboptimal filters using the canonical factorization of a certain indefinite transfer operator. Here we can take the same approach: the indefinite transfer operator will now be a finite indefinite matrix (or Gramian) and the canonical factorization will correspond to a (block) LDU (lower-diagonal-factorization). As we have seen in Chapter 2, when the Gramian has state-space structure (as it does in the $H^\infty$ state-space estimation problem under consideration) the triangular factorization can be obtained from the Krein space Kalman filter. This route will then lead us both to the solution of the $H^\infty$ filtering problems (via the Krein space Kalman filter, which incidentally also explains the connections with the conventional Kalman filter) and to the existence conditions for such filters (since an LDU factorization with certain inertia properties requires additional conditions). However, we shall here take a slightly different (and closer to the general spirit of Chapter 2) route to the solution, *i.e.*, one that is based upon associating an indefinite quadratic form (instead of an indefinite transfer operator) with the $H^\infty$ filtering problem. There are, of course, close connections between these two approaches and we shall comment on them wherever deemed appropriate.

As shown Chapter 2, the first step is to associate an indefinite quadratic form with each of the (level $\gamma$) a posteriori and a priori filtering problems. This will lead us to construct an appropriate (so-called partially equivalent) Krein space state-space model, the Kalman filter for which will allow us to compute the stationary points for the $H^\infty$ quadratic forms; conditions that these are actually minima will be deduced from the general results of Chapter 2, and shown to be just (3.2.8) and (3.2.13); simpler equivalent conditions will also be noted.

Therefore we begin by examining the structure of the $H^\infty$ problem in more detail. The goal will be to relate the problem to an indefinite quadratic form. We shall first consider the a posteriori filtering problem.

### 3.3.1 The Suboptimal $H^\infty$ Problem and Quadratic Forms

Referring to Problem 3.2.2, we first note that $\|T_i(\mathcal{F}_f)\|_\infty < \gamma_f$, implies that for all nonzero $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$

$$\frac{\sum_{j=0}^i |\check{s}_{j|j} - L_j x_j|^2}{(x_0 - \check{x}_0)^* \Pi_0^{-1}(x_0 - \check{x}_0) + \sum_{j=0}^i |u_j|^2 + \sum_{j=0}^i |y_j - H_j x_j|^2} < \gamma_f^2. \tag{3.3.1}$$

Moreover, (3.3.1) implies that for *all* $k \leq i$, we must have

$$\frac{\sum_{j=0}^k |\check{s}_{j|j} - L_j x_j|^2}{(x_0 - \check{x}_0)^* \Pi_0^{-1}(x_0 - \check{x}_0) + \sum_{j=0}^k |u_j|^2 + \sum_{j=0}^k |y_k - H_i x_i|^2} < \gamma_f^2. \tag{3.3.2}$$

We remark that if the $\{y_j\}_{j=0}^i$ are all zero then it is easy to see that the $\{\check{s}_{j|j}\}$ must all be zero as well. Therefore we need only consider the case where $\{y_j\}_{j=0}^i$ is a nonzero sequence. We shall then prove the following result, relating the condition $\|T_i(\mathcal{F}_f)\|_\infty < \gamma_f$ to the positivity of a certain indefinite quadratic form. Incidentally from now on, without loss of generality, we assume $\check{x}_0 = 0$; a nonzero $\check{x}_0 = 0$ will only change the initial condition of the filter.

**Lemma 3.3.1 (Indefinite Quadratic Form)** *Given $\gamma_f > 0$, then $\|T_i(\mathcal{F}_f)\|_\infty < \gamma_f$ if, and only if, there exists $\check{s}_{k|k} = \mathcal{F}_f(y_0, \ldots, y_k)$ (for all $0 \leq k \leq i$) such that for all complex vectors $x_0$, for all causal sequences $\{u_j\}_{j=0}^i$ and for all nonzero causal sequences $\{y_j\}_{j=0}^i$, the scalar quadratic form*

$$\begin{gathered} J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k) = x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^k u_j^* u_j + \\ \sum_{j=0}^k (y_j - H_j x_j)^*(y_j - H_j x_j) - \gamma_f^{-2} \sum_{j=0}^k (\check{s}_{j|j} - L_j x_j)^*(\check{s}_{j|j} - L_j x_j) \end{gathered} \tag{3.3.3}$$

*satisfies*

$$J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k) > 0 \quad \textit{for all} \quad 0 \leq k \leq i. \tag{3.3.4}$$

**Proof:** For one direction, assume there exists a solution $\check{s}_{k|k}$ (for all $k \leq i$) that achieves $\|T_i(\mathcal{F}_f)\|_\infty < \gamma_f$. Then if we multiply both sides of (3.3.2) by the positive denominator on the LHS, we obtain (3.3.4).

Conversely, if there exists a solution $\check{s}_{k|k}$ (for all $k \leq i$) that achieves (3.3.4), we can divide both sides of (3.3.4) by the positive quantity

$$x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{k} u_j^* u_j + \sum_{j=0}^{k} (y_j - H_j x_j)^* (y_j - H_j x_j)$$

to obtain (3.3.2), and thereby $\|T_i(\mathcal{F}_f)\|_\infty < \gamma_f$.

■

**Remark:** Lemma 3.3.1 is a straightforward restatement of the inequality (3.3.2) which is required of all suboptimal $H^\infty$ a posteriori filters with level $\gamma_f$. However, the statement of Lemma 3.3.2, given below, is a key result, since it shows how to check the conditions of Lemma 3.3.1 by computing the stationary point of the indefinite quadratic form $J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$ and checking its condition for a minimum. This is in the spirit of the approach taken in Chapter 2.

Note that since the $\check{s}_{k|k}$ are functions of the $\{y_j\}_{j=0}^{k}$, $J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$ is really a function of *only* $\{x_0, u_0, \ldots, u_k, y_0, \ldots, y_k\}$. Moreover, since the $\{y_j\}$ are fixed observations, the only *free variables* in $J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$ are the disturbances $\{x_0, u_0, \ldots, u_k\}$. We then have the following result.

**Lemma 3.3.2 (Positivity Condition)** *The scalar quadratic forms*

$$J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k),$$

*satisfy the conditions (3.3.4), if, and only if, for all* $0 \leq k \leq i$*,*

*(i)* $J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$ *has a minimum with respect to* $\{x_0, u_0, u_1, \ldots u_k\}$*.*

*(ii) The* $\{\check{s}_{k|k}\}_{k=0}^{i}$ *can be chosen such that the value of* $J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$ *at this minimum is positive, viz.,*

$$\min_{\{x_0, u_0, \ldots, u_k\}} J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k, \check{s}_{0|0}, \ldots, \check{s}_{k|k}) > 0.$$

**Proof:** Assume $J_{f,k}(x_0, \{u_j\}_{j=0}^{k}, \{y_j\}_{j=0}^{k}) > 0$, then condition (i) is clearly satisfied because if $J_{f,k}(x_0, \{u_j\}_{j=0}^{k}, \{y_j\}_{j=0}^{k})$ does not have a minimum over $\{x_0, u_0, \ldots, u_k\}$, then it is always possible to choose $\{x_0, \{u_j\}_{j=0}^{k}\}$ so as to make $J_{f,k}(x_0, \{u_j\}_{j=0}^{k}, \{y_j\}_{j=0}^{k})$ arbitrarily small and negative. Moreover, the existence of a minimum, along with $J_{f,k}(x_0, \{u_j\}_{j=0}^{k}, \{y_j\}_{j=0}^{k}) > 0$, guarantees condition (ii) since the value at the minimum must be positive.

Conversely, if (i) and (ii) hold, then it follows that $J_{f,k}(x_0, \{u_j\}_{j=0}^{k}, \{y_j\}_{j=0}^{k}) > 0$. ■

### 3.3.2 A Krein Space State-Space Model

To apply the methodology of Chapter 2, we first identify the indefinite quadratic form $J_{f,k}$ as a special case of the general form studied in Theorem 2.7.4 by rewriting it as

$$J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k) =$$
$$x_0^*\Pi_0^{-1}x_0+\sum_{j=0}^{k} u_j^* u_j+\sum_{j=0}^{k}\left(\begin{bmatrix} y_j \\ \check{s}_{j|j} \end{bmatrix} - \begin{bmatrix} H_j \\ L_j \end{bmatrix} x_j\right)^* \begin{bmatrix} I & 0 \\ 0 & -\gamma_f^{-2} I \end{bmatrix} \left(\begin{bmatrix} y_j \\ \check{s}_{j|j} \end{bmatrix} - \begin{bmatrix} H_j \\ L_j \end{bmatrix} x_j\right). \tag{3.3.5}$$

Then by Lemmas 2.6.1 and 2.6.2, we can introduce the following Krein space state-space model,

$$\left\{ \begin{array}{rcl} \mathbf{x}_{j+1} & = & F_j\mathbf{x}_j + G_j\mathbf{u}_j \\ \begin{bmatrix} \mathbf{y}_j \\ \check{\mathbf{s}}_{j|j} \end{bmatrix} & = & \begin{bmatrix} H_j \\ L_j \end{bmatrix} \mathbf{x}_j + \mathbf{v}_j \end{array} \right. \tag{3.3.6}$$

with

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_k \\ \mathbf{v}_k \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & I\delta_{jk} & 0 \\ 0 & 0 & \begin{bmatrix} I & 0 \\ 0 & -\gamma_f^2 I \end{bmatrix}\delta_{jk} \end{bmatrix} \tag{3.3.7}$$

Note that $Q_j = I$, $S_j = 0$, $\Pi_0 > 0$, and that we must consider a Krein space since

$$R_j = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -\gamma_f^2 I \end{bmatrix} \tag{3.3.8}$$

is indefinite.[3]

[3]We should also mention that the output Gramian of $\begin{bmatrix} \mathbf{y}_j \\ \check{\mathbf{s}}_{j|j} \end{bmatrix}$ (in the state-space model (3.3.5-3.3.7) is simply the indefinite Gramian of Theorem 1.4.1 whose canonical factorization is needed in

### 3.3.3 Proof of Theorem 3.2.1

In order to focus the discussion we briefly review the procedure of the proof.

- Referring to Lemma 3.3.2, we first need to check the whether $J_{f,k}(x_0, \{u_j, y_j\}_{j=0}^k)$ has a minimum with respect to $\{x_0, \{u_j\}_{j=0}^k\}$. This is done via the Krein space Kalman filter corresponding to (3.3.6-3.3.7), and yields the condition (3.2.8) along with several equivalent conditions.

- Next we need to choose the $\{\check{s}_{k|k}\}_{k=0}^i$ such that the value of $J_{f,k}(x_0, \{u_j, y_j\}_{j=0}^k)$ is positive at its minimum. Now according to Theorem 2.7.4, the value at the minimum is $J_{f,k}(\min) = \sum_{j=0}^k e_j R_{e,j}^{-1} e_j$, where $e_j$ is the innovations corresponding to (3.3.6-3.3.7). We can then compute the $\{e_j\}$ using the Krein space Kalman filter, and thereby choose the appropriate $\{\check{s}_{k|k}\}_{k=0}^i$ which yields the desired a posteriori filter.

**A remark on the strong regularity of the model (3.3.6-3.3.7):** In what follows we would like to use the Krein space Kalman filter corresponding to the state-space model (3.3.6-3.3.7). This of course requires the strong regularity of its output Gramian matrix, which we denote by $R_{\substack{y\\s}}$ (since the output of (3.3.6) consists of both a $\mathbf{y}$ and a $\mathbf{s}$ component).

If $R_{\substack{y\\s}}$ is strongly regular, then the Krein space Kalman filter may be applied to check for the positivity of $J_{f,k}$ for each $0 \le k \le i$. But what if $R_{\substack{y\\s}}$ is not strongly regular? Then it turns out that $J_{f,k}$ cannot be positive for all $0 \le k \le i$. To see why suppose that $J_{f,k} > 0$ for some arbitrary $k$. Then $J_{f,k}$ must have a minimum, and according to Lemma 2.6.4, the leading $k \times k$ block submatrices of $R_{\substack{y\\s}}$ and $R - S^*QS = R$ must have the same inertia. Now due to (3.3.8), all leading submatrices of $R$ are nonsingular, and since $k$ was arbitrary, the same will be true of $R_{\substack{y\\s}}$. Therefore $R_{\substack{y\\s}}$ will be strongly regular.

To summarize, we may use the Krein space Kalman filter to check the positivity of $J_{f,k}$. If one of the $R_{e,k}$ becomes singular (so that $R_{\substack{y\\s}}$ is no longer strongly regular),

the solution of the $H^\infty$ estimation problem. Since the Krein space Kalman filter corresponding to (3.3.5) can be used to perform this canonical factorization, we will shortly (albeit via different route) see that it plays a crucial role in the solution.

$J_{f,k}$ will lose its positivity by default.[4]

**Proof of existence condition (3.2.8):**

The Riccati recursion corresponding to (3.3.6) is the exact same Riccati recursion that was given by (3.2.9) in Theorem 3.2.1. We can now apply *any* of the conditions for a minimum developed in Chapter 2, to check whether a minimum exists for $J_{f,k}(x_0, u_0, \ldots, u_k, y_0 \ldots, y_k)$ for all $0 \leq k \leq i$. If we assume that the $\begin{bmatrix} F_k & G_k \end{bmatrix}$ have full rank, then according to Lemma 2.7.4, $J_{f,k}(x_0, u_0, \ldots, u_k, y_0 \ldots, y_k)$ will have a minimum for all $0 \leq k \leq i$, if, and only if,

$$P_{j|j}^{-1} = P_j^{-1} + \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -\gamma_f^2 I \end{bmatrix}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix} > 0$$

which yields the condition (3.2.8).

Since we still need to satisfy the second condition of Lemma 3.3.2, this, of course, only shows that (3.2.8) is a necessary condition for the existence of an $H^\infty$ a posteriori filter of level $\gamma_f$. However, we shall later show that if the minimum condition is satisfied then the second condition of Lemma 3.3.2 can also be satisfied. Therefore (3.2.8) is indeed necessary and sufficient for the existence of the filter.

∎

**Other Existence Conditions**

Using the results Chapter 2, we can obtain alternative conditions for the existence of $H^\infty$ a posteriori filters of level $\gamma_f$. If we use Lemma 2.7.3, we have the following condition.

[4]Another way to show the strong regularity of $R_{\underset{s}{y}}$ is to appeal to Theorem 1.4.1. There it was shown (in a more general setting) that an $H^\infty$ filter exists if, and only if, $R_{\underset{s}{y}}$ admits a canonical factorization with a certain prescribed inertia. It is easy to check that the existence of this canonical factorization requires, at the very least, the strong regularity of $R_{\underset{s}{y}}$.

**Lemma 3.3.3 (Alternative Test for Existence)** *The condition (3.2.8) can be replaced by the condition that*

$$R_j = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -\gamma_f^2 I \end{bmatrix} \quad and \quad R_{e,j} = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & -\gamma_f^2 I \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix}$$

*have the same inertia for all $0 \leq j \leq i$. We no longer require that $\begin{bmatrix} F_j & G_j \end{bmatrix}$ have full rank, and the size of the matrices involved is generally smaller than in (3.2.8).*

Using a block triangular factorization of $R_{e,j}$, and the fact that when we have a minimum, $P_j$ is positive definite, we can show the following result.

**Corollary 3.3.1 (Alternative Test for Existence)** *The condition of Lemma 3.3.3 is equivalent to*

$$I + H_j P_j H_j^* > 0 \quad and \quad -\gamma_f^2 I + L_j (P_j^{-1} + H_j^* H_j)^{-1} L_j^* < 0, \tag{3.3.9}$$

*for all $0 \leq j \leq i$.*

The test of Lemma 3.3.3 has various advantages over (3.2.8) that are mentioned in the discussions following Lemma 2.7.4. In particular, Lemma 3.3.3 allows us to go to a square-root form of the $H^\infty$ filtering algorithm, where there is no need to explicitly check for the existence condition - these conditions are built into the the square-root recursions themselves, so that a solution exists if, and only if, the algorithm can be performed (see [HSK94c] and Chapter 5).

Many alternative existence conditions can also be obtained. Here is one that follows Lemma 2.7.5.

**Lemma 3.3.4 (Alternative Test for Existence)** *If the $\{F_j\}$ are nonsingular, an $H^\infty$ a posteriori filter of level $\gamma_f$ exists, if, and only if,*

$$P_{i+1} > 0,$$

*and*

$$I - G_j^* P_{j+1}^{-1} G_j > 0 \quad j = 0, 1, \ldots, i.$$

### Construction of the $H^\infty$ a posteriori filters

To complete the proof of Theorem 3.2.1 we still need to show that if a minimum over $\{x_0, u_0, \ldots, u_k\}$ exists for all $0 \leq k \leq i$, then we can find the estimates $\{\check{s}_{k|k}\}_{k=0}^{i}$ such that the value of $J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$ at its minimum is positive.

According to Theorem 2.7.4, the minimum value of $J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$ is

$$\sum_{j=0}^{k} \begin{bmatrix} e_{y,j}^* & e_{s,j}^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} e_{y,j} \\ e_{s,j} \end{bmatrix} =$$

$$\sum_{j=0}^{k} \begin{bmatrix} y_j - \hat{y}_{j|j-1} \\ \check{s}_{j|j} - \hat{s}_{j|j-1} \end{bmatrix}^* \begin{bmatrix} I + H_j P_j H_j^* & H_j P_j L_j^* \\ L_j P_j H_j^* & -\gamma_f^2 I + L_j P_j L_j^* \end{bmatrix}^{-1} \begin{bmatrix} y_j - \hat{y}_{j|j-1} \\ \check{s}_{j|j} - \hat{s}_{j|j-1} \end{bmatrix}$$

where $\hat{y}_{j|j-1}$ and $\hat{s}_{j|j-1}$ are obtained from the Krein space projections of $\mathbf{y}_j$ and $\check{\mathbf{s}}_{j|j}$ onto $\mathcal{L}\left\{\{\mathbf{y}_l\}_{l=0}^{j-1}, \{\check{\mathbf{s}}_{l|l}\}_{l=0}^{j-1}\right\}$, respectively. Thus $\hat{s}_{j|j-1}$ is a linear function of $\{y_l\}_{l=0}^{j-1}$.

Using the block triangular factorization of the $R_{e,j}$ we may rewrite the above as

$$\sum_{j=0}^{k} \begin{bmatrix} y_j - \hat{y}_{j|j-1} \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix}^* \begin{bmatrix} I + H_j P_j H_j^* & 0 \\ 0 & -\gamma_f^2 I + L_j (P_j^{-1} + H_j^* H_j)^{-1} L_j^* \end{bmatrix}^{-1} \begin{bmatrix} y_j - \hat{y}_{j|j-1} \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix}$$

where

$$\begin{bmatrix} y_j - \hat{y}_{j|j-1} \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix} \triangleq \begin{bmatrix} I & 0 \\ -L_j P_j H_j^* (I + H_j P_j H_j^*)^{-1} & I \end{bmatrix} \begin{bmatrix} y_j - \hat{y}_{j|j-1} \\ \check{s}_{j|j} - \hat{s}_{j|j-1} \end{bmatrix}. \tag{3.3.10}$$

Or in other words,

$$\hat{s}_{j|j} = \hat{s}_{j|j-1} + L_j P_j H_j^* (I + H_j P_j H_j^*)^{-1} (y_j - \hat{y}_{j|j-1}). \tag{3.3.11}$$

Note that $\hat{s}_{j|j}$ is obtained from the Krein space projection of $\check{\mathbf{s}}_{j|j}$ onto $\mathcal{L}\left\{\{\mathbf{y}_l\}_{l=0}^{j}, \{\check{\mathbf{s}}_{l|l}\}_{l=0}^{j-1}\right\}$, and is therefore a linear function of $\{y_l\}_{l=0}^{j}$. Recall from Corollary 3.3.1 that

$$I + H_j P_j H_j^* > 0 \quad \text{and} \qquad -\gamma_f^2 I + L_j (P_j^{-1} + H_j^* H_j)^{-1} L_j^* < 0. \tag{3.3.12}$$

Therefore all we must do is choose some $\check{s}_{j|j}$ such that

$$\begin{aligned} &\textstyle\sum_{j=0}^{k} (y_j - \hat{y}_{j|j-1})^* (I + H_j P_j H_j^*)^{-1} (y_j - \hat{y}_{j|j-1}) \\ &+ \textstyle\sum_{j=0}^{k} (\check{s}_{j|j} - \hat{s}_{j|j})^* \left(-\gamma_f^2 I + L_j (P_j^{-1} + H_j H_j^*)^{-1} L_j^*\right)^{-1} (\check{s}_{j|j} - \hat{s}_{j|j}) > 0. \end{aligned} \tag{3.3.13}$$

There are many such choices, but in view of (3.3.12), the simplest is

$$\begin{aligned}\check{s}_{j|j} = \hat{s}_{j|j} &= L_j \hat{x}_{j|j} \\ &= L_j \hat{x}_j + L_j P_j H_j^* (I + H_j P_j H_j^*)^{-1} (y_j - H_j \hat{x}_j)\end{aligned}$$

where $\hat{x}_j$ is the usual predicted state-variable of the Krein space Kalman filter, and where $\hat{x}_{j|j}$ is given by the Krein space projection of the state $\mathbf{x_j}$ onto $\left\{\{\mathbf{y}_l\}_{l=0}^{j}, \{\check{\mathbf{s}}_{l|l}\}_{l=0}^{j-1}\right\}$. Using the (predicted form) of the Krein space Kalman filter allows us to write

$$\left\{\begin{aligned}\hat{x}_{j+1} &= F_j \hat{x}_j + K_{p,j} \left[\begin{array}{c} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{array}\right] \\ \check{s}_{j|j} &= L_j \hat{x}_j + L_j P_j H_j^* (I + H_j P_j H_j^*)^{-1} (y_j - H_j \hat{x}_j)\end{aligned}\right. . \tag{3.3.14}$$

Replacing the second of the above equations into the first will allow us to obtain a more explicit recursion for $\hat{x}_j$, and hence a more explicit formula for $\check{s}_{j|j}$. However, an easier route is to utilize the filtered form of the Krein space Kalman filter corresponding to the state-space model (3.3.6) to recursively compute $\hat{x}_{j|j}$ (see Corollary 2.7.1),

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + P_{j+1} \left[\begin{array}{cc} H_{j+1}^* & L_{j+1}^* \end{array}\right] R_{e,j+1}^{-1} \left[\begin{array}{c} y_{j+1} - \hat{y}_{j+1|j} \\ \check{s}_{j+1|j+1} - \hat{s}_{j+1|j} \end{array}\right] \tag{3.3.15}$$

Using $\hat{y}_{j+1|j} = H_{j+1} F_j \hat{x}_{j|j}$ and the above mentioned triangular factorization of $R_{e,j+1}$ we have

$$\begin{aligned}\hat{x}_{j+1|j+1} &= F_j \hat{x}_{j|j} + P_{j+1} \left[\begin{array}{cc} H_{j+1}^* & L_{j+1}^* \end{array}\right] \left[\begin{array}{cc} I & -(I + H_{j+1} P_{j+1} H_{j+1}^*)^{-1} H_{j+1} P_{j+1} L_{j+1}^* \\ 0 & I \end{array}\right] \\ & \left[\begin{array}{cc} I + H_{j+1} P_{j+1} H_{j+1}^* & 0 \\ 0 & -\gamma_f^2 I + L_{j+1} (P_{j+1}^{-1} + H_{j+1}^* H_{j+1})^{-1} L_{j+1}^* \end{array}\right]^{-1} \left[\begin{array}{c} y_{j+1} - H_{j+1} F_j \hat{x}_{j|j} \\ \check{s}_{j+1|j+1} - \hat{s}_{j+1|j+1} \end{array}\right]\end{aligned}$$

Choosing $\check{s}_{j+1|j+1} = \hat{s}_{j+1|j+1}$ yields the desired recursion of Theorem 3.2.1

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + P_{j+1} H_{j+1}^* (I + H_{j+1} P_{j+1} H_{j+1}^*)^{-1} (y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}).$$

■

We mention in passing that using the canonical factorization of $R_{\underset{s}{y}}$ (via the Krein space Kalman filter), it is straightforward to show that the transfer operators $\mathcal{L}_{ij}$ of Theorem 1.4.1 have the following state-space representations

$$\left[\begin{array}{cc} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{array}\right] : \quad \left\{\begin{aligned} X_{i+1} &= A_i X_i + B_i U_i \\ Y_i &= C_i X_i + D_i U_i \end{aligned}\right. \tag{3.3.16}$$

where

$$\left\{\begin{array}{lcl} A_i & = & F_i \\ G_i & = & K_{p,i}\left[\begin{array}{cc} (I_p+H_iP_iH_i^*)^{1/2} & 0 \\ -L_iP_iH_i^*(I_p+H_iP_iH_i^*)^{-*/2} & (\gamma^2 I_q - L_iP_i(I+H_i^*H_iP_i)^{-1}L^*)^{1/2} \end{array}\right] \\ C_i & = & \left[\begin{array}{c} H_i \\ -L_i \end{array}\right] \\ D_i & = & \left[\begin{array}{cc} I_p & 0 \\ 0 & I_q \end{array}\right] \end{array}\right. . \tag{3.3.17}$$

Using the above state-space representation, it is straightforward to show (though we shall not provide the algebraic manipulations here) that the central filter, $\mathcal{K}_{cen} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1}$, has state-space model given by the recursions of Theorem 3.2.1. It is also possible to use the $\mathcal{L}_{ij}$ (obtained above) to parametrize all possible $H^\infty$ filters of level $\gamma$ along the lines of Theorem 1.4.1. Here, however, we shall take a different route to this parametrization which has the benefit of allowing for nonlinear estimators.

### 3.3.4 Parametrization of All $H^\infty$ A Posteriori Filters

The filter of Theorem 3.2.1 is one among *many possible* filters with level $\gamma$. All filters that guarantee $J_{f,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k) > 0$ are represented by (3.3.12) and (3.3.13). It is useful to formalize this representation in the following result.

**Lemma 3.3.5 (All $H^\infty$ A Posteriori Estimators)** *All $H^\infty$ a posteriori estimators that achieve a level $\gamma_f$ (assuming they exist) are given by any $\check{s}_{j|j} = \mathcal{F}_{f,j}(y_0, \ldots y_j)$ that satisfy*

$$\sum_{j=0}^{k}\left[\begin{array}{c} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{array}\right]^* R_{e,j}^{-1}\left[\begin{array}{c} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{array}\right] \geq 0, \quad 0 \leq k \leq i \tag{3.3.18}$$

*where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{p,j}\left[\begin{array}{c} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{array}\right], \qquad \hat{x}_0 \tag{3.3.19}$$

*with*

$$K_{p,j} = F_j P_j \left[ \begin{array}{cc} H_j^* & L_j^* \end{array} \right] R_{e,j}^{-1} \quad \textit{and} \quad R_{e,j} = \left[ \begin{array}{cc} I_p & 0 \\ 0 & -\gamma_f^2 I_q \end{array} \right] + \left[ \begin{array}{c} H_j \\ L_j \end{array} \right] P_j \left[ \begin{array}{cc} H_j^* & L_j^* \end{array} \right] \tag{3.3.20}$$

*and*

$$P_{j+1} = F_j P_j F_j^* + G_j Q_j G_j^* - K_{p,j} R_{e,j} K_{p,j}^*, \quad P_0 = \Pi_0. \tag{3.3.21}$$

We may also use these expressions to obtain a more explicit characterization of all possible estimators. Similar results appear in [KN91, ST92, GL95].

**Theorem 3.3.1 (All $H^\infty$ A Posteriori Estimators)** *All $H^\infty$ a posteriori estimators that achieve a level $\gamma_f$ (assuming they exist) are given by*

$$\begin{aligned} \check{s}_{j|j} \quad = \quad & L_j \hat{x}_{j|j} + [\gamma_f^2 I - L_j (P_j^{-1} + H_j^* H_j)^{-1} L_j^*]^{\frac{1}{2}} \\ & \mathcal{S}_j \left( (I + H_j P_j H_j^*)^{\frac{1}{2}} (y_j - H_j \hat{x}_{j|j}), \ldots, (I + H_0 P_0 H_0^*)^{\frac{1}{2}} (y_0 - H_0 \hat{x}_{0|0}) \right) \end{aligned} \tag{3.3.22}$$

*where $\hat{x}_{j|j}$ satisfies the recursion*

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{s,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}) - K_{c,j}(\check{s}_{j|j} - L_j \hat{x}_{j|j}) \tag{3.3.23}$$

*with $K_{s,j+1}$ the same as in Theorem 3.2.1,*

$$K_{c,j} = (I + P_{j+1} H_{j+1} H_{j+1}^*)^{-1} F_j (P_j^{-1} + H_j H_j^* - \gamma_f^{-2} L_j L_j^*)^{-1} L_j^*, \tag{3.3.24}$$

*and*

$$\mathcal{S}(a_j, \ldots, a_0) = \left[ \begin{array}{c} \mathcal{S}_0(a_0) \\ \mathcal{S}_1(a_1, a_0) \\ \vdots \\ \mathcal{S}_j(a_j, \ldots, a_0) \end{array} \right]$$

*is any (possibly nonlinear) contractive causal mapping, i.e.,*

$$\sum_{j=0}^{k} |\mathcal{S}_j(a_j, \ldots, a_0)|^2 < \sum_{j=0}^{k} |a_j|^2 \quad \textit{for all} \quad k = 0, 1, \ldots, i.$$

**Remark:** Note that when the contraction of Theorem 3.3.1 is chosen as $\mathcal{S}=0$, then we have $\check{s}_{j|j}=L_j\hat{x}_{j|j}$, and (3.3.23) reduces to the recursion of Theorem 3.2.1.

**Proof of Theorem 3.3.1:** Expression (3.3.13) may be rewritten as:

$$\begin{array}{c}\sum_{j=0}^k (y_j-H_j\hat{x}_j)^*(I+H_jP_jH_j^*)^{-1}(y_j-H_j\hat{x}_j) \\ +\sum_{j=0}^k(\check{s}_{j|j}-L\hat{x}_{j|j})^*\left(-\gamma_f^2 I+L_j(P_j^{-1}+H_jH_j^*)^{-1}L_j^*\right)^{-1}(\check{s}_{j|j}-L_j\hat{x}_{j|j})>0\end{array} \tag{3.3.25}$$

where $\hat{x}_j$ and $\hat{x}_{j|j}$ denote the Krein space projections of $\mathbf{x_j}$ onto $\left\{\{\mathbf{y_l}\}_{l=0}^{j-1},\{\check{\mathbf{s}}_{\mathbf{l|l}}\}_{l=0}^{j-1}\right\}$ and $\left\{\{\mathbf{y_l}\}_{l=0}^{j},\{\check{\mathbf{s}}_{\mathbf{l|l}}\}_{l=0}^{j-1}\right\}$, respectively. Therefore $\hat{x}_j$ and $\hat{x}_{j|j}$ are related through one additional projection onto $y_j$, and we may write

$$\hat{x}_{j|j}=\hat{x}_j+P_jH_j^*(I+H_jP_jH_j^*)^{-1}(y_j-H_j\hat{x}_j). \tag{3.3.26}$$

Therefore

$$\begin{aligned} y_j-H_j\hat{x}_{j|j} &= \left(I-H_jP_jH_j^*(I+H_jP_jH_j^*)^{-1}\right)(y_j-H_j\hat{x}_j) \\ &= (I+H_jP_jH_j^*)^{-1}(y_j-H_j\hat{x}_j)\end{aligned}$$

so that

$$(y_j-H_j\hat{x}_j)=(I+H_jP_jH_j^*)(y_j-H_j\hat{x}_{j|j}).$$

Now (3.3.25) can be written as:

$$\begin{array}{c}\sum_{j=0}^k (y_j-H_j\hat{x}_{j|j})^*(I+H_jP_jH_j^*)(y_j-H_j\hat{x}_{j|j}) \\ +\sum_{j=0}^k(\check{s}_{j|j}-L_j\hat{x}_{j|j})^*\left(-\gamma_f^2 I+L_j(P_j^{-1}+H_jH_j^*)^{-1}L_j^*\right)^{-1}(\check{s}_{j|j}-L_j\hat{x}_{j|j})>0\end{array}$$

or equivalently,

$$\|\,(\gamma_f^2I-L_j(P_j^{-1}+H_jH_j^*)^{-1}L_j^*)^{-\frac{1}{2}}(\check{s}_{j|j}-\hat{s}_{j|j})\,\|_2^2 \;<\; \|\,(I+H_jP_jH_j^*)^{\frac{1}{2}}(y_j-H_j\hat{x}_{j|j})\,\|_2^2\,.$$

Since $\check{s}_{j|j}$ is a causal function of the observations $y_j$, then $(\check{s}_{j|j}-\hat{s}_{j|j})$ will also be a causal function of $(y_j-H_j\hat{x}_{j|j})$. Therefore using the above expression, we can write

$$\begin{bmatrix} (\gamma_f^2I-L_0(P_0^{-1}+H_0H_0^*)^{-1}L_0^*)^{-\frac{1}{2}}(\check{s}_{0|0}-\hat{s}_{0|0}) \\ \vdots \\ (\gamma_f^2I-L_i(P_i^{-1}+H_iH_i^*)^{-1}L_i^*)^{-\frac{1}{2}}(\check{s}_{i|i}-\hat{s}_{i|i})\end{bmatrix} = \mathcal{S}\begin{bmatrix}(I+H_0P_0H_0^*)^{\frac{1}{2}}(y_0-H_0\hat{x}_{0|0}) \\ \vdots \\ (I+H_iP_iH_i^*)^{\frac{1}{2}}(y_i-H_i\hat{x}_{i|i})\end{bmatrix}$$

for some causal contractive mapping $\mathcal{S}$. Eq. (3.3.22) now readily follows.

Finally, we must show (3.3.23). To this end, recall from the proof of Theorem 3.2.1 (see (3.3.15)) that the recursion for $\hat{x}_j$ is given by

$$\hat{x}_{j+1|j+1} = F_j\hat{x}_{j|j} + P_{j+1}\begin{bmatrix} H_{j+1}^* & L_{j+1}^* \end{bmatrix} R_{e,j+1}^{-1}\begin{bmatrix} y_{j+1} - \hat{y}_{j+1|j} \\ \check{s}_{j+1|j+1} - \hat{s}_{j+1|j} \end{bmatrix}.$$

Using (3.3.26) to replace $\hat{x}_j$ by $\hat{x}_{j|j}$ yields, after some algebra, the desired recursion (3.3.23).

■

Note that although the filter obtained in Theorem 3.2.1 is linear, the full parametrization of all $H^\infty$ filters with level $\gamma_f$ is given by a *nonlinear* causal contractive mapping $\mathcal{S}$.[5] The filter of Theorem 3.2.1 is known as the *central* filter and as we have seen, corresponds to $\mathcal{S} = 0$. This central filter has a number of other interesting properties. It corresponds, as we shall see in the next chapter, to the risk-sensitive optimal filter [Whi90], and, as we have seen in Chapter 1, to the *maximum entropy* filter [GM89]. Moreover, in the game theoretic formulation of the $H^\infty$ problem, the central filter corresponds to the solution of the game [BB95]. In our context, the central filter is recognized as the Krein space Kalman filter corresponding to the state-space model (3.3.6).

### 3.3.5 Derivation of the A Priori $H^\infty$ Filter

We shall now turn to the $H^\infty$ a priori filter of Problem 3.2.2 and our main goal will be to prove the results of Theorem 3.2.2. Our approach will follow the one used for the a posteriori case, namely we will relate an indefinite quadratic form to the a priori problem, construct its corresponding Krein space state-space model, and use

[5]We should remark that, from the outset, we did not restrict ourselves to linear estimators. Therefore Theorem 3.2.1 states that whenever the $H^\infty$ filtering problem of level $\gamma$ is feasible there exists a linear filter that achieves it. This has led various authors to claim that nonlinear estimators and controllers have no advantage over linear ones when estimating or controlling linear plants (see *e.g.*, [GL95] page 216). [Incidentally, LQG-optimal controllers are also linear.] However, including nonlinear estimators in the above parametrization may prove to be of use when attempting to optimize some additional cost criterion over the set of all possible $H^\infty$ filters achieving a certain level $\gamma$.

the Krein space Kalman filter to obtain the solution. Since our derivations parallel the ones given earlier, we shall omit several details.[6]

### The Suboptimal $H^\infty$ A Priori Problem and Quadratic Forms

Referring to Problem 3.2.2 we first note that $\|T_i(\mathcal{F}_p)\|_\infty < \gamma_p$, implies that for all nonzero $\{x_0, \{u_j\}_{j=0}^{i-1}, \{v_j\}_{j=0}^{i-1}\}$

$$\frac{\sum_{j=0}^{i} |\check{s}_j - L_j x_j|^2}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{i-1} |u_j|^2 + \sum_{j=0}^{i-1} |y_j - H_j x_j|^2} < \gamma_p^2, \tag{3.3.28}$$

where, without loss of generality, we have assumed $\check{x}_0 = 0$. Moreover, (3.3.28) implies that for *all* $k \leq i$, we must have

$$\frac{\sum_{j=0}^{k} |\check{s}_j - L_j x_j|^2}{x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{k-1} |u_j|^2 + \sum_{j=0}^{k-1} |y_j - H_j x_j|^2} < \gamma_p^2. \tag{3.3.29}$$

As before, we may easily show the following result.

**Lemma 3.3.6 (Indefinite Quadratic Form)** *Given $\gamma_p > 0$, then $\|T_i(\mathcal{F}_p)\|_\infty < \gamma_p$ if, and only if, there exists $\check{s}_k = \mathcal{F}_p(y_0, \ldots, y_{k-1})$ (for all $0 \leq k \leq i$) such that for*

[6] Returning to the general estimation problem of Sec. 1.2, and using the canonical factorization approach of Sec. 1.4, it is possible to show that a *strictly* causal estimator of level $\gamma$ exists if, and only if, there exists a canonical factorization of the form

$$\begin{bmatrix} I + \mathcal{H}\mathcal{H}^* & -\mathcal{H}\mathcal{L}^* \\ -\mathcal{L}\mathcal{H}^* & -\gamma^2 I + \mathcal{L}\mathcal{L}^* \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}^* & \mathcal{L}_{21}^* \\ \mathcal{L}_{12}^* & \mathcal{L}_{22}^* \end{bmatrix},$$

with $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix}$ and $\mathcal{L}_{11}$ causal and causally invertible, and $\mathcal{L}_{21}$ strictly causal. [Recall that in Theorem 1.4.1 we required $\mathcal{L}_{21}$ to be strictly causal.] If this is the case, then all possible strictly causal $H^\infty$ estimators of level $\gamma$ are given by

$$\mathcal{K} = (\mathcal{L}_{22}\mathcal{Q} - \mathcal{L}_{21})(\mathcal{L}_{11} - \mathcal{L}_{12}\mathcal{Q})^{-1}, \tag{3.3.27}$$

where $\mathcal{Q}$ is any strictly causal and strictly contractive operator. The central solution, $\mathcal{K}_{cen} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1}$, results from the choice, $\mathcal{Q} = 0$.

As before, when we have state-space structure, the canonical factorization can be performed via an appropriate Krein space Kalman filter. We shall not present this approach to the $H^\infty$ a priori filtering problem here, but shall instead present the (closely related) approach that relies on indefinite quadratic forms.

*all complex vectors $x_0$, for all causal sequences $\{u_j\}_{j=0}^{i-1}$ and for all nonzero causal sequences $\{y_j\}_{j=0}^{i-1}$ the scalar quadratic form*

$$\begin{array}{c} J_{p,k}(x_0, u_0, \ldots, u_{k-1}, y_0, \ldots, y_{k-1}) = x_0^* \Pi_0^{-1} x_0 + \sum_{j=0}^{k-1} u_j^* u_j + \\ \sum_{j=0}^{k-1} (y_j - H_j x_j)^* (y_j - H_j x_j) - \gamma_p^{-2} \sum_{j=0}^{k} (\check{s}_j - L_j x_j)^* (\check{s}_j - L_j x_j) \end{array} \tag{3.3.30}$$

*satisfies*

$$J_{p,k}(x_0, u_0, \ldots, u_{k-1}, y_0, \ldots, y_{k-1}) > 0 \quad \textit{for all} \quad 0 \le k \le i. \tag{3.3.31}$$

We can also readily obtain the analog of Lemma 3.3.2.

**Lemma 3.3.7 (Positivity Condition)** *The scalar quadratic forms*

$$J_{p,k}(x_0, u_0, \ldots, u_{k-1}, y_0, \ldots, y_{k-1}),$$

*satisfy the conditions (3.3.31), if, and only if, for all $0 \le k \le i$,*

*(i)* $J_{p,k}(x_0, u_0, \ldots, u_{k-1}, y_0, \ldots, y_{k-1})$ *has a minimum with respect to* $\{x_0, u_0, u_1, \ldots u_{k-1}\}$.

*(ii) The $\{\check{s}_k\}_{k=0}^{i}$ can be chosen such that the value of $J_{p,k}(x_0, u_0, \ldots, u_{k-1}, y_0, \ldots, y_{k-1})$ at this minimum is positive, viz.,*

$$\min_{\{x_0, u_0, \ldots, u_{k-1}\}} J_{p,k}(x_0, u_0, \ldots, u_{k-1}, y_0, \ldots, y_{k-1}, \check{s}_0, \ldots, \check{s}_k) > 0.$$

### A Krein Space State-Space Model

Due to the fact that the summations in $J_{p,k}$ go up to both $k$ and $k-1$ (see (3.3.30)), it is slightly more difficult to come up with a Krein state-space model whose corresponding quadratic form is $J_{p,k}$. However, with some effort, we see that the appropriate Krein state-space model is

$$\begin{array}{l} \left\{ \begin{array}{rcl} \xi_{2j+1} & = & \xi_{2j}, \qquad \xi_0 = \mathbf{x}_0 \\ \hat{\mathbf{s}}_j & = & L_j \xi_{2j} + \mathbf{v}_{2j} \end{array} \right. \\ \left\{ \begin{array}{rcl} \xi_{2j+2} & = & F_j \xi_{2j+1} + G_j \bar{\mathbf{u}}_{2j+1} = \mathbf{x}_{2i+2} \\ \mathbf{y}_j & = & H_j \xi_{2j+1} + \mathbf{v}_{2j+1} \end{array} \right. \end{array} \qquad j \le 0 \tag{3.3.32}$$

where $\Pi_0 > 0$, $Q_{2j} = 0$, $Q_{2j+1} = I$, $R_{2j} = -\gamma_p^2 I$, $R_{2j+1} = I$, and $S_j = 0$. To see why, let us construct the deterministic quadratic form corresponding to (3.3.32). Thus:

$$\begin{aligned} J_{\xi,2k} &= \xi_0^* \Pi_0^{-1} \xi + \sum_{j=0}^{2k} \bar{u}_j^* Q_j^{-1} \bar{u}_j + \sum_{j=0}^{2k} v_j^* R_j^{-1} v_j \\ &= \xi_0^* \Pi_0^{-1} \xi + \sum_{j=0}^{k-1} \bar{u}_{2j+1}^* \bar{u}_{2j+1} + \sum_{j=0}^{k-1} v_{2j+1}^* R_{2j+1}^{-1} v_{2j+1} + \sum_{j=0}^{k} v_{2j}^* R_{2j}^{-1} v_{2j} \\ &= \xi_0^* \Pi_0^{-1} \xi + \sum_{j=0}^{k-1} \bar{u}_{2j+1}^* \bar{u}_{2j+1} + \sum_{j=0}^{k-1} |y_j - H_j \xi_{2j+1}|^2 - \gamma_p^{-2} \sum_{j=0}^{k} |\check{s}_j - L_j \xi_{2j}|^2. \end{aligned}$$

From (3.3.32) we see that $\xi_{2j} = \xi_{2j+1} = x_j$. Using this fact, and defining $\bar{u}_{2j+1} = u_j$, we readily see that $J_{\xi,2k} = J_{p,k}$.

Note also that the Riccati recursion for the model (3.3.32) is

$$\left\{ \begin{array}{lcl} \Sigma_{2j+1} &=& \Sigma_{2j} - \Sigma_{2j} L_j^* (-\gamma_p^2 I + L_j \Sigma_{2j} L_j^*)^{-1} L_j \Sigma_{2j} \\ \Sigma_{2j+2} &=& F_j \Sigma_{2j+1} F_j^* + G_j G_j^* - F_j \Sigma_{2j+1} H_j^* (I + H_j \Sigma_{2j+1} H_j^*)^{-1} H_j \Sigma_{2j+1} F_j^* \end{array} \right. \quad \Sigma_0 = \Pi_0. \tag{3.3.33}$$

## Existence Conditions

Using Lemma 2.7.3, the condition for a minimum is that $R_{e,j}$ and $R_j$ should have the same inertia for all $j = 0, 1, \ldots, 2i$ (since each two time steps in (3.3.32) correspond to one time step in $J_{p,k}$). Thus the condition for a minimum is

$$-\gamma_p^2 I + L_j \Sigma_{2j} L_j^* < 0 \quad \text{and} \quad I + H_j \Sigma_{2j+1} H_j^* > 0. \tag{3.3.34}$$

The second of the above conditions is obvious since when we have a minimum $\Sigma_j$ is positive definite. If the $\begin{bmatrix} F_j & G_j \end{bmatrix}$ have full rank, then using Lemma 2.7.4, the condition for a minimum is

$$\Sigma_{2j}^{-1} - \gamma_p^{-2} L_j^* L_j > 0 \quad \text{and} \quad \Sigma_{2j+1}^{-1} + H_j^* H_j > 0 \tag{3.3.35}$$

where once more the second of the above conditions is redundant.

In order to connect with the results of Theorem 3.2.2, we may note that by defining $P_j = \Sigma_{2j}$ and combining the coupled pair of Riccati recursions in (3.3.33), we can write the following Riccati recursion for $P_j$

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - F_j P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j F_j^*, \quad P_0 = \Pi_0 \tag{3.3.36}$$

with

$$R_{e,j} = \begin{bmatrix} -\gamma_p^2 I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix}. \tag{3.3.37}$$

But this is the same Riccati as (3.2.9) in Theorem 3.2.2. Thus the condition for a minimum (3.3.35) becomes

$$P_j^{-1} - \gamma_p^{-2} L_j^* L_j > 0, \tag{3.3.38}$$

which is the condition (3.2.13) of Theorem 3.2.2.

To express the condition (3.3.34) in a form that is more similar to that of Lemma 3.3.3, we introduce the Krein space state-space model

$$\begin{cases} \mathbf{x}_{j+1} & = & F_j \mathbf{x}_j + G_j \mathbf{u}_j \\ \begin{bmatrix} \check{\mathbf{s}}_{j|j} \\ \mathbf{y}_j \end{bmatrix} & = & \begin{bmatrix} L_j \\ H_j \end{bmatrix} \mathbf{x}_j + \mathbf{v}_j \end{cases} \tag{3.3.39}$$

where $\Pi_0 > 0$, $Q_j = I$, $S_j = 0$ and

$$R_j = \begin{bmatrix} -\gamma_p^2 I & 0 \\ 0 & I \end{bmatrix}.$$

Note that the only difference between the state-space models (3.3.6) and (3.3.39) is that the order of the output equations has been reversed.

We can now use the state-space model (3.3.39) to express the condition (3.3.34) in the form of the following Lemma.

**Lemma 3.3.8 (Alternative Test for Existence)** *The condition (3.2.13) can be replaced by the condition that all leading submatrices of*

$$R_j = \begin{bmatrix} -\gamma_p^2 I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \quad \textit{and} \quad R_{e,j} = \begin{bmatrix} -\gamma_p^2 I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} + \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix}$$

*have the same inertia for all* $0 \leq j \leq i$. *In other words,*

$$-\gamma_p^2 I + L_j P_j L_j^* < 0 \quad \text{and} \quad I + H_j \tilde{P}_j H_j^* > 0$$

*where* $\tilde{P}_j^{-1} = P_j^{-1} - \gamma_p^{-2} L_j^* L_j$. *We no longer require that* $\begin{bmatrix} F_j & G_j \end{bmatrix}$ *have full rank, and the size of the matrices involved is generally smaller than in (3.2.8).*

Note that compared to Lemma 3.3.3, the condition in Lemma 3.3.8 is more stringent since it requires that all leading submatrices of $R_j$ and $R_{e,j}$ have the same inertia. This distinction is especially important in square-root implementations of the $H^\infty$ filters (see [HSK94c] and Chapter 5).

### Construction of the $H^\infty$ a priori filters

To complete the proof of Theorem 3.2.2 we still need to show that if a minimum over $\{x_0, u_0, \ldots, u_{k-1}\}$ exists for all $0 \leq k \leq i$, then we can find the estimates $\{\check{s}_k\}_{k=0}^{i}$ such that the value of $J_{p,k}(x_0, u_0, \ldots, u_{k-1}, y_0, \ldots, y_{k-1})$ at its minimum is positive.

According to Theorem 2.7.4, the minimum value of $J_{p,k}(x_0, u_0, \ldots, u_{k-1}, y_0, \ldots, y_{k-1})$ is

$$\sum_{j=0}^{k-1} \begin{bmatrix} e_{s,j}^* & e_{y,j}^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} e_{s,j} \\ e_{y,j} \end{bmatrix} + e_{s,k}^*(-\gamma_p^2 I + L_k P_k L_k^*)^{-1} e_{s,k} =$$

$$\sum_{j=0}^{k-1} \begin{bmatrix} \check{s}_j - \hat{s}_{j|j-1} \\ y_j - \hat{y}_{j|j-1} \end{bmatrix}^* \begin{bmatrix} -\gamma_p^2 I + L_j P_j L_j^* & L_j P_j H_j^* \\ H_j P_j L_j^* & I + H_j P_j H_j^* \end{bmatrix}^{-1} \begin{bmatrix} \check{s}_j - \hat{s}_{j|j-1} \\ y_j - \hat{y}_{j|j-1} \end{bmatrix}$$
$$+ (\check{s}_k - \hat{s}_{k|k-1})^*(-\gamma_p^2 I + L_k P_k L_k^*)^{-1}(\check{s}_k - \hat{s}_{k|k-1}) > 0$$

where $\hat{s}_{j|j-1}$ and $\hat{y}_{j|j-1}$ are obtained from the Krein space projections of $\check{\mathbf{s}}_j$ and $\mathbf{y}_j$ onto $\mathcal{L}\left\{\{\check{\mathbf{s}}_l\}_{l=0}^{j-1}, \{\mathbf{y}_l\}_{l=0}^{j-1}\right\}$, respectively. Thus $\hat{s}_{j|j-1}$ is a linear function of $\{y_l\}_{l=0}^{j-1}$. Using the block triangular factorization of the $R_{e,j}$ we may rewrite the above as

$$\begin{aligned} &\sum_{j=0}^{k} (\check{s}_j - \hat{s}_{j|j-1})^*(-\gamma_p^2 I + L_j P_j L_j^*)^{-1}(\check{s}_j - \hat{s}_{j|j-1}) + \\ &\sum_{j=0}^{k-1} (y_j - \bar{y}_{j|j-1})^*(I + H_j \tilde{P}_j H_j^*)^{-1}(y_j - \bar{y}_{j|j-1}) > 0 \end{aligned} \tag{3.3.40}$$

where

$$\begin{bmatrix} \check{s}_j - \hat{s}_{j|j-1} \\ y_j - \bar{y}_{j|j-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ -H_j P_j L_j^*(-\gamma_p^2 I + L_j P_j L_j^*)^{-1} & I \end{bmatrix} \begin{bmatrix} \check{s}_j - \hat{s}_{j|j-1} \\ y_j - \hat{y}_{j|j-1} \end{bmatrix}. \tag{3.3.41}$$

Or, in other words,

$$\bar{y}_{j|j-1} = \hat{y}_{j|j-1} + H_j P_j L_j^* (-\gamma_p^2 I + L_j P_j L_j^*)^{-1} (\check{s}_j - \hat{s}_{j|j-1}).$$

Note that $\bar{y}_{j|j-1}$ is given by the Krein space projection of $\mathbf{y}_j$ onto $\left\{ \{\check{\mathbf{s}}_l\}_{l=0}^{j}, \{\mathbf{y}_l\}_{l=0}^{j-1} \right\}$. Recall from Lemma 3.3.8 that

$$-\gamma_p^2 I + L_j P_j L_j^* < 0 \quad , \quad I + H_j \tilde{P}_j H_j^* > 0.$$

Any choice of $\check{s}_{j|j-1}$ that renders (3.3.40) positive will do, and the simplest choice is $\check{s}_{j|j-1} = \hat{s}_{j|j-1} = L_j \hat{x}_{j|j-1}$, where $\hat{x}_{j|j-1}$ is given by the Krein space projection of $\mathbf{x}_j$ onto $\left\{ \{\check{\mathbf{s}}_l\}_{l=0}^{j-1}, \{\mathbf{y}_j\}_{l=0}^{j-1} \right\}$. We may now utilize the Krein space Kalman filter corresponding to the state-space model (3.3.39) to recursively compute $\hat{x}_{j|j-1}$, *viz.*,

$$\hat{x}_{j+1|j} = F_j \hat{x}_{j|j-1} + F_j P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} \check{s}_j - L_j \hat{x}_{j|j-1} \\ y_j - H_j \hat{x}_{j|j-1} \end{bmatrix} \tag{3.3.42}$$

Setting $\check{s}_j - L_j \hat{x}_{j|j-1} = 0$ and simplifying we get the desired recursion for $\hat{x}_{j+1|j}$.

∎

We should also mention here that in the a priori case the $\mathcal{L}_{ij}$ of the canonical spectral factorization are given by the following state-space representations,

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{bmatrix} : \left\{ \begin{array}{rcl} X_{i+1} & = & A_i X_i + B_i U_i \\ Y_i & = & C_i X_i + D_i U_i \end{array} \right. \tag{3.3.43}$$

where

$$\left\{ \begin{array}{rcl} A_i & = & F_i \\ G_i & = & K_{p,i} \begin{bmatrix} (I_p + H_i \tilde{P}_i H_i^*)^{1/2} & -H_i P_i L_i^* (-\gamma^2 I_q + L_i P_i L_i^*)^{-*/2} \\ 0 & (\gamma^2 I_q - L_i P_i L_i^*)^{1/2} \end{bmatrix} \\ C_i & = & \begin{bmatrix} H_i \\ -L_i \end{bmatrix} \\ D_i & = & \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} \end{array} \right. . \tag{3.3.44}$$

Note now that the transfer operator, $\mathcal{L}_{21}$ is strictly causal. Using the above state-space representation, it is straightforward to show that the central filter, $\mathcal{K}_{cen} = -\mathcal{L}_{21}\mathcal{L}_{11}^{-1}$, has state-space model given by the recursions of Theorem 3.2.2. It is also possible to use the $\mathcal{L}_{ij}$ (obtained above) to parametrize all possible $H^\infty$ filters of level $\gamma$, however, we shall take a route similar to the one taken in Sec. 3.3.4.

### 3.3.6 All $H^\infty$ A Priori Filters

The positivity condition (3.3.40) gives a full parametrization of all $H^\infty$ a priori estimators. We thus have the following result.

**Lemma 3.3.9 (All $H^\infty$ A Priori Estimators)** *All $H^\infty$ a priori estimators that achieve a level $\gamma_p$ (assuming they exist) are given by any $\check{s}_j = \mathcal{F}_{p,j}(y_0, \ldots y_{j-1})$ that satisfy*

$$\begin{array}{c}\sum_{j=0}^{k-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_j - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_j - L_j\hat{x}_j \end{bmatrix} \qquad 0 \leq k \leq i \\ -(\check{s}_k - L_k\hat{x}_k)^*(\gamma_p^2 I_q - L_k P_k L_k^*)^{-1}(\check{s}_k - L_k\hat{x}_k) \geq 0, \end{array} \tag{3.3.45}$$

*where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_j - L_j\hat{x}_j \end{bmatrix}, \quad \hat{x}_0 \tag{3.3.46}$$

*with*

$$K_{p,j} = F_j P_j \begin{bmatrix} H_j^* L_j^* \end{bmatrix} R_{e,j}^{-1} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_p^2 I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \tag{3.3.47}$$

*and*

$$P_{j+1} = F_j P_j F_j^* + G_j Q_j G_j^* - K_{p,j} R_{e,j} K_{p,j}^*, \quad P_0 = \Pi_0. \tag{3.3.48}$$

We can also give a more explicit parametrization as follows.

**Theorem 3.3.2 (All $H^\infty$ A Priori Estimators)** *All $H^\infty$ a priori estimators that achieve a level $\gamma_p$ (assuming they exist) are given by*

$$\begin{aligned} \check{s}_j &= L_j\hat{x}_j + (\gamma_p^2 I - L_j P_j L_j^*)^{\frac{1}{2}} \\ &\quad \mathcal{S}_j\left((I + H_{j-1}\tilde{P}_{j-1}H_{j-1}^*)^{-\frac{1}{2}}(y_{j-1} - H_{j-1}\bar{x}_{j-1}), \ldots, (I + H_0\tilde{P}_0 H_0^*)^{-\frac{1}{2}}(y_0 - H_0\bar{x}_0)\right) \end{aligned} \tag{3.3.49}$$

*where*

$$\bar{x}_k = \hat{x}_k + P_k L_k^*(-\gamma_p^2 I + L_k L_k^*)^{-1}(\check{s}_k - L_k \hat{x}_k), \tag{3.3.50}$$

$\hat{x}_j$ *satisfies the recursion*

$$\hat{x}_{j+1|j} = F_j \hat{x}_{j|j-1} + F_j P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} \check{s}_j - L_j \hat{x}_{j|j-1} \\ y_j - H_j \hat{x}_{j|j-1} \end{bmatrix}, \tag{3.3.51}$$

*with* $P_j$, $\tilde{P}_j$ *and* $R_{e,j}$ *given by Theorem 3.2.2, and* $\mathcal{S}$ *is any (possibly nonlinear) contractive causal mapping.*

**Proof:** Referring to (3.3.41), we see that the $\bar{y}_{j|j-1} = H_j \bar{x}_j$ differ from $\hat{y}_{j|j-1} = H_j \hat{x}_j$, via the additional projection onto $\check{s}_{j|j-1}$. Thus we can write

$$\bar{x}_{j|j-1} = \hat{x}_{j|j-1} + P_j L_j^*(-\gamma_p^2 I + L_j P_j L_j^*)^{-1}(\check{s}_{j|j-1} - \hat{s}_{j|j-1}),$$

which proves (3.3.50). Moreover from the proof of Theorem 3.2.2 (see (3.3.42)), the recursion for $\hat{x}_j$ is given by (3.3.51). Condition (3.3.40) can now be rewritten as

$$\sum_{j=0}^{k}(\check{s}_{j|j-1} - \hat{s}_{j|j-1})^*(-\gamma_p^2 I + L_j P_j L_j^*)^{-1}(\check{s}_{j|j-1} - \hat{s}_{j|j-1}) + \\ \sum_{j=0}^{k-1}(y_j - \bar{y}_{j|j-1})^*(I + H_j \tilde{P}_j H_j^*)^{-1}(y_j - \bar{y}_{j|j-1}) > 0 \tag{3.3.52}$$

and an argument similar to the one given in the proof of Theorem 3.3.1 will yield the desired result.

■

## 3.4 The $H^\infty$ Smoother

If instead of $e_{f,k}$ and $e_{p,k}$, which correspond to the a posteriori and a priori filters, respectively, we consider the *smoothed* error

$$e_{s,k} = \check{s}_{k|i} - L_k x_k, \qquad k \le i$$

where $\check{s}_{k|i} = \mathcal{F}_s(y_0, y_1, \ldots, y_i)$ is the estimate of $s_k$ given all observations $\{y_j\}$ from time 0 until time $i$, we are led to the so-called $H^\infty$ smoothers. Such estimators guarantee that the maximum energy gain from the disturbances $\left\{\Pi_0^{-1/2}(x_0 - \check{x}_0), \{u_j, v_j\}_{j=0}^i\right\}$ to the smoothing errors $\{e_{s,j}\}_{j=0}^i$ is bounded by $\gamma_s$, *i.e.*,

$$\sup_{x_0, u\in h^2, v\in h^2} \frac{\sum_{j=0}^i e_{s,j}^* e_{s,j}}{(x_0 - \check{x}_0)^* {\Pi_0}^{-1}(x_0 - \check{x}_0) + \sum_{j=0}^i u_j^* u_j + \sum_{j=0}^i v_j^* v_j} < \gamma_s^2. \tag{3.4.1}$$

Using an argument similar to the ones given before, we are led to the following quadratic form

$$\begin{gathered} J_{s,i}(x_0, u_0, \ldots, u_i, y_0, \ldots, y_i) = x_0 \Pi_0^{-1} x_0^* + \sum_{k=0}^i u_k^* u_k + \\ \sum_{k=0}^i (y_k - H_k x_k)^*(y_k - H_k x_k) - \gamma^{-2} \sum_{k=0}^i (\check{s}_{k|i} - L_k x_k)^*(\check{s}_{k|i} - L_k x_k). \end{gathered} \tag{3.4.2}$$

Note that the only difference between $J_{s,i}$ and $J_{f,i}$ is that $\check{s}_{k|k}$ has been replaced by $\check{s}_{k|i}$ (*i.e.,* filtered estimates have been replaced by smoothed estimates). Once more it can be shown that an $H^\infty$ smoother of level $\gamma_s$ will exist if, and only if, there exists some $\check{s}_{k|i}$ such that $J_{s,i} \geq 0$. The rather interesting result shown below, and which has already been pointed out in the literature (see *e.g.,* [KN91, ST92, SFB92]), is that one $H^\infty$ smoother is given by the conventional $H^2$ smoother (which does not even depend on the value of $\gamma_s$).

**Theorem 3.4.1** (**$H^\infty$ Smoother**) *For a given $\gamma_s > 0$, an $H^\infty$ smoother that achieves level $\gamma_s$ exists, if, and only if, the block diagonal matrix*

$$R_e = R_{e,0} \oplus R_{e,1} \oplus \ldots \oplus R_{e,i},$$

*where*

$$R_{e,j} = \begin{bmatrix} I & 0 \\ 0 & -\gamma_s^2 \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix}$$

*and $P_j$ is the same as in Theorem 3.2.1, has $(i+1)p$ positive eigenvalues and $(i+1)q$ negative eigenvalues. In other words, if, and only if,*

$$In\,[R_e] = \begin{bmatrix} (i+1)p & 0 & (i+1)q \end{bmatrix}.$$

*If this is the case, one possible $H_\infty$ smoother is given by the $H^2$ smoother.*

**Proof:** The condition for $J_{s,i}(x_0, u_0, \ldots, u_i, y_0, \ldots, y_i)$ to have a minimum is slightly different than the earlier cases, since we do *not* require that $J_{s,k}(x_0, u_0, \ldots, u_k, y_0, \ldots, y_k)$ have a minimum over the disturbances for all past values $k < i$. Thus using Lemma 2.6.4, the condition for a minimum over $\{x_0, u_0, \ldots, u_i\}$ is that the matrices

$$R_e \quad \text{and} \quad R = R_0 \oplus R_1 \oplus \ldots \oplus R_i$$

have the same inertia, where $R_j = I_p \oplus (-\gamma_s^2 I_q)$. But this is precisely the inertia condition given in the statement of the Theorem.

The value of $J_{s,i}$ at its minimum is (see Chapter 2)

$$\begin{bmatrix} y^* & \check{s}_{|i}^* \end{bmatrix} \begin{bmatrix} R_y & R_{y\check{s}} \\ R_{\check{s}y} & R_{\check{s}} \end{bmatrix}^{-1} \begin{bmatrix} y \\ \check{s}_{|i} \end{bmatrix}$$

where we have defined

$$\begin{bmatrix} R_y & R_{y\check{s}} \\ R_{\check{s}y} & R_{\check{s}} \end{bmatrix} = \langle \begin{bmatrix} \mathbf{y} \\ \check{\mathbf{s}}_{|i} \end{bmatrix}, \begin{bmatrix} \mathbf{y} \\ \check{\mathbf{s}}_{|i} \end{bmatrix} \rangle \tag{3.4.3}$$

with

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_0 \\ \vdots \\ \mathbf{y}_i \end{bmatrix}, \quad \check{\mathbf{s}}_{|i} = \begin{bmatrix} \check{\mathbf{s}}_{0|i} \\ \vdots \\ \check{\mathbf{s}}_{i|i} \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ \vdots \\ y_i \end{bmatrix}, \quad \check{s}_{|i} = \begin{bmatrix} \check{s}_{0|i} \\ \vdots \\ \check{s}_{i|i} \end{bmatrix},$$

and where the $\{\mathbf{y}_j\}$ and $\{\check{\mathbf{s}}_{j|i}\}$ satisfy the Krein state-space model (3.3.6). In this case all the entries in $\check{s}_{|i}$ are unknown and there is no causal dependence between the $\{\check{s}_{j|k}\}$ and the $\{y_j\}$. Using a block triangular factorization, or a completion of squares argument, the value at the minimum can be rewritten as

$$y^* R_y^{-1} y + (\check{s}_{|i} - R_{\check{s}y} R_y^{-1} y)^* (R_{\check{s}} - R_{\check{s}y} R_y^{-1} R_{y\check{s}})^{-1} (\check{s}_{|i} - R_{\check{s}y} R_y^{-1} y).$$

But $R_y > 0$ (since it is the covariance of a Hilbert space state-space model), and hence one possible choice of $\check{s}_{|i}$ to guarantee $J_{s,i} > 0$ is to choose $\check{s}_{|i} = R_{\check{s}y} R_y^{-1} y = \hat{s}_{|i}$, which is clearly the $H^2$ smoothed estimate of $\check{s}$.

■

The following result is now straightforward.

**Theorem 3.4.2 (All $H^\infty$ Smoothers)** *All $H^\infty$ smoothers that achieve a level $\gamma_s$ (assuming they exist) are given by*

$$\check{s}_{|i} = \hat{s}_{|i} + (-R_{\check{s}} + R_{\check{s}y} R_y^{-1} R_{y\check{s}})^{1/2} \mathcal{S}(R_y^{-1/2} y) \tag{3.4.4}$$

*where $\mathcal{S}$ is any (not necessarily causal) contractive mapping, $\hat{s}_{|i}$ is the usual $H^2$ smoothed estimate, and $R_y$ and $R_{\check{s}} - R_{\check{s}y} R_y^{-1} y R_{y\check{s}}$ are defined in (3.4.3).*

It is clear from the discussions so far in this chapter that the Krein space estimation formalism provide simple derivations of $H^\infty$ estimators. These estimators turn out to be certain Krein space Kalman filters, and show that Krein space estimation yields a unified approach to $H^2$ and $H^\infty$ problems. To derive such filters, and to solve other related problems as discussed ahead, all one essentially needs is to identify an indefinite quadratic form, and to construct a convenient auxiliary state-space model with the appropriate Gramians. Before considering further applications in the next chapter, it will be useful to consider the problem of $l$-step $H^\infty$ prediction.

## 3.5 $l$-step $H^\infty$ Prediction

Consider once more the standard state-space model,

$$\begin{cases} x_{i+1} &= F_i x_i + G_i u_i, \quad x_0 \\ y_i &= H_i x_i + v_i, \quad i \geq 0 \\ s_i &= L_i x_i \end{cases} \tag{3.5.1}$$

where, as before, the initial condition $x_0$ and the disturbances $\{u_i, v_i\}$ are unknown quantities, $y_i$ is the observed output, and $s_i$ is the signal we intend to estimate. Suppose now that we would like to find $l$-step predictions of $s_i$, *i.e.*, estimates

$$\check{s}_{i|i-l} = \mathcal{F}_{l,i}(y_0. \ldots, y_{i-l}), \tag{3.5.2}$$

that can only make use of the observations from time 0 to time $i-l$, where $l > 0$ is some given value. [Obviously, $l = 1$ corresponds to the a priori filtering problem just considered.] In this case, our estimation errors will be the $l$-step prediction errors,

$$e_{l,i} = L_i x_i - \check{s}_{i|i-l}. \tag{3.5.3}$$

We can now define the problem of finding $l$-step $H^\infty$ predictors. Such predictors will have the property that the maximum energy gain from the disturbances

$$\left\{\Pi_0^{-1/2}(x_0 - \check{x}_0), \{u_j\}_{j=0}^{i}, \{v_j\}_{j=0}^{i-l}\right\},$$

to the prediction errors

$$\{e_{l,j}\}_{j=0}^{i},$$

is bounded by some given $\gamma_l$, *i.e.*,

$$\sup_{x_0, u\in h^2, v\in h^2} \frac{\sum_{j=0}^{i} e_{l,j}^* e_{l,j}}{(x_0 - \check{x}_0)^* {\Pi_0}^{-1}(x_0 - \check{x}_0) + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i-l} v_j^* v_j} < \gamma_l^2. \tag{3.5.4}$$

Note that in the above definition and inequality we have only included the measurement disturbances $v_j$ until time $i-l$ since measurement disturbances after time $i-l$ only affect the observations, $\{y_j, j > i-l\}$, which the estimator is not allowed to use.

Before attempting to solve this $H^\infty$ prediction problem, let us see what happens in the $H^2$ case. There to obtain, $\hat{x}_{i|i-l}$, one needs to project $x_i$ onto $\mathcal{L}\{y_j\}_{j=0}^{i-l}$, *i.e.*, the linear space spanned by the random variables, $\{y_0, \ldots, y_{i-l}\}$. Using the state equation in (3.5.1) this leads to

$$\hat{x}_{j+1|i-l} = F_j \hat{x}_{j|i-l} + G_j \hat{u}_{j|i-l}. \tag{3.5.5}$$

Now if we assume $j > i-l$, then $\hat{u}_{j|i-l} = 0$ (since $u_j$ is independent of the observations, $\{y_k\}_{k=0}^{i-l}$). This implies that

$$\hat{x}_{j+1|i-l} = F_j \hat{x}_{j|i-l}, \quad j > i-l. \tag{3.5.6}$$

In other words, the predictions $\hat{x}_{j|i-l}$ satisfy an "unforced" version of the same recursion that the states, $x_j$ satisfy. Therefore to find $\hat{x}_{i|i-l}$, all one needs to do is propagate the above recursion starting with the predicted estimate, $\hat{x}_{i-l+1}$, as the initial condition. Thus, we may write

$$\hat{x}_{i|i-l} = F_{i-1} \ldots F_{i-l+1} \hat{x}_{i-l+1}. \tag{3.5.7}$$

We shall presently see that the above expression is no longer true for $l$-step $H^\infty$ predictions. It turns out that the recursion that needs to be propagated is slightly more complicated.

To obtain the required $H^\infty$ estimates let us first note that (3.5.4) implies the inequalities,

$$\sup_{x_0, u \in h^2, v \in h^2} \frac{\sum_{j=0}^{k} e_{l,j}^* e_{l,j}}{(x_0 - \check{x}_0)^* \Pi_0{}^{-1} (x_0 - \check{x}_0) + \sum_{j=0}^{k} u_j^* u_j + \sum_{j=0}^{k-l} v_j^* v_j} < \gamma_l^2, \tag{3.5.8}$$

for all $k = 0, \ldots, i$. This in turn implies that for all $x_0$ and for all $\{u_j\}_{j=0}^{i}$ (that are not simultaneously zero), we must be able to choose the $\check{s}_{j|j-l} = \mathcal{F}_{l,i}(y_0, \ldots, y_{j-l})$ such that

$$J_{l,k} = (x_0 - \check{x}_0)^* \Pi_0{}^{-1} (x_0 - \check{x}_0) + \sum_{j=0}^{k} u_j^* u_j + \sum_{j=0}^{k-l} v_j^* v_j - \gamma_l^{-2} \sum_{j=0}^{k} e_{l,j}^* e_{l,j} > 0, \tag{3.5.9}$$

for all $k = 0, \ldots, i$. Of course, the above is true if, and only if,

(i) $J_{l,k}$ has a minimum over the free variables $\{x_0, \{u_j\}_{j=0}k\}$, for all $k = 0, \ldots, i$.

(ii) The $\check{s}_{j|j-l} = \mathcal{F}_{l,i}(y_0, \ldots, y_{j-l})$ can be chosen such that the value of $J_{l,k}$ at this minimum is positive (for all $k = 0, \ldots, i$).

Let us now proceed to check and construct the above two conditions recursively. At time $i = 0$, we have need to guarantee

$$J_{l,k} = (x_0 - \check{x}_0)^* \Pi_0{}^{-1} (x_0 - \check{x}_0) + u_0^* u_0 - \gamma_l^{-2} (\check{s}_{0|-l} - L_0 x_0)^* (\check{s}_{0|-l} - L_0 x_0) > 0. \tag{3.5.10}$$

Now since $\Pi_0 > 0$, it is straightforward to see that we will have a minimum over the free variables $\{x_0, u_0\}$ if, and only if,

$$-\gamma_l^2 I_q + L_0 \Pi_0 L_0^* < 0. \tag{3.5.11}$$

Moreover, the value at the minimum is given by

$$(\check{s}_{0|-l} - L_0 \check{x}_0)^* \left[ -\gamma_l^2 I_q + L_0 \Pi_0 L_0^* \right]^{-1} (\check{s}_{0|-l} - L_0 \check{x}_0), \tag{3.5.12}$$

from which we readily infer that the (here) *unique* prediction is

$$\check{s}_{0|-l} = L_0 \check{x}_0. \tag{3.5.13}$$

Assume now that we have solved the problem until time $k-1$, *i.e.*, we have checked the condition for a minimum for all $J_{l,j}$ from time $j = 0$ to time $j = k - 1$, and have chosen the $\{\check{s}_{j|j-l}\}_{j=0}^{k-1}$ such that the values of the $J_{l,j}$ at their minima are positive. We would now like to solve the problem for time $k$.

To this end, let us write $J_{l,k}$ as

$$J_{l,k} = J_{l,k}^1 + J_{l,k}^2, \tag{3.5.14}$$

where

$$\begin{gathered} J_{l,k}^1 = (x_0 - \check{x}_0)^* \Pi_0{}^{-1}(x_0 - \check{x}_0) + \textstyle\sum_{j=0}^{k-l} u_j^* u_j + \\ \textstyle\sum_{j=0}^{k-l} \begin{bmatrix} y_j - H_j x_j \\ \check{s}_{j|j-l} - L_j x_j \end{bmatrix}^* \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_l^{-2} I_q \end{bmatrix} \begin{bmatrix} y_j - H_j x_j \\ \check{s}_{j|j-l} - L_j x_j \end{bmatrix} \end{gathered} \tag{3.5.15}$$

and

$$J_{l,k}^2 = \sum_{j=k+1-l}^{k} u_j^* u_j - \sum_{j=k+1-l}^{k} \gamma_l^{-2} (\check{s}_{j|j-l} - L_j x_j)^* (\check{s}_{j|j-l} - L_j x_j). \tag{3.5.16}$$

The reason why we have decomposed $J_{l,k}$ in this fashion is that $J_{l,k}^1$ is a function of $\{x_0, u_0, \ldots, u_{k-l}\}$, or equivalently, $\{u_0, \ldots, u_{k-l}, x_{k-l+1}\}$, *i.e.*,

$$J_{l,k}^1 = J_{l,k}^1(u_0, \ldots, u_{k-l}, x_{k-l+1}), \tag{3.5.17}$$

and $J_{l,k}^2$ is a function of $\{x_{k-l+1}, u_{k-l+1}, \ldots, u_k\}$, *i.e.*,

$$J_{l,k}^2 = J_{l,k}^2(x_{k-l+1}, u_{k-l+1}, \ldots, u_k). \tag{3.5.18}$$

Thus, in effect, the above decomposition decouples the effects of $\{u_j, j \leq k-l\}$ and $\{u_j, j > k-l\}$ and allows us to minimize over the two sets separately.

Now the condition that $J_{l,k}^1$ have a minimum over $\{u_j, j \leq k-l\}$ is readily seen to be that the matrices

$$R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_l^2 I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \tag{3.5.19}$$

have the same inertia for all $j = 0, \ldots k-l$, where $P_j$ satisfies the Riccati recursion

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix}, \quad P_0 = \Pi_0. \tag{3.5.20}$$

Moreover the minimum value is

$$J^1_{l,k}(\min) = \sum_{j=0}^{k-l} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix} + \tilde{x}^*_{k+1-l} P^{-1}_{k+1-l} \tilde{x}^*_{k+1-l}, \tag{3.5.21}$$

where $\hat{x}_j$ satisfies the Krein space Kalman filter recursions,

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}, \quad \hat{x}_0 = \check{x}_0 \tag{3.5.22}$$

with

$$K_{p,j} = F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1}. \tag{3.5.23}$$

Since the first term (*i.e.*, the summation) in $J^1_{l,k}(\min)$ is independent of $x_{k+1-l}$, to minimize $J_{l,k}$ over the remaining variables $\{x_{k-l+1}, u_{k-l+1}, \ldots, u_k\}$ it suffices to minimize

$$\begin{gathered} J^2_{l,k} + \tilde{x}^*_{k+1-l} P^{-1}_{k+1-l} \tilde{x}^*_{k+1-l} = \\ \tilde{x}^*_{k+1-l} P^{-1}_{k+1-l} \tilde{x}^*_{k+1-l} + \textstyle\sum_{j=k+1-l}^{k} u_j^* u_j - \sum_{j=k+1-l}^{k} \gamma_l^{-2} (\check{s}_{j|j-l} - L_j x_j)^* (\check{s}_{j|j-l} - L_j x_j) \end{gathered} \tag{3.5.24}$$

over those variables. But the condition for this minimum is readily seen to be

$$R_{e,j}^{k+1-l} \triangleq -\gamma_l^2 I_q + L_j P_j^{k+1-l} L_j^* < 0, \quad j = k+1-l, \ldots, k \tag{3.5.25}$$

where $P_{j,k+1-l}$ satisfies the Riccati recursion

$$P_{j+1}^{k+1-l} = F_j P_j^{k+1-l} F_j^* + G_j G_j^* - F_j P_j^{k+1-l} L_j^* \left[R_{e,j}^{k+1-l}\right]^{-1} L_j P_j^{k+1-l} F_j^*, \quad P_{k+1-l}^{k+1-l} = P_{k+1-l}. \tag{3.5.26}$$

Moreover, in this case the value at the minimum is given by

$$\sum_{j=k+1-l}^{k} (\check{s}_{j|j-l} - L_j \hat{x}_j^{k+1-l})^* \left[R_{e,j}^{k+1-l}\right]^{-1} (\check{s}_{j|j-l} - L_j \hat{x}_j^{k+1-l}), \tag{3.5.27}$$

where $\hat{x}_j^{k+1-l}$ satisfies the Krein space Kalman filter recursions,[7]

$$\hat{x}_{j+1}^{k+1-l} = F_j \hat{x}_j^{k+1-l} + K_{p,j}^{k+1-l} (\check{s}_{j|j-l} - L_j \hat{x}_j^{k+1-l}), \quad j = k+1-l, \ldots k \tag{3.5.28}$$

[7] Note that at time $k$ all the $\check{s}_{j|j-l}$ (for $j = k+1-l, \ldots k$) in the given recursion are known.

where

$$K_{p,j}^{k+1-l} = F_j P_j^{k+1-l} L_j^* \left[R_{e,j}^{k+1-l}\right]^{-1}. \tag{3.5.29}$$

Therefore the value of $J_{l,k}$ at it its minimum is given by

$$\begin{array}{l} J_{l,k}(\min) = \sum_{j=0}^{k-l} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix} + \\ \qquad \sum_{j=k+1-l}^{k} (\check{s}_{j|j-l} - L_j\hat{x}_j^{k+1-l})^* \left[R_{e,j}^{k+1-l}\right]^{-1} (\check{s}_{j|j-l} - L_j\hat{x}_j^{k+1-l}). \end{array} \tag{3.5.30}$$

Now if we define

$$A \triangleq J_{l,k}(\min) - (\check{s}_{k|k-l} - L_j\hat{x}_k^{k+1-l})^* \left[R_{e,k}^{k+1-l}\right]^{-1} (\check{s}_{k|k-l} - L_j\hat{x}_k^{k+1-l}), \tag{3.5.31}$$

then it possible to show that $A > 0$. Indeed,

$$\begin{aligned} A &= \sum_{j=0}^{k-l} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix} + \\ &\quad \sum_{j=k-l}^{k-1} (\check{s}_{j|j-l} - L_j\hat{x}_j^{k+1-l})^* \left[R_{e,j}^{k+1-l}\right]^{-1} (\check{s}_{j|j-l} - L_j\hat{x}_j^{k+1-l}) \\ &= \sum_{j=0}^{k-l-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix} + \\ &\quad \min_{x_{k-l}, u_{k-l}, \ldots, u_{k-1}} \left[\tilde{x}_{k-l}^* P_{k-l}^{-1} \tilde{x}_{k-l} + v_{k-l}^* v_{k-l} + \sum_{j=k-l}^{k-1} u_j^* u_j - \sum_{j=k-l}^{k-1} \gamma_l^{-2} e_{l,j}^* e_{l,j}\right] \\ &\geq \sum_{j=0}^{k-l-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix} + \\ &\quad \min_{x_{k-l}, u_{k-l}, \ldots, u_{k-1}} \left[\tilde{x}_{k-l}^* P_{k-l}^{-1} \tilde{x}_{k-l} + \sum_{j=k-l}^{k-1} u_j^* u_j - \sum_{j=k-l}^{k-1} \gamma_l^{-2} e_{l,j}^* e_{l,j}\right] \\ &= J_{f,k-1}(\min) > 0. \end{aligned}$$

[Note that in the last step we have made use of the fact that the $\{\check{s}_{j|j-l}\}_{j=0}^{k-1}$ were chosen such that $J_{f,k-1}(\min) > 0$.

Thus writing,

$$J_{l,k}(\min) = A + (\check{s}_{k|k-l} - L_j\hat{x}_k^{k+1-l})^* \left[R_{e,k}^{k+1-l}\right]^{-1} (\check{s}_{k|k-l} - L_j\hat{x}_k^{k+1-l}),$$

we note that any choice of $\check{s}_{k|k-l}$ that renders $J_{l,k}(\min) > 0$ is an acceptable estimate. Since $A > 0$ and $R_{e,k}^{k+1-l} < 0$ there are obviously many choices of which the choice

$$\check{s}_{k|k-l} = L_j \hat{x}_k^{k+1-l}, \tag{3.5.32}$$

seems the most natural (and is referred to as the central filter).

We can summarize the results obtained so far in the following theorem.

**Theorem 3.5.1 ($l$-Step $H^\infty$ Predictor)** *Given $\gamma_l > 0$, an $l$-step $H^\infty$ predictor, $\check{s}_{j|j-l} = \mathcal{F}_{l,j}(y_0, \ldots y_{j-l})$ that achieves*

$$\sup_{x_0, u \in h^2, v \in h^2} \frac{\sum_{j=0}^{i} (s_j - \check{s}_{j|j-l})^*(s_j - \check{s}_{j|j-l})}{(x_0 - \check{x}_0)^* \Pi_0^{-1} (x_0 - \check{x}_0) + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i-l} v_j^* v_j} < \gamma_l^2, \tag{3.5.33}$$

*exists if, and only if, for each $j = 0, \ldots, i-l$,*

*(i) the two matrices*

$$R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma_l^2 I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$$

*have the same inertia, where $P_j$ satisfies the Riccati recursion*

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix}, \quad P_0 = \Pi_0. \tag{3.5.34}$$

*(ii) the sequence of matrices*

$$R_{e,j+m}^{j+1} \triangleq -\gamma_l^2 I_q + L_{j+m} P_{j+m}^{j+1} L_{j+m}^*, \quad m = 1, \ldots l$$

*are negative definite, where $P_{j+m}^{j+1}$ satisfies the Riccati recursion*

$$\begin{array}{rcl} P_{j+m+1}^{j+1} & = & F_{j+m} P_{j+m}^{j+1} F_{j+m}^* + G_{j+m} G_{j+m}^* - \\ & & F_{j+m} P_{j+m}^{j+1} L_{j+m}^* \left[ R_{e,j+m}^{j+1} \right]^{-1} L_{j+m} P_{j+m}^{j+1} F_{j+m}^* \end{array}, \quad P_{j+1}^{j+1} = P_{j+1}. \tag{3.5.35}$$

*If this is the case then one possible $l$-step $H^\infty$ predictor is given by*[8]

$$\check{s}_{j+l|j} = L_{j+l}\hat{x}_{j+l}^{j+1}, \tag{3.5.36}$$

*where $\hat{x}_{j+m}^{j+1}$ satisfies the recursion,*

$$\hat{x}_{j+m+1}^{j+1} = F_{j+m}\hat{x}_{j+m}^{j+1} + K_{p,j+m}^{j+1}(\check{s}_{j+m|j+m-l} - L_{j+m}\hat{x}_{j+m}^{j+1}), \quad \hat{x}_{j+1}^{j+1} = \hat{x}_{j+1} \tag{3.5.37}$$

*with*

$$K_{p,j+m}^{j+1} = F_{j+m}P_{j+m}^{j+1}L_{j+m}^* \left[R_{e,j+m}^{j+1}\right]^{-1}, \tag{3.5.38}$$

*and where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{p,j}\begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}, \quad \hat{x}_0 = \check{x}_0 \tag{3.5.39}$$

*with*

$$K_{p,j} = F_jP_j\begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1}. \tag{3.5.40}$$

**Remark:** It will be useful to further explain the structure of the $l$-step $H^\infty$ filter given in the above theorem.

1. Set $\check{s}_{k|k-l} = L_kF_{k-1}\ldots F_0\check{x}_0$ for $k = 0, \ldots, l-1$.

2. Set $j = 0$. ($j$ is the observation index that runs from 0 to $i - l$.)

3. Check inertia condition (i). Compute $P_{j+1}$ (using (3.5.34)) and $\hat{x}_{j+1}$ (using (3.5.39)).

4. Perform the $l$-step propagation of the Riccati variable, $P_{j+m}^{j+1}$, (using (3.5.35)) and check the negativity conditions (ii).

5. Perform the $l$-step propagation of $\hat{x}_{j+m}^{j+1}$ (using (3.5.37)) to find the estimate, $\check{s}_{j+l|j} = L_{j+l}\hat{x}_{j+l}^{j+1}$.

[8] We also have $\check{s}_{k|k-l} = L_kF_{k-1}\ldots F_0\check{x}_0$ for $k = 0, \ldots, l-1$.

6. Set $j = j + 1$ and go to 3.

The important feature of the $l$-step $H^\infty$ predictor is that at each iteration (*i.e.*, after each observation) in addition to propagating one step of the usual Riccati recursion (3.5.34) and state estimate recursion (3.5.39)) (which are the same recursions as those of the a posteriori filter) we need to propagate $l$ steps of the auxiliary Riccati recursion for $P_{i+m}^{i+1}$ and $l$ steps of the auxiliary state estimate, $\hat{x}_{i+m}^{i+1}$. This is clearly different from the case of $l$-step $H^2$ prediction.

It is also straightforward to parametrize all possible $l$-step $H^\infty$ predictors of a given level $\gamma_l$.

**Lemma 3.5.1 (All $l$-Step $H^\infty$ Predictors)** *All $l$-step $H^\infty$ predictors that achieve a level $\gamma_l$ (assuming they exist) are given by any $\check{s}_{j|j-l} = \mathcal{F}_{l,j}(y_0, \ldots y_{j-l})$ that satisfy*

$$\begin{array}{c}\sum_{j=0}^{k-l} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix} + \\ \sum_{j=k+1-l}^{k} (\check{s}_{j|j-l} - L_j\hat{x}_j^{k+1-l})^* \left[R_{e,j}^{k+1-l}\right]^{-1} (\check{s}_{j|j-l} - L_j\hat{x}_j^{k+1-l}) > 0 \end{array} \qquad k = 0, \ldots, i \tag{3.5.41}$$

*where $\hat{x}_{j+m}^{j+1}$ and $\hat{x}_j$ satisfy the recursions*

$$\hat{x}_{j+m+1}^{j+1} = F_{j+m}\hat{x}_{j+m}^{j+1} + K_{p,j+m}^{j+1}(\check{s}_{j+m|j+m-l} - L_{j+m}\hat{x}_{j+m}^{j+1}), \quad \hat{x}_{j+1}^{j+1} = \hat{x}_{j+1} \tag{3.5.42}$$

*and*

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j-l} - L_j\hat{x}_j \end{bmatrix}, \quad \hat{x}_0 = \check{x}_0 \tag{3.5.43}$$

*respectively, and where $R_{e,j}^{k+1-l}$, $R_{e,j}$, $K_{p,j+m}^{j+1}$ and $K_{p,j}$ as in Theorem 3.5.1.*

## 3.6 Conclusion

Certain studies in least-squares estimation, adaptive filtering and $H^\infty$ filtering motivated us to develop a theory for linear estimation in certain indefinite metric spaces, called Krein spaces. The main difference from the conventional Hilbert space framework for Kalman filtering and LQG control are that projections in Krein spaces may not necessarily exist or be unique, and that quadratic forms may have stationary points that are not necessarily extreme points (*i.e.*, minima or maxima). We showed that these simple but fundamental differences explain both the unexpected similarities and differences between the well-known Kalman filter solution for stochastic state-space systems and the solution for the completely nonstochastic $H^\infty$ filtering problem.

The main points are the following. There are many problems whose solution can be reduced to the recursive minimization of some indefinite quadratic form. A stationary point, when it exists, of the quadratic form can be computed as follows: set up a (partially equivalent) problem of projecting a vector in a Krein space onto a certain subspace. The advantage is that when there is state-space structure, this projection can be recursively computed by using the innovations approach to derive a Krein space Kalman filter. The equivalence is only partial because the Krein space projection only defines the stationary point of the quadratic form and further conditions need to be checked to determine if this point is also a minimum. It turns out that this checking can also be done recursively using quantities arising in the Kalman filtering algorithms.

Apart from quite straightforward derivations of known results in $H^2$, $H^\infty$ and risk-sensitive estimation and control, the above approach allows us to extend to the $H^\infty$ setting some of the huge body of results and insights developed over the last three decades in the field of Kalman filtering (and LQG control). A first bonus is the derivation (see [HSK94c] and Chapter 5) of square-root and (fast) Chandrasekhar algorithms for $H^\infty$ estimation and control, a possibility that is much less obvious in current approaches. These square-root algorithms, which are now increasingly standard in $H^2$ Kalman filtering, have two advantages over the earlier $H^\infty$ algorithms: they eliminate the need for explicitly checking the existence conditions of the filters, and have various potential numerical and implementational advantages.

Application of the Krein space formulation to adaptive filtering arises from the approach in [SK94b] where it was shown how to recast adaptive filtering problems as state-space estimation problems. If we further allow the elements of the state-space model to belong to a Krein space, then we can solve finite memory and $H^\infty$ adaptive filtering problems. In the finite memory case, this allows us to consider general sliding patterns with windows of varying lengths. In the $H^\infty$ adaptive case, this has allowed us to establish that the famed LMS (or stochastic gradient) algorithm is an optimal $H^\infty$ filter [HSK96a] (see also Chapter 9).

We also remark that, although not pursued here, it is also possible to construct *dual* (rather than partially equivalent) Krein state-space models (via the concept of a dual basis), which can be used to extend the methods to the solution of $H^2$ and $H^\infty$ control problems. This will be done in Chapter 6.

# Chapter 4

# Further Applications

In this chapter we show that several interesting problems in risk-sensitive estimation and control, dynamic quadratic game theory, and finite memory adaptive filtering follow as special cases of the Krein space linear estimation theory developed in Chapter 2. The major point is that all these problems can be cast into the problem of calculating the stationary point of certain indefinite quadratic forms, and that by considering the appropriate state space models and error Gramians, we can use the Krein space estimation theory to calculate these stationary points and study their properties. The approach discussed here allows for interesting generalizations, such as finite memory adaptive filtering with varying sliding patterns and suboptimal recursive total least-squares algorithms.

## 4.1 Introduction

The classical Kalman filter can be viewed as a recursive procedure that minimizes the expected value of a certain quadratic cost function. Recently there has also been increasing interest in an alternative so-called exponential-quadratic cost function [Jac73, SDJ74, Whi90, SFB92], and estimators (and controllers) that minimize its expected value. The ensuing theory is sometimes called LEQG (linear-exponential-quadratic-Gaussian) theory to reflect the facts that the resulting optimal estimators (and controllers) are linear, the cost function is the exponential of a quadratic, and the disturbances are assumed to be Gaussian random variables. It is also called (following Whittle) risk-sensitive estimation and control, since the criterion is risk-sensitive, in the sense that it depends on a real parameter that determines whether more or less weight should be given to higher or smaller errors. [Roughly speaking when more weight is given to smaller errors the criterion is risk-seeking, and when more weight is given to large errors the criterion is risk-averse.] The filters obtained within this framework are termed risk-sensitive and include the conventional Kalman filter as a (so-called risk-neutral) special case.

Following some pioneering work in game theory (motivated primarily by the field of economics [NM44, KT50]), since the mid 1960's there has been considerable interest in applying game-theoretic ideas and methods to estimation and control. In this framework, the problem of estimation (or control) is treated as a noncooperative two-player game, with one player (the opponent) being the exogenous signals, and the other player being the estimator (or controller). This approach, which treats the exogenous signals as malignant disturbances that compete against the estimator (or controller), is, of course, fundamentally different from the $H^2$ (or risk-sensitive) approach where the exogenous signals are simply taken to be random variables with known probability distributions. The class of games most often applied to estimation and control is the class of differential games with quadratic payoff [Isa65, Ber64, BH69], which, in the discrete-time case of interest to us, shall henceforth be referred to as a quadratic dynamic game. One reason for its pervasive use may be that the solutions to quadratic dynamic games bear many similarities to the solutions of $H^2$ estimation and control — estimators have an observer structure, full information controllers have state-feedback structure, the various observer and state-feedback gains are found from the solution of certain Riccati equations, etc.

In fact, it has recently been shown that there is a close connection between $H^\infty$ estimation and control, game theory, and risk-sensitive estimation and control [GD88, Bas89, LAKG92]. Indeed, it turns out the central $H^\infty$ estimators and controllers, as well as the risk-sensitive optimal estimators and controllers can be derived as solutions to a certain quadratic dynamic game. [This observation is the driving force of the game-theoretic approach to $H^\infty$ control.]

In this chapter we shall shed further light on the connections between these theories by using the Krein space approach of Chapter 2. Indeed, we shall see that, as with $H^\infty$ estimation and control, risk-sensitive estimation and control problems, quadratic games, and finite memory adaptive filtering problems lead almost by inspection to indefinite deterministic quadratic forms. Following Chapters 2 and 3, we can solve these problems by constructing the corresponding Krein space models. Once this is done, the Krein space Kalman filter solutions can be written down immediately, and the existence conditions for the problem can also be expressed in terms of quantities easily related to the basic Riccati equations of the Kalman filter.

The remainder of the chapter is organized as follows. In Section 4.2 we describe the problem of risk-sensitive estimation, and show that a risk-sensitive estimator is one that computes the stationary point of a certain (possibly indefinite) quadratic form, provided that this quadratic form has a minimum over a certain set of variables. By considering a corresponding Krein state space model, we use the results of Chapter 2 to derive conditions for the existence of the minimum, and to show that the Krein space projection also solves the risk-sensitive estimation problem. We then derive risk-sensitive a posteriori, a priori, smoothed, and $l$-step ahead filters parallel to what was done in Chapter 3. Sec. 4.3 gives a very brief introduction to the theory of quadratic dynamic games. We show that such games lend themselves to analysis by Krein space methods since the cost (or payoff) function in is an indefinite quadratic form. Moreover, we show that the (max-min, min-max, or saddle point) solutions to such games are found by stationarizing these quadratic forms and checking for certain minimizing and maximizing conditions. Some specific examples from state-space estimation and control are also provided. Finite memory adaptive filtering is studied in Sec. 4.4, where the Krein space approach is used to solve this problem, and to connect it to state-space approaches to adaptive filtering. An interesting byproduct of our analysis is the physical interpretation of innovations with negative Gramian as corresponding to the loss of information. The chapter is concluded in Sec. 4.5 with some remarks on further applications of the Krein space framework, *e.g.*, to recursive solutions of (suboptimal) total least-squares problems.

## 4.2 Risk-Sensitive Estimation

The so-called risk-sensitive (or exponential cost) criterion was introduced in [Jac73] and further studied in [SDJ74, Whi90, SFB92]. Glover and Doyle [GD88] noticed their close connection to the $H^\infty$ filters of Chapter 3. We shall make this connection in a different way by introducing an appropriate quadratic form.

### 4.2.1 The Exponential Cost Function

We again start with a state-space model of the form

$$\left\{\begin{array}{rcl} \mathbf{x}_{j+1} & = & F_j\mathbf{x}_j + G_j\mathbf{u}_j, \quad j \geq 0 \\ \mathbf{y}_j & = & H_j\mathbf{x}_j + \mathbf{v}_j \end{array}\right. \tag{4.2.1}$$

However, we now assume that the initial condition, $\mathbf{x}_0$, and the driving and measurement disturbances, $\{\mathbf{u}_j\}$, and $\{\mathbf{v}_j\}$ are independent zero mean Gaussian random variables with covariances with

$$E\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}\begin{bmatrix} \mathbf{x}_0^* & \mathbf{u}_j^* & \mathbf{v}_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & 0 \\ 0 & 0 & R_i\delta_{ij} \end{bmatrix}. \tag{4.2.2}$$

As we have seen in Sec. 1.3, conventional $H^2$ estimators, such as the Kalman filter, estimate the quantity $\mathbf{s}_j = L_j\mathbf{x}_j$ from the observations $\{\mathbf{y}_k\}_{k=0}^m$ (*i.e.*, $\check{s}_{j|m} = \mathcal{F}_i(\mathbf{y}_0, \ldots, \mathbf{y}_m)$) by performing the following minimization

$$\min_{\{\check{\mathbf{s}}_{j|m}\}} E\left[\mathbf{C}_i\right] \tag{4.2.3}$$

where $\mathbf{C}_i \triangleq \sum_{j=0}^{i}(\check{\mathbf{s}}_{j|m} - L_j\mathbf{x}_j)^*(\check{\mathbf{s}}_{j|m} - L_j\mathbf{x}_j)$ and $E\left[\cdot\right]$ denotes expectation. As we have seen earlier, the choices $m = j$, $m = j-1$, $m = i$, and $m = j - l$ correspond to the a posteriori, the a priori, the smoothed, and the $l$-step ahead, estimation problems, respectively. Moreover, the expectation is taken over the jointly Gaussian random variables $\{\mathbf{x}_0, \{\mathbf{u}_j\}_{j=0}^i, \{\mathbf{y}_j\}_{j=0}^i\}$.[1] Using the fact that $\mathbf{x}_0$ and the $\{\mathbf{u}_j\}$ and $\{\mathbf{v}_j\}$ are independent Gaussian random variables with covariances given by (4.2.2), and the

[1] Note that this set of random variables is equivalent to the set $\{\mathbf{x}_0, \{\mathbf{u}_j\}_{j=0}^i, \{\mathbf{v}_j\}_{j=0}^i\}$.

fact that $\mathbf{v}_j = \mathbf{y}_j - H_j\mathbf{x}_j$, allows us to write the joint probability distribution of the $\{\mathbf{x}_0, \{\mathbf{u}_j\}_{j=0}^{i}, \{\mathbf{y}_j\}_{j=0}^{i}\}$ as

$$p(x_0; u_0, \ldots, u_i; y_0, \ldots, y_i) \;\propto\; \exp\left[-\frac{1}{2}J_i(x_0; u_0, \ldots, u_i; y_0, \ldots, y_i)\right] \tag{4.2.4}$$

where the symbol $\propto$ stands for "proportional to" and $J_i(x_0; u_0, \ldots, u_i; y_0, \ldots, y_i)$ is given by

$$x_0^*\Pi_0^{-1}x_0 + \sum_{j=0}^{i} u_j^* Q_j^{-1} u_j + \sum_{j=0}^{i}(y_j - H_jx_j)^* R_j^{-1}(y_j - H_jx_j). \tag{4.2.5}$$

In the terminology of [Whi90], any filter that minimizes (4.2.3) is known as a *risk-neutral* filter.

An alternative criterion that is *risk-sensitive* has been extensively studied in [Jac73, SDJ74, Whi90, SFB92] and corresponds to the minimization problem

$$\min_{\{\check{\mathbf{s}}_{j|m}\}} \mu_i(\theta) = \min_{\{\check{\mathbf{s}}_{j|m}\}} \left(-\frac{2}{\theta}\log\left[E\exp(-\frac{\theta}{2}\mathbf{C}_i)\right]\right). \tag{4.2.6}$$

The criterion in (4.2.6) is known as an *exponential cost* criterion, and any filter that minimizes $\mu_i(\theta)$ is referred to as a *risk-sensitive* filter.[2] The scalar parameter $\theta$ is correspondingly called the *risk-sensitivity* parameter. Some intuition concerning the nature of this modified criterion is obtained by expanding $\mu_i(\theta)$ in terms of $\theta$ and writing,

$$\mu_i(\theta) = E(\mathbf{C}_i) - \frac{\theta}{4}Var(\mathbf{C}_i) + O(\theta^2).$$

The above equation shows that for $\theta = 0$, we have the risk-neutral case (*i.e.*, conventional $H^2$ estimation). When $\theta > 0$, we seek to maximize $E\exp(-\frac{\theta}{2}\mathbf{C}_i)$, which is convex and decreasing in $\mathbf{C}_i$. Such a criterion is termed *risk-seeking* (or optimistic) since larger weights are on small values of $\mathbf{C}_i$, and hence we are more concerned with the frequent occurrence of moderate values of $\mathbf{C}_i$ than with the occasional occurrence of large values. When $\theta < 0$, we seek to minimize $E\exp(-\frac{\theta}{2}\mathbf{C}_i)$, which is convex and increasing in $\mathbf{C}_i$. Such a criterion is termed *risk-averse* (or pessimistic) since large weights are on large values of $\mathbf{C}_i$, and hence we are more concerned with the

[2]Such filters are also called LEQG (linear-exponential-quadratic-Gaussian) filters to reflect the facts that, as we shall momentarily see, the optimum filter is linear, the cost is the exponential of a quadratic, and the disturbances are Gaussian.

occasional occurrence of large values than with the frequent occurrence of moderate ones. In what follows, we shall see that in the risk-averse case $\theta < 0$, the limit at which minimizing (4.2.6) makes sense corresponds to the *optimal* $H^\infty$ criterion. [To further compare the risk neutral and risk-averse approaches to estimation we have plotted the cost functions (whose expected values are to be minimized) for these two approaches in Fig. 4.1 (the risk-averse case corresponds to $\theta = 2$). As can be seen in the risk-averse case we incur (exponentially) larger costs for large values of $\mathbf{C}_i$.]

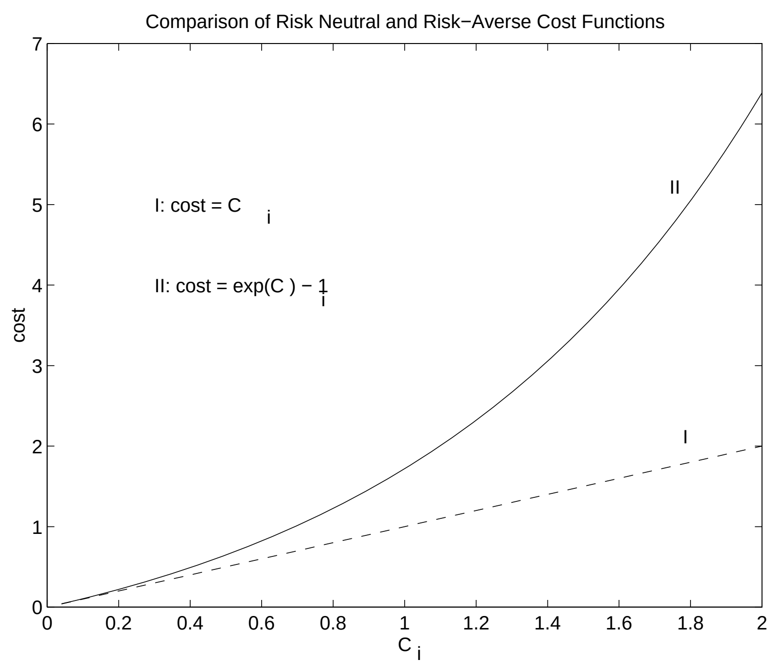


Figure 4.1: Comparison of risk neutral and risk-averse cost functions.

## 4.2.2 Minimizing the Risk-Sensitive Criterion

Using the probability density function (4.2.4) we can easily verify that

$$E\left[\exp(-\frac{\theta}{2}\mathbf{C}_i)\right] \;\propto\; \int \exp(-\frac{\theta}{2}C_i)\exp\left[-\frac{1}{2}J_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)\right] dx_0 \prod_{k=0}^{i} du_k \prod_{k=0}^{i} dy_k,$$

which, upon a change in the order of the integrations, becomes

$$E(\exp(-\frac{\theta}{2}\mathbf{C}_i)) \;\propto\; \int \prod_{k=0}^{i} dy_k \int \exp\left[-\frac{1}{2}\left\{J_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)+\theta C_i\right\}\right] dx_0 \prod_{k=0}^{i} du_k. \tag{4.2.7}$$

Since the $\{\check{z}_{j|m}\}$ are functions of the observations $\{y_k\}_{k=0}^{m}$, the above relation shows that the risk-sensitive criterion (4.2.6) can be alternatively written as

(i) For $\theta > 0$:

$$\max_{\{\check{s}_{j|m}\}} \int \exp\left[-\frac{1}{2}\left\{J_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)+\theta C_i\right\}\right] dx_0 \prod_{k=0}^{i} du_k,$$

(ii) For $\theta < 0$:

$$\min_{\{\check{s}_{j|m}\}} \int \exp\left[-\frac{1}{2}\left\{J_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)+\theta C_i\right\}\right] dx_0 \prod_{k=0}^{i} du_k.$$

This suggests that we define the second-order scalar form

$$\bar{J}_i(x_0,u_0,\ldots,u_{i,0},\ldots,y_i) \triangleq J_i(x_0,u_0,\ldots,u_{i,0},\ldots,y_i)+\theta C_i =$$

$$x_0^*\Pi_0^{-1}x_0+\sum_{j=0}^{i} u_j^*Q_j^{-1}u_j+\sum_{j=0}^{i}(y_j-H_jx_j)^*R_j^{-1}(y_j-H_jx_j)+\theta\sum_{j=0}^{i}(\check{s}_{j|j-1}-L_jx_j)^*(\check{s}_{j|j-1}-L_jx_j)$$

Before proceeding with the extremizations in (i) and (ii), we need to ensure that the integrals in (i) and (ii) be finite. The condition is given by the following Lemma, which is easy to prove.

**Lemma 4.2.1 (Finiteness Condition)** *The integral*

$$\int \exp\left[-\frac{1}{2}\bar{J}_i(x_0;u_0,\ldots,u_i;y_0,\ldots,y_i)\right] dx_0 \prod_{k=0}^{i} du_k,$$

*is finite if, and only if,* $\bar{J}_i(x_0,u_0,\ldots,u_i;y_0,\ldots,y_i)$ *has a minimum over* $\{x_0,u_0,\ldots,u_i\}$. *In this case the value of the integral is proportional to*

$$\exp\left\{-\frac{1}{2}\min_{x_0,u}\bar{J}_i(x_0,u_0,\ldots,u_i;y_0,\ldots,y_i)\right\}.$$

The above Lemma thus reduces the risk-sensitive problem to one of finding the minimum of a scalar quadratic form. More precisely, the criterion becomes

(i) For $\theta > 0$:

$$\min_{\{\check{s}_{j|l}\}} \min_{x_0,u} \bar{J}_i(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i),$$

(ii) For $\theta < 0$:

$$\max_{\{\check{s}_{j|l}\}} \min_{x_0,u} \bar{J}_i(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i).$$

It is noteworthy that the second of the above problems is a quadratic dynamic game (see *e.g.*, [Isa65, BB95]). We shall (very) briefly study quadratic game theory in Sec. 4.3 and shall show that they can also be solved using the Krein space approaches developed in Chapters 2 and 3.

For the time being, however, let us return to the above risk-sensitive problems. In order to solve them, we can introduce the following *auxiliary* Krein state-space model that corresponds to the (possibly indefinite) quadratic form $\bar{J}_i(x_0, u_0, \ldots, u_i, _0, \ldots, y_i)$.

$$\left\{ \begin{array}{rcl} \mathbf{x}_{j+1} & = & F_j\mathbf{x}_j + G_j\mathbf{u}_j \quad j \geq 0 \\ \begin{bmatrix} \mathbf{y}_i \\ \check{\mathbf{s}}_{i|l} \end{bmatrix} & = & \begin{bmatrix} H_i \\ L_i \end{bmatrix} \mathbf{x}_i + \mathbf{v}_i \end{array} \right. \tag{4.2.8}$$

with

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_k \\ \mathbf{v}_k \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & 0 \\ 0 & 0 & \begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} \delta_{jk} \end{bmatrix}. \tag{4.2.9}$$

We can now readily use the state-space model (4.2.8), and the results of the earlier Chapters 2 and 3, to check for the condition of a minimum over $\{x_0, u_0, \ldots, u_i\}$, and to compute the value at the minimum. Since we have provided the details in those earlier chapters, here we shall only briefly outline the steps for the a posteriori risk-sensitive estimation problem.[3]

[3]Indeed the derivations are almost identical to those of the $H^\infty$ filters of Chapter 3, since the quadratic form $\bar{J}_i(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$ is exactly the same as the quadratic forms $J_{f,i}(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$, $J_{p,i}(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$ and $J_{s,i}(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$, when we choose $\theta = \gamma_f^{-2}$, $\theta = \gamma_p^{-2}$ and $\theta = \gamma_s^{-2}$, and when the estimate is chosen as a filtered, predicted and smoothed estimate, respectively.

First note that when $\theta > 0$, the quadratic form $\bar{J}_i(x_0, u_0, \ldots, u_{i,0}, \ldots, y_i)$ is non-negative so that it always has a minimum. Thus risk-seeking optimal filters (corresponding to $\theta > 0$) always exist. On the other hand, when $\theta < 0$, the quadratic form $\bar{J}_i(x_0, u_0, \ldots, u_{i,0}, \ldots, y_i)$ is, in general, indefinite, so that further conditions are required for a minimum to exist. This condition is given by the requirement that the matrices,

$$\begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1} I \end{bmatrix} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1} I \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \tag{4.2.10}$$

have the same inertia for all $j = 0, \ldots i$, where $P_0 = \Pi_0$, and

$$P_{j+1} = F_j P_j F_j^* + G_j Q_j G_j^* - F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_i F_i^*. \tag{4.2.11}$$

In both cases, the value of $\bar{J}_i(x_0, u_0, \ldots, u_i; y_0, \ldots, y_i)$ at its minimum is given by

$$\bar{J}_i(\min) = \sum_{j=0}^{i} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix}, \tag{4.2.12}$$

where $\hat{x}_j$ obeys the Krein space Kalman filter recursion,

$$\hat{x}_{j+i} = F_j \hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix}, \quad \hat{x}_0 = 0 \tag{4.2.13}$$

with $K_{p,j} = F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1}$. Using a block LDU (lower-diagonal-upper) triangular factorization of the matrix, $R_{e,j}$, and defining the Schur complement,

$$\Delta_j = \theta^{-1} I + L_j P_j L_j^* - L_j P_j H_j^* (R_j + H_j P_j H_j^*)^{-1} H_j P_j L_j^*, \tag{4.2.14}$$

and the estimate,

$$\hat{s}_{j|j} = L_j \hat{x}_j + L_j P_j H_j^* (R_j + H_j P_j H_j^*)^{-1} (y_j - H_j \hat{x}_j), \tag{4.2.15}$$

allows us to write

$$\bar{J}_i(\min) = \sum_{j=0}^{i} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix}^* \begin{bmatrix} R_j^{-1} & 0 \\ 0 & \Delta_j^{-1} \end{bmatrix} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix}. \tag{4.2.16}$$

Thus in the risk-seeking case ($\theta > 0$), our problem becomes,

$$min_{\{\check{s}_{j|j}\}} \sum_{j=0}^{i} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix}^* \begin{bmatrix} R_j^{-1} & 0 \\ 0 & \Delta_j^{-1} \end{bmatrix} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}, \tag{4.2.17}$$

and in the risk-averse case ($\theta < 0$),

$$max_{\{\check{s}_{j|j}\}} \sum_{j=0}^{i} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - \hat{s}_{j|j} \end{bmatrix}^* \begin{bmatrix} R_j^{-1} & 0 \\ 0 & \Delta_j^{-1} \end{bmatrix} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}. \tag{4.2.18}$$

But when $\theta > 0$, we have $R_{e,j} > 0$, and thus $\Delta_j > 0$. Therefore the quadratic form in (4.2.17) has a minimum over the $\{\check{s}_{j|j}\}$ which is readily given by

$$\check{s}_{j|j} = \hat{s}_{j|j}. \tag{4.2.19}$$

Likewise, when $\theta < 0$, due to the inertia condition on $R_{e,j}$, we have $\Delta_j < 0$. Thus, in this case, the quadratic form in (4.2.18) has a maximum over the $\{\check{s}_{j|j}\}$ which is also given by

$$\check{s}_{j|j} = \hat{s}_{j|j}. \tag{4.2.20}$$

As in Chapter 3 it is also possible to give a recursion for the $\hat{x}_{j|j}$ which yields $L_j\hat{x}_{j|j} = \hat{s}_{j|j}$. This leads to the following result.

**Theorem 4.2.1 (A Posteriori Risk-Sensitive Filter)** *Consider the a posteriori risk-sensitive estimation problem,*

$$\min_{\{\check{\mathbf{s}}_{j|j}\}} \; -\frac{2}{\theta}\log\left[E\exp\left(-\frac{\theta}{2}\sum_{j=0}^{i}(\mathbf{s}_j - \hat{\mathbf{s}}_{j|j})^*(\mathbf{s}_j - \hat{\mathbf{s}}_{j|j})\right)\right],$$

*for some given $\theta$. When $\theta > 0$, the risk-sensitive estimation problem always has a solution. When $\theta < 0$, a solution exists if, and only if, the matrices*

$$\begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix}$$

*have the same inertia for all $j = 0, 1, \ldots i$, where $P_0 = \Pi_0$ and*

$$P_{j+1} = F_jP_jF_j^* + G_jQ_jG_j^* - F_jP_j\begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_iF_i^*.$$

*In both cases the optimal risk-sensitive filter with parameter $\theta$ is given by*

$$\check{s}_{j|j} = L_j \hat{x}_{j|j},$$

*where $\hat{x}_{j|j}$ satisfies the recursion,*

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{s,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}) \quad , \quad \hat{x}_{-1|-1} = 0$$

*and*

$$K_{s,j+1} = P_{j+1} H_{j+1}^* (I + H_{j+1} P_{j+1} H_{j+1}^*)^{-1}.$$

Similar results hold for the a priori, smoothed, and $l$-step ahead, estimation problems, as given below. The proofs are similar and are omitted for brevity.

**Theorem 4.2.2 (A Priori Risk-Sensitive Filter)** *Consider the a priori risk-sensitive estimation problem,*

$$\min_{\{\check{s}_j\}} \; -\frac{2}{\theta} \log \left[ E \exp \left( -\frac{\theta}{2} \sum_{j=0}^{i} (\mathbf{s}_j - \hat{\mathbf{s}}_j)^* (\mathbf{s}_j - \hat{\mathbf{s}}_j) \right) \right],$$

*for some given $\theta$. When $\theta > 0$, the a priori risk-sensitive estimation problem always has a solution. When $\theta < 0$, a solution exists if, and only if, all leading submatrices of*

$$\begin{bmatrix} \theta^{-1} I & 0 \\ 0 & R_j \end{bmatrix} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} \theta^{-1} I & 0 \\ 0 & R_j \end{bmatrix} + \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix}$$

*have the same inertia for all $j = 0, 1, \ldots i$, where $P_j$ is the same as in the a posteriori problem. In both cases the a priori risk-sensitive filter with parameter $\theta$ is given by*

$$\check{s}_{j|j-1} = L_j \hat{x}_{j|j-1}$$

*where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1|j} = F_j \hat{x}_{j|j-1} + K_{a,j}(y_j - H_j \hat{x}_{j|j-1}) \quad , \quad \hat{x}_{0|-1} = 0$$

*with*

$$K_{a,j} = F_j \tilde{P}_j H_j^* (R_j + H_j \tilde{P}_j H_j^*)^{-1}, \quad \text{and} \quad \tilde{P}_j = P_j - P_j L_j^* (\theta^{-1} I + L_j P_j L_j^*)^{-1} L_j P_j.$$

**Theorem 4.2.3 (Risk-Sensitive Smoother)** *Consider the smoothed risk-sensitive estimation problem,*

$$\min_{\{\check{\mathbf{s}}_{j|i}\}} \; -\frac{2}{\theta}\log\left[E\exp\left(-\frac{\theta}{2}\sum_{j=0}^{i}(\mathbf{s}_j-\hat{\mathbf{s}}_{j|i})^*(\mathbf{s}_j-\hat{\mathbf{s}}_{j|i})\right)\right],$$

*for some given $\theta$. When $\theta > 0$, the smoothed risk-sensitive problem always has a solution. When $\theta < 0$, a solution exists if, and only if, the block diagonal matrix*

$$R_e = R_{e,0} \oplus R_{e,1} \oplus \ldots \oplus R_{e,i}$$

*where*

$$R_{e,j} = \begin{bmatrix} R_j & 0 \\ 0 & \theta^{-1}I \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix}$$

*and $P_j$ is the same as in the a posteriori problem, has $(i+1)p$ positive eigenvalues and $(i+1)q$ negative eigenvalues. In other words, if, and only if,*

$$In[R_e] = \begin{bmatrix} (i+1)p & 0 & (i+1)q \end{bmatrix}.$$

*In both cases the risk-sensitive smoother is the same as the $H^2$ smoother.*

**Theorem 4.2.4 (Risk-Sensitive $l$-Step Predictor)** *Consider the $l$-step ahead risk-sensitive estimation problem,*

$$\min_{\{\check{\mathbf{s}}_{j|j-l}\}} \; -\frac{2}{\theta}\log\left[E\exp\left(-\frac{\theta}{2}\sum_{j=0}^{i}(\mathbf{s}_j-\hat{\mathbf{s}}_{j|j-l})^*(\mathbf{s}_j-\hat{\mathbf{s}}_{j|j-l})\right)\right],$$

*for some given $\theta$ and $l > 0$. When $\theta > 0$, the $l$-step ahead risk-sensitive problem always has a solution. When $\theta < 0$, a solution exists if, and only if, for each $j = 0, \ldots, i-l$,*

*(i) the two matrices*

$$R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & \theta^{-1}I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_p & 0 \\ 0 & \theta^{-1}I_q \end{bmatrix}$$

*have the same inertia, where $P_j$ is the same as in the a posteriori problem.*

*(ii) the sequence of matrices*

$$R_{e,j+m}^{j+1} \triangleq \theta^{-1} I_q + L_{j+m} P_{j+m}^{j+1} L_{j+m}^*, \quad m = 1, \ldots l$$

*are negative definite, where* $P_{j+m}^{j+1}$ *satisfies the Riccati recursion*

$$\begin{array}{rcl} P_{j+m+1}^{j+1} & = & F_{j+m} P_{j+m}^{j+1} F_{j+m}^* + G_{j+m} Q_{j+m} G_{j+m}^* - \\ & & F_{j+m} P_{j+m}^{j+1} L_{j+m}^* \left[ R_{e,j+m}^{j+1} \right]^{-1} L_{j+m} P_{j+m}^{j+1} F_{j+m}^* \end{array}, \quad P_{j+1}^{j+1} = P_{j+1}.$$

*In either case, the risk-sensitive $l$-step predictor is given by*[4]

$$\check{s}_{j+l|j} = L_{j+l} \hat{x}_{j+l}^{j+1},$$

*where* $\hat{x}_{j+m}^{j+1}$ *satisfies the recursion,*

$$\hat{x}_{j+m+1}^{j+1} = F_{j+m} \hat{x}_{j+m}^{j+1} + K_{p,j+m}^{j+1} (\check{s}_{j+m|j+m-l} - L_{j+m} \hat{x}_{j+m}^{j+1}), \quad \hat{x}_{j+1}^{j+1} = \hat{x}_{j+1}$$

*with*

$$K_{p,j+m}^{j+1} = F_{j+m} P_{j+m}^{j+1} L_{j+m}^* \left[ R_{e,j+m}^{j+1} \right]^{-1},$$

*and where* $\hat{x}_j$ *satisfies the recursion,*

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j-l} - L_j \hat{x}_j \end{bmatrix}, \quad \hat{x}_0 = \check{x}_0$$

*with*

$$K_{p,j} = F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1}.$$

We can now state the striking resemblances between the $H^\infty$ and the risk-sensitive filters. The *central* $H^\infty$ filters obtained earlier are essentially risk-sensitive filters with parameter $\theta = -\gamma^{-2}$.[5] Note, however, that at each level $\gamma$, the $H^\infty$ filters are not

[4] We also have $\check{s}_{k|k-l} = L_k F_{k-1} \ldots F_0 \check{x}_0$ for $k = 0, \ldots, l-1$.

[5] This leads to a stochastic interpretation of the central $H^\infty$ filters (and controllers). In contrast to $H^2$ filters (and controllers) that minimize an expected quadratic cost, the central $H^\infty$ filters (and controllers) minimize an expected exponential-quadratic cost. We shall have more to say about this interpretation in Chapter 9. Here instead let us recall that we had earlier, *i.e.*, in Chapter 1, obtained another stochastic interpretation of the central $H^\infty$ filters and controllers — namely that they are maximum entropy solutions.

unique, whereas for each $\theta$, the risk-sensitive filters are unique. Also, the risk-sensitive filters generalize to the $\theta > 0$ case. It is also noteworthy that the *optimal* $H_\infty$ filter corresponds to the risk-sensitive filter with $\bar{\theta} = -\gamma_{opt}^{-2}$, and that $\bar{\theta}$ is that value for which the minimizing property of $\bar{J}_i$ breaks down and $\mu_i(\theta)$ becomes infinite. This relationship between the optimal $H_\infty$ filter and the corresponding risk-sensitive filter was first noted in [GD88].

### 4.2.3 Risk-Sensitive Control

We should also mention that the risk-sensitive framework can be used to study control problems. To this end, consider the (possibly time-varying) state-space model,

$$\left\{ \begin{array}{rcl} \mathbf{x}_{i+1} & = & F_i\mathbf{x}_i + G_{1,i}\mathbf{w}_i + G_{2,i}\mathbf{u}_i \\ \mathbf{s}_i & = & L_i\mathbf{x}_i \end{array} \right. ,0 \leq i \leq N \tag{4.2.21}$$

where $\{\mathbf{w}_i\}$ is the exogenous input, which is assumed to be a zero-mean Gaussian stochastic process with variance,

$$E\mathbf{w}_i\mathbf{w}_j^* = Q_i\delta_{ij}, \tag{4.2.22}$$

$\mathbf{s}_i$ is the signal we intend to regulate, and $\mathbf{u}_i$ is the control signal used to influence the dynamics of the system. Moreover, the initial state, $\mathbf{x}_0$, is also a zero-mean Gaussian random variable, independent of the $\{\mathbf{w}_i\}$, with variance, $\Pi_0$.

Recall that in the $H^2$ case, the control signals, $\{\mathbf{u}_i\}$ were chosen so as to minimize the expected value of the quadratic cost,

$$\mathbf{J}_N^c = \sum_{j=0}^{N} \mathbf{s}_i^* R_i^c \mathbf{s}_i + \sum_{j=0}^{N} \mathbf{u}_i^* Q_i^c \mathbf{u}_i + \mathbf{x}_{N+1} P_{N+1}^c \mathbf{x}_{N+1}, \tag{4.2.23}$$

where the $\{Q_i^c, R_i^c\}$ and $P_{N+1}^c$ are given positive semidefinite weighting matrices.

In risk-sensitive control, as in risk-sensitive estimation, the objective is to choose the controls so as to minimize the expected value of an exponential-quadratic cost, *i.e.*,

$$\min_{\{\mathbf{u}_i\}} \left( -\frac{2}{\theta}\log\left[ E\exp(-\frac{\theta}{2}\mathbf{J}_N^c) \right] \right), \tag{4.2.24}$$

where $\theta$ is, once more, referred to as the risk-sensitivity parameter. The remarks presented at the end of Sec. 4.2.1 on risk-neutral, risk-seeking, and risk-averse estimators apply to the controllers that satisfy (4.2.24) in the exact same way, and therefore will not be repeated here.

Moreover, we should stress that, as was the case in Sec. 1.5, we can have either full information or measurement feedback risk-sensitive controllers. When the control signals that satisfy (4.2.24) are allowed to be causal functions of the states and exogenous inputs, *i.e.*, the $\{\mathbf{x}_i, \mathbf{w}_i\}$, then the controller is referred to as a full information controller. In the measurement feedback case, $\mathbf{u}_i$ is allowed only to be a causal function of a certain observation process,

$$\mathbf{y}_i = H_i\mathbf{x}_i + \mathbf{v}_i, \tag{4.2.25}$$

where $\{\mathbf{v}_i\}$ is a zero-mean Gaussian random process such that

$$E\begin{bmatrix} \mathbf{w}_i \\ \mathbf{v}_i \\ \mathbf{x}_0 \end{bmatrix}\begin{bmatrix} \mathbf{w}_j^* & \mathbf{v}_j^* & \mathbf{x}_0^* \end{bmatrix} = \begin{bmatrix} Q_i\delta_{ij} & 0 & 0 \\ 0 & R_i\delta_{ij} & 0 \\ 0 & 0 & \Pi_0 \end{bmatrix}. \tag{4.2.26}$$

We shall not go any further into the details of risk-sensitive control here. We just mention, in passing, that its solution is also amenable to the Krein space techniques studied so far. Control problems (especially of the $H^\infty$ type) will be studied in Chapter 6.

## 4.3 Quadratic Dynamic Games

Game theory is a vast subject with numerous applications in economics and the social sciences [NM44, KT50, Mor94, Col95]. More recently, there has also been considerable use of game-theoretic ideas and methods in estimation and control [Isa65, Ber64, BH69, KS88, Pon90, BO82]. Even a brief glimpse at this field is well beyond the intentions and scope of this thesis. However, a small subset of game-theoretical problems, namely the class of quadratic dynamic games, is closely related to the problems studied in this thesis and lends itself to analysis by the methods that we have used so far.[6] The reason for this is that quadratic games can be immediately related to certain indefinite quadratic forms whose properties can be studied via Krein space methods. Indeed the (min-max) solutions to such games are found by stationarizing these quadratic forms and checking for certain minimizing and maximizing conditions.

### 4.3.1 General Remarks

To gain an understanding of quadratic games, consider the quadratic cost function,

$$J(a,b) = \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{4.3.1}$$

where $a \in \mathcal{C}^n$ and $b \in \mathcal{C}^m$ are arbitrary vectors and $\{A, B, C\}$ are given matrices (with $A$ and $C$ Hermitian). Suppose now that there are two players, say player $I$ and player $II$, where player $I$ can choose $a$ and would like to do so to minimize $J(a,b)$, and where player $II$ would like to maximize $J(a,b)$ through its choice of $b$. The central question here (and of game theory in general) is what the optimal strategies of the players should be.

To this end, suppose that player $I$ has access to player $II$'s choice of $b$. Therefore player $I$ will choose $a$ such that $J(a,b)$ is minimized for that value of $b$. In this case, player $II$'s best strategy will be to choose $b$ such that this minimum value of $J(a,b)$ is maximized. This then leads to the following (so-called max-min) problem,

$$\max_b \ \min_a \ \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{4.3.2}$$

Now clearly the condition for having a (unique) minimum over $a$ is that $A > 0$. Once this is the case it easy to see that

$$\min_a \ J(a,b) = b^*(C - B^*A^{-1}B)b \quad \text{and} \quad \arg\min_a \ J(a,b) = A^{-1}Bb. \tag{4.3.3}$$

Now $b^*(C - B^*A^{-1}B)b$ will have a maximum over $b$ if, and only if, $C - B^*A^{-1}B < 0$.

[6] Incidentally, quadratic dynamic games are the ones that have been most studied in control theory, and have turned out to be closely related to $H^\infty$ and risk-sensitive control.

Thus, we conclude that

$$\max_b \min_a \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0, \tag{4.3.4}$$

and that the optimal max-min strategy is $a = 0$ and $b = 0$. Moreover the condition for the existence of a max-min solution is that

$$A > 0 \quad \text{and} \quad C - B^*A^{-1}B < 0. \tag{4.3.5}$$

Of course, we can also reverse the situation (and assume that player $II$ has access to player $I$'s choice) so that we are led to the (so-called min-max) problem,

$$\min_a \max_b \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{4.3.6}$$

Using a similar argument, we can see that

$$\min_a \max_b \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0, \tag{4.3.7}$$

and that the optimal min-max strategy is $a = 0$ and $b = 0$. The condition for the existence of a min-max strategy, however, now is

$$C < 0 \quad \text{and} \quad A - BC^{-1}B^* > 0. \tag{4.3.8}$$

Note that although the max-min and min-max strategies have turned out to be the same (which is generally the case, since they are both found from stationarizing the quadratic form, $J(a, b)$), the existence conditions for the two solutions are different (since one problem requires first a minimum over $a$ and then a maximum over $b$, and the other problem the vice versa). The condition that the max-min and min-max solutions exist simultaneously is called the *saddle point* condition, which using (4.3.5) and (4.3.8) is readily seen to be

$$A > 0 \quad \text{and} \quad C < 0. \tag{4.3.9}$$

[The above two conditions are obviously necessary for (4.3.5) and (4.3.8) to hold. They are also sufficient, since they imply $C - B^*A^{-1}B < 0$ and $A - BC^{-1}B^* > 0$.]

In this case the optimal strategy, say $(a_{opt}, b_{opt})$, is called the saddle point strategy, and has the property that,

$$J(a, b_{opt}) \geq J(a_{opt}, b_{opt}) \geq J(a_{opt}, b), \tag{4.3.10}$$

for all possible strategies, $(a, b)$. Indeed it is straightforward to see,

$$J(a, b_{opt}) = J(a, 0) = a^* A a \geq 0 = J(a_{opt}, b_{opt}) = 0 \geq b^* C b \geq J(0, b) = J(a_{opt}, b).$$

In other words, either player may lose by choosing a strategy differing from the saddlepoint strategy.

The above problem is, of course, very simplistic[7]. However, it serves to illustrate the major points of quadratic game theory. Before listing these major points, it will be useful to consider a slight generalization of the above problem, in which the cost function is replaced by

$$J(a, b, y) = \begin{bmatrix} a^* & b^* & y^* \end{bmatrix} \begin{bmatrix} A & B & D \\ B^* & C & E \\ D^* & E^* & F \end{bmatrix} \begin{bmatrix} a \\ b \\ y \end{bmatrix}, \tag{4.3.11}$$

in which $y \in \mathcal{C}^p$ is a vector which players $I$ and $II$ both have access to, and $\{D, E, F\}$ are given matrices (with $F$ Hermitian). In this case, it is straightforward to see that the stationary point of $J(a, b, y)$ over $(a, b)$ is given by

$$\begin{bmatrix} a_{opt} \\ b_{opt} \end{bmatrix} = - \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}^{-1} \begin{bmatrix} D \\ E \end{bmatrix} y, \tag{4.3.12}$$

and that the value of $J(a, b, y)$ at the stationary point is,

$$J(a_{opt}, b_{opt}, y) = y^* \left\{ F - \begin{bmatrix} D^* & E^* \end{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}^{-1} \begin{bmatrix} D \\ E \end{bmatrix} \right\} y. \tag{4.3.13}$$

The condition for this stationary point to be a max-min, min-max, or saddle point solution is still given by (4.3.5), (4.3.8) or (4.3.9), respectively.

Using the above examples, we can now list the major points of quadratic game theory as follows:

[7] It is called a *static* quadratic game.

(i) The cost function is an *indefinite* quadratic form.

(ii) The solution to the game (be it a max-min, min-max, or saddle point solution) is found by stationarizing the indefinite quadratic form over the strategies of the two players (since max-min points, min-max points, and saddle points are all special cases of a stationary point).

(iii) The condition for the existence of a solution of a certain kind (such as a max-min, min-max, or saddle point solution) is given by a certain inertia condition (such as (4.3.5), (4.3.8) or (4.3.9)) on the coefficient matrix of the game's indefinite quadratic form.

The above remarks indicate the connections between quadratic game theory and the Krein space approach developed in this thesis. To solve a quadratic game, using this approach, all we need to do is identify a Krein space model with the indefinite quadratic form of the game, and to then compute the stationary point via a Krein space projection. The conditions for the existence of a max-min, min-max, or saddle point solution is then given by various inertia conditions on the Gramians of certain Krein space variables. [For example, if the problem has state-space structure, the projection (hence the stationary point) can be computed via the Krein space Kalman filter, and the inertia conditions can be checked from variables that are by-products of the Kalman recursions (see Sec. 2.7).]

To illustrate the above remarks, consider the cost function, $J(a,b,y)$, of (4.3.11). To identify a Krein space model with it, we can use the approach of Sec. 2.5. To this end, let us define $s \triangleq \begin{bmatrix} a \\ b \end{bmatrix}$ and the Krein space variables, $\{\mathbf{s},\mathbf{y}\}$, such that

$$\left\langle \begin{bmatrix} \mathbf{s} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{s} \\ \mathbf{y} \end{bmatrix} \right\rangle = \begin{bmatrix} R_s & R_{sy} \\ R_{ys} & R_y \end{bmatrix} \triangleq \begin{bmatrix} \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} & \begin{bmatrix} D \\ E \end{bmatrix} \\ \begin{bmatrix} D^* & E^* \end{bmatrix} & F \end{bmatrix}^{-1}. \tag{4.3.14}$$

This then allows us to write,

$$J(a,b,y) = J(s,y) = \begin{bmatrix} s^* & y^* \end{bmatrix} \begin{bmatrix} R_s & R_{sy} \\ R_{ys} & R_y \end{bmatrix}^{-1} \begin{bmatrix} s \\ y \end{bmatrix}. \tag{4.3.15}$$

Using Theorem 2.5.2, the stationary point of $J(s,y)$ over $s$, which we denote by $\hat{s}$, is given by the Krein space projection of $\mathbf{s}$ on $\mathcal{L}(\mathbf{y})$, *i.e.*,

$$\hat{s} = R_{sy} R_y^{-1} y. \tag{4.3.16}$$

Moreover, the conditions for the above stationary point to be a max-min, min-max, or saddlepoint solution are related to the inertia of the matrix,

$$(R_s - R_{sy} R_y^{-1} R_{ys})^{-1} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}. \tag{4.3.17}$$

[Recall that methods to determine this inertia have been developed in Secs. 2.5.3 and 2.6.1.]

We close this section with one more remark. The quadratic games considered so far were all static. When the vectors $a$, $b$ and $y$ are aggregate vectors of some time series, $\{a_i\}$, $\{b_i\}$ and $\{y_i\}$, *i.e.*,

$$a = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix} \quad , \quad b = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_N \end{bmatrix} \quad , \quad y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix} \tag{4.3.18}$$

that are dynamically related (say, via a state-space model), then the game is referred to as a quadratic *dynamic* game. In such games, both players (or occasionally one of the two) have only causal access to the observations, $\{y_i\}$, or to the other player's strategy. In other words, the strategy chosen at time $i$ by, say, player $I$, (which we recall is just $a_i$) is only allowed to be a function of current and past observations, $\{y_j, j \le i\}$ (and occasionally of current and past strategies of player $II$, $\{b_j, j \le i\}$). This causality restriction affects both the structure of the optimal strategies and the actual conditions for the existence of a max-min, min-max or saddle point solution. Although the solutions in this case are somewhat more complex, they can either be obtained via a canonical factorization approach[8] of the matrix, $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$, (along

[8] The relationship between game theory and indefinite factorizations were apparently first studied by Yakubovich [Yak70b] and Banker [Ban73].

the lines of Secs. 1.3.1 and 1.6.2) or by performing a recursive stationarization of the quadratic form $J(a, b, y)$ (along the lines of Sec. 2.7). In summary, there is no essential difference between the methods for solving dynamic games with a causality restriction and the methods for finding causal estimators and controllers that have been studied so far.

### 4.3.2 Specific Examples

We now very briefly turn to some special cases of the general quadratic games described above. The examples we consider are in the contexts of (state-space) estimation and control and are intimately related to the $H^\infty$ and risk-sensitive formulations studied earlier. In essence, we shall see that the central $H^\infty$ filters and controllers, and the risk-sensitive filters and controllers, can be considered as the solutions to certain quadratic dynamic games.

#### Estimation Problems

Consider the standard state-space model

$$\left\{\begin{array}{rcl} x_{i+1} & = & F_i x_i + G_i u_i, \quad x_0 \\ y_i & = & H_i x_i + v_i \\ s_i & = & L_i x_i \end{array}\right. \qquad 0 \leq i \leq N \tag{4.3.19}$$

where, as usual, the initial condition $x_0$, and the disturbances, $\{u_i, v_i\}$, are unknown, $\{y_i\}$ is the observations sequence, and $\{s_i\}$ is the signal we intend to estimate (using the observations). In the game-theoretic approach to estimation, the unknown disturbances $\{x_0, \{u,v_i\}_{i=0}^N\}$ are considered to be an adversary whose objective is to disrupt our attempt at estimating the $\{s_i\}$. To be more specific, consider the a posteriori filtering problem where the estimates are given by $\check{s}_{i|i} = \mathcal{F}_{f,i}(y_0, \ldots, y_i)$, with $\mathcal{F}_f(\cdot)$ being the estimation strategy that must be determined. Now one measure of how well our estimation scheme is performing is the estimation error energy,

$$\sum_{i=0}^{N} (s_i - \check{s}_{i|i})^* (s_i - \check{s}_{i|i}). \tag{4.3.20}$$

Roughly speaking, our objective is to choose the estimator, $\mathcal{F}_f(\cdot)$ so as to minimize the estimation error energy, whereas nature's (our adversary's) objective is to choose the disturbances so as to maximize it. To put the situation on an equal footing, one must include a penalty on the choice of large disturbances (otherwise nature could make the estimation error arbitrarily large by choosing arbitrarily large disturbances). This is achieved by adding a negative definite quadratic term to the above cost that penalizes large disturbances. Thus we are led to the following cost function,

$$J_N = \sum_{i=0}^{N}(s_i - \check{s}_{i|i})^*(s_i - \check{s}_{i|i}) - \gamma^2\left(x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N}u_i^*Q_i^{-1}u_i + \sum_{i=0}^{N}v_i^*R_i^{-1}v_i\right), \quad (4.3.21)$$

where $\Pi_0$, $Q_i$ and $R_i$ are given positive definite weighting matrices, and where $\gamma$ is a scalar that determines the respective contributions of the estimation error and disturbance energies to the cost function $J_N$.

The game-theoretic estimation problem can be thus formulated as follows,

$$\min_{\{\check{s}_{i|i}\}} \max_{x_0,\{u_i\},\{v_i\}} \sum_{i=0}^{N}(s_i - \check{s}_{i|i})^*(s_i - \check{s}_{i|i}) - \gamma^2\left(x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N}u_i^*Q_i^{-1}u_i + \sum_{i=0}^{N}v_i^*R_i^{-1}v_i\right), \quad (4.3.22)$$

where $\check{s}_{i|i} = \mathcal{F}_{f,i}(y_0, \ldots, y_i)$ and the $\{x_0, \{u_i\}, \{v_i\}, \{y_i\}, \{s_i\}\}$ are related via the state-space model (4.3.19).

Now the quadratic form, $J_N$, is an indefinite quadratic form of the standard type that has been studied since Chapter 2. Therefore we can readily use the Krein space Kalman filter to compute the stationary point of $J_N$ and to check whether the stationary point has the desired (maximum over $\{x_0, \{u_i\}, \{v_i\}\}$, minimum over $\{\check{s}_{i|i}\}$) properties (via certain inertia conditions obtained from the Kalman filter recursions). Indeed further inspection of $J_N$ reveals that, apart from a minus sign, it is the exact same indefinite quadratic form that was studied in the $H^\infty$ and risk-sensitive a posteriori estimation problems. Following through with the arguments presented there, (which we will not repeat here) shows that the central $H^\infty$ and risk-sensitive a posteriori filters are, in fact, given by the solution to the game (4.3.22).[9] Similar statements hold for the a priori, smoothed, and $l$-step ahead central $H^\infty$ and risk-sensitive filters as well.

[9]This observation is the whole premise of the game-theoretic approach to $H^\infty$ estimation and control (see *e.g.*, [BB95, LAKG92]).

The above remarks conclude our brief study of game-theoretic estimation and we now turn our attention to control.

### Control Problems

Consider the state-space model,

$$\left\{ \begin{array}{rcll} x_{i+1} & = & F_i x_i + G_{1,i} w_i + G_{2,i} u_i, & x_0 \\ s_i & = & L_i x_i & \end{array} \right. \quad 0 \leq i \leq N \tag{4.3.23}$$

where $\{w_i\}$ is the exogenous input, $\{s_i\}$ is the signal we want to regulate, and $\{u_i\}$ is the control input. In the full information problem, the control signal $u_i$ is allowed to be a function of the initial state, $x_0$, and of the current and past exogenous inputs, $\{w_j, j \leq i\}$. In the measurement feedback problem, all we have access to is the measurement signal,

$$y_i = H_i x_i + v_i, \tag{4.3.24}$$

where $\{v_i\}$ is an (unknown) measurement disturbance, and therefore the control signal $u_i$ is only allowed to be a function of current and past observations, $\{y_j, j \leq i\}$. In either case, the control objective is to minimize the quadratic cost,

$$\sum_{j=0}^{N} s_i^* R_i^c s_i + \sum_{j=0}^{N} u_i^* Q_i^c u_i + x_{N+1} P_{N+1}^c x_{N+1}. \tag{4.3.25}$$

In the game-theoretic approach to control the disturbances $x_0$, $\{w_i\}$, and in the case of measurement feedback control, $\{v_i\}$, are considered to be adversaries that choose to maximize the above cost. As in estimation, to level the playing field, we must penalize the choice of large values of disturbances on the part of nature (our opponent). This is achieved by adding a negative definite quadratic term to the above cost that penalizes large disturbances. We are thus led to the following cost functions: for the full information problem,

$$J_N^c = \sum_{j=0}^{N} s_i^* R_i^c s_i + \sum_{j=0}^{N} u_i^* Q_i^c u_i + x_{N+1} P_{N+1}^c x_{N+1} - \gamma^2 \left( x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \right), \tag{4.3.26}$$

and for the measurement feedback problem,

$$J_N^c = \sum_{j=0}^{N} s_i^* R_i^c s_i + \sum_{j=0}^{N} u_i^* Q_i^c u_i + x_{N+1} P_{N+1}^c x_{N+1} - \gamma^2 \left( x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i \right), \tag{4.3.27}$$

where $\Pi_0$, $Q_i$ and $R_i$ are given positive definite weighting matrices, and where $\gamma$ is a scalar that determines the respective contributions of the original quadratic cost and disturbance energies to the cost function $J_N^c$.

The game-theoretic estimation problem can be thus formulated as follows: for the full information problem,

$$\min_{\{u_i\}} \max_{x_0, \{u_i\}} J_N^c, \tag{4.3.28}$$

where $u_i = \mathcal{F}_i(x_0, w_0, \ldots, w_i)$, $J_N^c$ is given by (4.3.26), and the $\{s_i\}$, and $\{x_0, \{w_i, u_i\}\}$ are related via (4.3.23); for the measurement feedback problem,

$$\min_{\{u_i\}} \max_{x_0, \{u_i\}, \{v_i\}} J_N^c, \tag{4.3.29}$$

where $u_i = \mathcal{F}_i(y_0, \ldots, y_i)$, $J_N^c$ is given by (4.3.27), and the $\{s_i, y_i\}$ and $\{x_0, \{w_i, u_i, v_i\}\}$ are related via (4.3.23) and (4.3.24).

The quadratic form, $J_N^c$, is an indefinite quadratic form, but not of the type that has been studied since Chapter 2. In Chapter 6 we shall show that this quadratic form can be motivated, and introduced, via the concept of a dual basis, and that its stationary point can be obtained by certain projections, not in the original spaces, but in the resulting dual spaces. [This is yet another manifestation of the duality between estimation and (full information) control, that we first noted in Sec. 1.5.1. In fact, the aforementioned duality is geometric while the one presented in Sec. 1.5.1 was purely algebraic.] We shall not give the details of how to compute these stationary points (or how to verify the min-max or saddle point conditions) here, since that topic will be studied in length in Chapter 6. Instead, we close this section by remarking that the indefinite quadratic form $J_N^c$ is exactly the same (apart from a minus sign) as the indefinite quadratic form that is obtained from $H^\infty$ and risk-sensitive control. In particular, it turns out that the full information and measurement feedback $H^\infty$ and risk-sensitive controllers coincide with the solutions to the quadratic dynamic games (4.3.28) and (4.3.29), respectively.

## 4.4 Finite Memory Adaptive Filtering

We now consider an application of the Krein space Kalman filter to the problem of finite memory (or sliding window) adaptive filtering.[10] It has been recently shown [SK94b] that a unified derivation of adaptive filtering algorithms, and their corresponding fast versions, can be obtained by properly recasting the adaptive problem into a standard state-space estimation problem. We now verify that if we further allow the elements of the state-space model to belong to a Krein space, then the so-called sliding window or finite-memory problem can also be handled within the same framework. In fact, this framework also allows us to easily consider more general sliding patterns with windows of varying lengths, as explained ahead. Moreover, we shall obtain a physical interpretation of innovations with negative Gramian, and see that it corresponds to the loss of information.

### 4.4.1 The Standard Problem

The finite memory adaptive filtering problem can be formulated as follows: given the input-output pairs $\{h_j, d_j\}$ where $h_j \in \mathcal{C}^{1\times n}$ is a known input vector and $d_j \in \mathcal{C}$ is a known output scalar, recursively determine estimates of an unknown weight vector $w \in \mathcal{C}^n$, such that the scalar quadratic form

$$J_i(w, d_{i-l_i+1}, \ldots, d_i, h_{i-l_i+1}, \ldots, h_i) = w^*\Pi_0^{-1}w + \sum_{j=i-l_i+i}^{i}(d_j - h_j w)^*(d_j - h_j w), \quad (4.4.1)$$

with $\Pi_0 > 0$ a given weighting matrix and the $\{l_i\}$ given (time-varying) window lengths, is minimized for each $i$.

Since $J_i$ is a function of the pairs $\{h_j, d_j\}_{j=i-l_i+i}^{i}$, at each time instant $i$, we are interested in determining the estimate of $w$ using only the data given over an interval of length $l_i$. The quantity $l_i \geq 0$ is therefore referred to as the length (or memory) of the sliding window.

Note that we allow for a time-variant window length. To clarify this point, consider the example of Fig. 4.2 where at time $i$ we have a window of length $l_i = l$. At the next

[10] Refer back to Sec. 1.2.1 for a brief introduction to adaptive filtering. The topic will be further studied in Chapters 9 and 11.

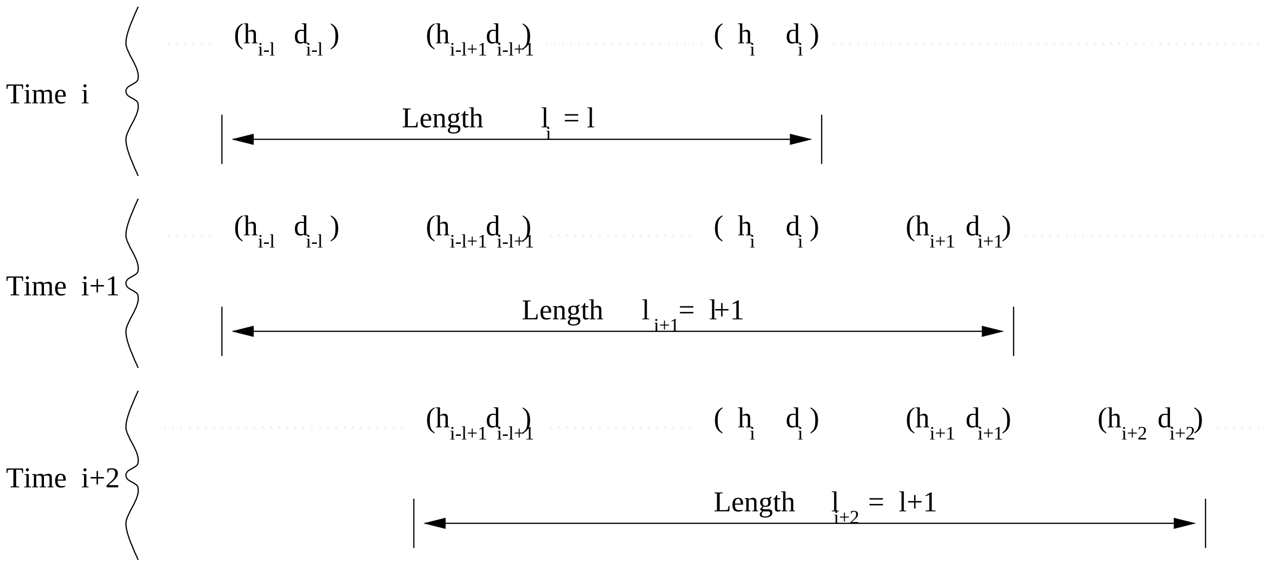


Figure 4.2: Sliding window with varying window length.

time instant we add the data point $\{h_{i+1}, d_{i+1}\}$, so that the window length changes to $l_{i+1} = l + 1$. At time $i + 2$ we add the data point $\{h_{i+2}, d_{i+2}\}$ and drop the data point $\{h_{i-l}, d_{i-l}\}$ so that the window length remains $l_{i+2} = l+1$. In a similar fashion, more general sliding window patterns can be considered as well.

To recast expression (4.4.1) into the usual quadratic form considered in this chapter, the lower index of the summation term needs to start at the fixed time 0. For this purpose, we rewrite $J_i$ as follows

$$J_i(w, d_{i-l_i+1}, \ldots, d_i, h_{i-l_i+1}, \ldots, h_i) = w^*\Pi_0^{-1}w + \sum_{j=i-l_i+1}^{i}(d_j - h_jw)^*(d_j - h_jw)$$

$$+\sum_{j=0}^{i-l_i}(d_j - h_jw)^*(d_j - h_jw) - \sum_{j=0}^{i-l_i}(d_j - h_jw)^*(d_j - h_jw) \qquad (4.4.2)$$

where we have added and subtracted identical terms. We now invoke a change of variables and substitute the time index $i$ by another time index $k$ that allows us to replace $J_i$ by a $\bar{J}_k$. The new index $k$ has the property whenever a new data point is added (*i.e.* , $i$ is incremented) then $k$ is incremented. However, whenever a data point is discarded from the window, $k$ is incremented as well. Thus if at time $i$ the length of the window is $l_i$, then the index $k$ will run from 0 to $2i - l_i + 1$ (since there will have been $i$ data points added and $i - l_i + 1$ data points removed). To be more specific, the change of variables is as follows.

(a) At each time $i$, since the data point $\{h_i, d_i\}$ is added, we increment the index $k$ and define

$$\bar{d}_k = d_i, \quad \bar{h}_k = h_i \quad \text{and} \quad \bar{R}_k = 1. \tag{4.4.3}$$

(b) If at time $i$ the data point $\{h_{i-l_i}, d_{i-l_i}\}$ is removed, we increment the index $k$ once more and define

$$\bar{d}_k = d_{i-l_i}, \quad \bar{h}_k = h_{i-l_i} \quad \text{and} \quad \bar{R}_k = -1. \tag{4.4.4}$$

With this convention we may write the quadratic form

$$J_i(w, d_{i-l_i+1}, \ldots, d_i, h_{i-l_i+1}, \ldots, h_i),$$

as

$$J_i = \bar{J}_k(w, d_0, \ldots, d_k, h_0, \ldots, h_k) = w^* \Pi_0^{-1} w + \sum_{j=0}^{k} (\bar{d}_j - \bar{h}_j w)^* (\bar{d}_j - \bar{h}_j w), \tag{4.4.5}$$

which is of the form that we have been considering in this thesis since Chapter 2. Note that the quadratic form $\bar{J}_k(w, d_0, \ldots, d_k, h_0, \ldots, h_k)$ is indefinite, since whenever a data point is dropped we have $\bar{R}_k = -1$. We can therefore use Krein space methods to solve the problem.

Using the same approach that we have used so far, we now construct the partially equivalent state-space model to the indefinite quadratic form $\bar{J}_k$. Thus

$$\left\{ \begin{array}{rcl} \mathbf{x}_{j+1} & = & \mathbf{x}_j, \quad \mathbf{x}_0 = \mathbf{w} \quad 0 \le j \le k \\ \bar{d}_j & = & \bar{h}_j \mathbf{x}_k + \mathbf{v}_k \end{array} \right. \tag{4.4.6}$$

with $\Pi > 0$, $Q_j = 0$, $S_j = 0$ and $\bar{R}_j$ as in (4.4.3) and (4.4.4).

We can now state the following result.

**Theorem 4.4.1 (Finite Memory Adaptive Filter)** *The finite memory adaptive filter is given by the following recursions:*

*(a) For updating, the data point* $\{h_i, d_i\}$ *at time* $i$ *we have*

$$\hat{w}_{|i:i-l_{i-1}} = \hat{w}_{|i-1:i-l_{i-1}} + K_{p,k}(d_i - h_i \hat{w}_{|i-1:i-l_{i-1}}) \tag{4.4.7}$$

*where* $\hat{w}_{|i:j}$ *is the estimate when the sliding window encompasses all the data from time* $j$ *to time* $i$*, and*

$$K_{p,k} = P_k h_i^* R_{e,k}^{-1} \quad , \quad R_{e,k} = 1 + h_i P_k h_i^* \tag{4.4.8}$$

*and where* $P_k$ *satisfies the recursion*

$$P_{k+1} = P_k - K_{p,k} R_{e,k} K_{p,k}^*, \quad P_0 = \Pi_0. \tag{4.4.9}$$

*(b) For downdating, the data point* $\{h_{i-l_i}, d_{i-l_i}\}$ *at time* $i$ *we have*

$$\hat{w}_{|i:i-l_i+1} = \hat{w}_{|i:i-l_i} + K_{p,k}(d_{i-l_i} - h_{i-l_i} \hat{w}_{|i:i-l_i}) \tag{4.4.10}$$

*where*

$$K_{p,k} = P_k h_{i-l_i}^* R_{e,k}^{-1} \quad , \quad R_{e,k} = -1 + h_{i-l_i} P_k h_{i-l_i}^* \tag{4.4.11}$$

*and* $P_k$ *satisfies the recursion*

$$P_{k+1} = P_k - K_{p,k} R_{e,k} K_{p,k}^*. \tag{4.4.12}$$

*Moreover, the above solutions for* $\hat{w}_{|i:j}$ *always correspond to a minimum, and in particular*

$$R_{e,k} = 1 + h_i P_k h_i^* > 0, \tag{4.4.13}$$

*when we are updating, and*

$$R_{e,k} = -1 + h_{i-l_i} P_k h_{i-l_i}^* < 0, \tag{4.4.14}$$

*when we are downdating.*

**Proof:** The solutions given by (a) and (b) in the above Theorem are simply the Krein space Kalman filter recursions for the state-space model (4.4.6) which we know computes the stationary point of $\bar{J}_k$ over $w$. However, this stationary point is always a minimum since

$$\frac{\partial^2 \bar{J}_k}{\partial w^2} = \frac{\partial^2 J_i}{\partial w^2} = \Pi_0^{-1} + \sum_{j=i-l_i+1}^{i} h_j^* h_j > 0,$$

(recall that $\Pi_0^{-1} > 0$).

Using Lemma 2.7.3, having a minimum means that $R_{e,k}$ and $R_k$ have the same inertia for all $k$. Thus the statements (4.4.13) and (4.4.14) readily follow.

■

The fact that whenever we drop data we have $R_{e,k} < 0$ has an interesting interpretation. Consider the equation

$$P_{k+1} = P_k - K_{p,k} R_{e,k} K_{p,k}$$

If we drop data at step $k$ we would expect $P_{k+1}$ to get larger (more positive-definite) than $P_k$. This can only happen if $R_{e,k} < 0$. Thus, we may infer that innovations with negative Gramian correspond to a loss of information.

The above discussion puts the problem of finite memory adaptive filtering into the same state-space estimation framework as conventional adaptive filtering techniques (see [SK94b]). Therefore the various algorithmic extensions discussed there may be applied to finite memory problems, albeit that we now need to consider a Krein space. We shall not give the details here, but shall just mention that when the elements of the input vectors $\{h_i\}$ form a time sequence, viz.,

$$h_i = \begin{bmatrix} u_i & u_{i-1} & \ldots & u_{i-n+1} \end{bmatrix},$$

and when the window length is constant, *i.e.* , $l_i \equiv l$, then the state-space model (4.4.6) is periodic with period $T = 2$, and we may speed up the the estimation algorithm by a so-called Chandrasekhar-type recursion. Similar results, obtained via a different approach, have been reported in [Hou92].

## 4.5 Conclusion

In this chapter we studied the problems of risk-sensitive estimation and control, quadratic game theory, and finite memory adaptive filtering, using the framework of linear estimation in Krein spaces that was developed in Chapter 2. The theme that unites all these problems is that they can be related, albeit in an indirect fashion, to certain indefinite quadratic forms. We anticipate that there should be other problems that are amenable to this approach, and mention, in passing, an interesting application to recursive algorithms for (suboptimal) total least-squares problems that has been given in [SHK96a].

# Chapter 5

# Square-Root Arrays and Chandrasekhar Recursions

Using the observation that $H^\infty$ filtering coincides with Kalman filtering in Krein space, in this chapter we develop square-root arrays and Chandrasekhar recursions for $H^\infty$ filtering problems. These are the generalizations of the conventional square-root arrays and Chandrasekhar recursions to the Krein space setting. The $H^\infty$ square-root algorithms involve propagating the indefinite square-root of the quantities of interest and have the property that the appropriate inertia of these quantities is preserved. For systems that are constant, or whose time-variation is structured in a certain way, the Chandrasekhar recursions allow a reduction in the computational effort per iteration from $O(n^3)$ to $O(n^2)$, where $n$ is the number of states. The $H^\infty$ square-root and Chandrasekhar recursions both have the interesting feature that one does not need to explicitly check for the positivity conditions required of the $H^\infty$ filters. These conditions are built into the algorithms themselves so that an $H^\infty$ estimator of the desired level exists if, and only if, the algorithms can be executed. [The results of this chapter were first reported in [HSK94c].]

## 5.1 Introduction

In Chapter 2 we presented a self-contained theory for linear estimation in Krein spaces with the objective of unifying the $H^2$, $H^\infty$, risk-sensitive and game-theoretic approaches to estimation and control. There we claimed that the major bonus of this approach is that, apart from rather more transparent derivations of existing results (as done in Chapters 3 and 4), it shows a way to apply to the $H^\infty$ (and these other) settings many of the results developed for Kalman filtering and LQG control over the last three decades. In this chapter we intend to support this claim by showing how the Krein space approach can be used to naturally extend the (numerically superior) square-root and (fast) Chandrasekhar array recursions of conventional Kalman filtering to $H^\infty$ filtering.

The (so-called) square-root array algorithms were devised in the late 1960's [DM69, KBS71, Har72], and, for several reasons, are currently more often used to implement the conventional Kalman filter. These algorithms are closely related to the (so-called) QR method for solving systems of linear equations [BG66, GL89, HJ90, Str93], and have the properties of better conditioning, reduced dynamical range, and the use of orthogonal transformations, which typically leads to stabler algorithms.

Furthermore for constant systems, or in fact for systems where the time-variation is structured in a certain way, the Riccati recursions and the square-root recursions, both of which take $O(n^3)$ elementary computations (flops) per iteration (where $n$ is the dimension of the state-space), can be replaced by more efficient recursions, which require only $O(n^2)$ flops per iteration [Kai72, MSK74, SK94a]. These algorithms are analogous to certain equations invented in 1943-47 by the astrophysicists Ambartsumian [Amb43] and Chandrasekhar [Cha48] (hence the name Chandrasekhar recursions), are also closely related to the concept of displacement structure [KKM79, Say92].

One immediate fall-out of our observation that $H^\infty$ filtering coincides with Kalman filtering in Krein space is that it allows us to generalize these square-root arrays and Chandrasekhar recursions to the $H^\infty$ setting. Both these algorithms involve propagating (indefinite) square-roots of the quantities of interest and guarantee that the proper inertia of these quantities is preserved. Furthermore, the condition required for the existence of the $H^\infty$ filters is built into the algorithms — if the algorithms can be carried out, then an $H^\infty$ filter of the desired level exists, and if they cannot be executed then such $H^\infty$ filters do not exist. This can be a significant simplification of the existing algorithms.

The remainder of the chapter is organized as follows.

The conventional square-root array algorithms are introduced in Sec. 5.2 along with some of their properties. In Sec. 5.3 we begin the development of the $H^\infty$ square-root array algorithms and mention why they are natural extensions of their conventional counterparts. We initially encounter some difficulties in generalizing these arrays to the Krein space setting, and in order to alleviate them we then introduce the concept of indefinite square-roots, and study the inertia properties of the Gramian matrices in the $H^\infty$ filtering problem in some detail. These inertia properties are related to the triangularization of matrices via $J$-unitary transformations and will be crucial for the development of the $H^\infty$ square-root array and Chandrasekhar recursions. Finally, the general form of the $H^\infty$ a posteriori and a priori filters are given in Sec. 5.3.1 and the central filters in Sec. 5.3.2.

The conventional (fast) Chandrasekhar recursions, along with several of their properties, is given in Sec. 5.4. Sec. 5.5.1 extends these recursions to the $H^\infty$ setting, and Sec. 5.5.2 gives the corresponding central $H^\infty$ filters.

In closing this introduction we note that there are many variations to the conventional square-root arrays and Chandrasekhar recursions, of which only a few have been considered here. However, the approach adopted here is of sufficient generality that it should allow a reader to extend any other variation of these algorithms to the $H^\infty$ setting.

A brief remark on the notation used in this chapter. To avoid confusion between the various gain vectors used in this chapter, we shall employ the following convention: $K_{p,i}$ will denote the gain vector in the usual Krein space (or Hilbert space) Kalman filter, $K_{f,i}$ the gain vector in the filtered form of the Krein space Kalman filter, and $K_{s,i}$ and $K_{a,i}$ will denote the gain vectors in the $H^\infty$ a posteriori and a priori filters, respectively.

## 5.2 $H^2$ Square-Root Array Algorithms

In state-space estimation problems we begin with a (possibly) time-varying state-space model of the form

$$\begin{cases} x_{j+1} = F_j x_j + G_j u_j, & x_0 \\ y_j = H_j x_j + v_j \end{cases} \tag{5.2.1}$$

where the $\{u_j, v_j\}$ are disturbances whose nature depends on the criterion being used, and where the $\{y_j\}$ are the observed outputs. We shall be, typically, interested in obtaining estimates of some given linear combination of the states, say $s_i \triangleq L_i x_i$, and, most frequently, filtered and predicted estimates, denoted by $\hat{s}_{j|j}$ and $\hat{s}_j$, respectively, that each use the observations $\{y_k, k \le j\}$ and $\{y_k, k < j\}$.

In conventional Kalman filtering the $\{x_0, u_j, v_j\}$ are assumed to be zero-mean random variables with

$$E \begin{bmatrix} x_0 \\ u_i \\ v_i \end{bmatrix} \begin{bmatrix} x_0^* & u_j^* & v_j^* \end{bmatrix} = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i \delta_{ij} & \\ 0 & 0 & R_i \delta_{ij} \end{bmatrix} \ge 0.$$

Moreover, the output covariance of (5.2.1) is assumed to be positive-definite, *i.e.* $R_y > 0$, where $[R_y]_{ij} = E y_i y_j^*$.[1]

Using the $H^2$ criterion, the predicted and filtered estimates are given by $\hat{s}_j = L_j \hat{x}_j$ and $\hat{s}_{j|j} = L_j \hat{x}_{j|j}$, respectively, where $\hat{x}_j$ satisfies the predicted form of the conventional Kalman filter recursions

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j}(y_j - H_j \hat{x}_j), \quad \hat{x}_0 = 0 \tag{5.2.2}$$

and $\hat{x}_{j|j}$ satisfies its filtered form,

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{f,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}), \quad \hat{x}_{-1|-1} = 0 \tag{5.2.3}$$

[$\hat{x}_j$ denotes the predicted estimate of $x_j$, given $\{y_0, \ldots, y_{j-1}\}$, and $\hat{x}_{j|j}$ denotes its filtered estimate, given $\{y_0, \ldots, y_j\}$.] The gain vectors $K_{p,j}$ and $K_{f,j}$ can be computed

[1] One way to ensure the positive definiteness of the output covariance, $R_y$, is to assume that the measurement noise covariance matrix is full rank, *i.e.*, $R_i > 0$. This is often a very reasonable assumption.

in several ways. The most common method uses a certain Riccati recursion, viz.,

$$K_{f,j+1} = P_{j+1}H_{j+1}R_{e,j+1}^{-1} \quad , \quad K_{p,j} = F_jK_{f,j-1} \quad , \quad R_{e,j} = R_j + H_jP_jH_j^* \tag{5.2.4}$$

where $P_j$ satisfies the Riccati recursion,

$$P_{j+1} = F_jP_jF_j^* + G_jQ_jG_j^* - K_{p,j}R_{e,j}K_{p,j}^*, \quad P_0 = \Pi_0. \tag{5.2.5}$$

[The invertibility of the $R_{e,j}$ is guaranteed by the positivity assumption on $R_y$.]

The matrix $P_j$ appearing in this Riccati recursion has the physical meaning of being the variance of the state prediction error, $\tilde{x}_j = x_j - \hat{x}_j$, and therefore has to be positive (semi)definite. Round-off errors can cause a loss of positive-definiteness, thus throwing all the obtained results into doubt. For this, and other reasons (reduced dynamic range, better conditioning, stabler algorithms, etc.) attention has moved in the Kalman filtering community to the so-called square-root array (or factorized) estimation algorithms [DM69, KBS71] that propagate square-root factors of $P_j$, *i.e.* a matrix, $P_j^{1/2}$ say, with positive diagonal entries, and such that

$$P_j = P_j^{1/2}(P_j^{1/2})^* = P_j^{1/2}P_j^{*/2}.$$

Square roots can be similarly defined for the system covariances $\{Q_j, R_j\}$. Then it is in fact not hard to show the following.

Find any orthogonal transformation, say $\Theta_j$,[2] that triangularizes the pre-array shown below

$$\begin{bmatrix} R_i^{1/2} & H_jP_j^{1/2} & 0 \\ 0 & F_jP_j^{1/2} & G_jQ_j^{1/2} \end{bmatrix} \Theta_j = \begin{bmatrix} X & 0 & 0 \\ Y & Z & 0 \end{bmatrix}. \tag{5.2.6}$$

The resulting post-array entries can be checked, by taking squares and using the orthogonality of $\Theta_j$, to obey

$$\begin{aligned} XX^* &= R_j + H_jP_jH_j^* = R_{e,j} \\ YX^* &= F_jP_jH_j^* \\ ZZ^* &= F_jP_jF_j^* + G_jQ_jG_j^* - YY^* \\ &= F_jP_jF_j^* + G_jQ_jG_j^* - YX^*(XX^*)^{-1}XY^* \end{aligned}$$

[2] By an orthogonal transformation, $\Theta$, we mean one for which, $\Theta\Theta^* = \Theta^*\Theta = I$.

$$
\begin{aligned}
&= F_j P_j F_j^* + G_j Q_j G_j^* - F_j P_j H_j^* R_{e,j}^{-1} H_j P_j F_j^* \\
&= P_{j+1}.
\end{aligned}
$$

Therefore we can identify

$$
Z = P_{j+1}^{1/2} \quad , \quad X = R_{e,j}^{1/2} \tag{5.2.7}
$$

and also

$$
Y = F_j P_j H_j^* R_{e,j}^{-*/2} = K_{p,j} R_{e,j}^{1/2}. \tag{5.2.8}
$$

Thus the square-root algorithm not only propagates the square-roots of the Riccati variable, $P_j$, but also gives us quantities useful for the state estimation recursion

$$
\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j} R_{e,j}^{-1/2} (y_j - H_j \hat{x}_j).
$$

The unitary transformation $\Theta_j$ is highly nonunique and can be computed in many ways, the simplest ones being to construct it as a sequence of elementary (Givens or plane) rotations nulling one entry at a time in the pre-array, or as a sequence of elementary (Householder) reflections nulling out a block of entries in each row. We refer to [Hou53, GL89, Str93] for more details.[3] The numerical advantages of the square-root transformations arise from the length preserving properties of unitary transformations, and from the fact that the dynamic range of the entries in $P_j^{1/2}$ is roughly the square-root of the dynamic range of those in $P_j$. Moreover, regular computational (systolic) arrays can be designed to implement sequences of elementary unitary transformations [ML92].

A final result will be useful before we summarize the above discussion in a theorem. Any unitary transformation $\Theta_j$ that triangularizes the pre-array in (5.2.6) also gives the, readily checked, identity,

$$
\begin{bmatrix} -R_j^{-1/2} y_j & P_j^{-1/2} \hat{x}_j & 0 \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j}^{-1/2} e_j & P_{j+1}^{-1/2} \hat{x}_{j+1} & \times \end{bmatrix} \tag{5.2.9}
$$

where $\times$ denotes an entry whose exact form is not relevant at the moment.

We can summarize the above discussion as follows.

[3]The above square-root method is closely related to the $QR$ (factorization) method for solving systems of linear equations.

**Algorithm 5.2.1 (Conventional Square-Root Algorithm)** *The gain vector $K_{p,j}$ necessary to obtain the state estimates in the conventional Kalman filter*

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j}(y_j - H_j \hat{x}_j), \quad \hat{x}_0 = 0,$$

*can be updated as follows*

$$\begin{bmatrix} R_j^{1/2} & H_j P_j^{1/2} & 0 \\ 0 & F_j P_j^{1/2} & G_j Q_j^{1/2} \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j}^{1/2} & 0 & 0 \\ K_{p,j} R_{e,j}^{1/2} & P_{j+1}^{1/2} & 0 \end{bmatrix}, \tag{5.2.10}$$

*where $\Theta_j$ is any unitary matrix that triangularizes the above pre-array. The algorithm is initialized with $P_0 = \Pi_0$.*

Note that the quantities necessary to update the square-root array, and to calculate the state estimates, may all be found from the triangularized post-array.

It will also be useful to quote the filtered form of the square-root array algorithm, that can be verified in a fashion similar to what was done above.

**Algorithm 5.2.2 (Conventional Square-Root Algorithm - Filtered Form)** *The gain vector $K_{f,j}$ necessary to obtain the state estimates in the filtered form of the conventional Kalman filter*

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{f,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}), \quad \hat{x}_{-1|-1} = 0,$$

*can be updated as follows*

$$\begin{bmatrix} R_j^{1/2} & H_j P_j^{1/2} \\ 0 & P_j^{1/2} \end{bmatrix} \Theta_j^{(1)} = \begin{bmatrix} R_{e,j}^{1/2} & 0 \\ K_{f,j} R_{e,j}^{1/2} & P_{j|j}^{1/2} \end{bmatrix}, \tag{5.2.11}$$

$$\begin{bmatrix} F_j P_{j|j}^{1/2} & G_j Q_j^{1/2} \end{bmatrix} \Theta_j^{(2)} = \begin{bmatrix} P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{5.2.12}$$

*where $\Theta_j^{(1)}$ and $\Theta_j^{(2)}$ are any unitary matrices that triangularize the above pre-arrays. The algorithm is initialized with $P_0 = \Pi_0$.*

## 5.3 $H^\infty$ Square-Root Array Algorithms

We now turn our attention to $H^\infty$ filtering. Our goal here is to investigate whether it is possible to construct square-root array implementations of $H^\infty$ filters, similar to what was done in the aforementioned $H^2$ case.

### 5.3.1 The General Case

Recall the parametrization of all possible $H^\infty$ a posteriori filters, $\check{s}_{j|j} = \mathcal{F}_{f,j}(y_0, \ldots y_j)$, as given by Lemma 3.3.5. There we saw that all such $\check{s}_{j|j}$ are given by any choices that render

$$\sum_{j=0}^{k} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix} \geq 0, \quad 0 \leq k \leq i \tag{5.3.1}$$

where $\hat{x}_j$ satisfies the recursion,

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j\hat{x}_j \\ \check{s}_{j|j} - L_j\hat{x}_j \end{bmatrix}, \qquad \hat{x}_0 = 0 \tag{5.3.2}$$

with

$$K_{p,j} = F_jP_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \tag{5.3.3}$$

and

$$P_{j+1} = F_jP_jF_j^* + G_jQ_jG_j^* - K_{p,j}R_{e,j}K_{p,j}^*, \quad P_0 = \Pi_0. \tag{5.3.4}$$

As repeatedly mentioned earlier, the above solution looks very much like the conventional Kalman filter. The essential difference is in the Riccati recursion where we now have indefinite covariance matrices (such as $R_{e,i}$). Nonetheless, let us persist to see whether we can come up with a square-root implementation of the $H^\infty$ Riccati recursion (5.3.4).

To this end, recall the conventional square-root array algorithm of Sec. 5.2,

$$\begin{bmatrix} R_j^{1/2} & H_jP_j^{1/2} & 0 \\ 0 & F_jP_j^{1/2} & G_jQ_j^{1/2} \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j}^{1/2} & 0 & 0 \\ K_{p,j}R_{e,j}^{1/2} & P_{j+1}^{1/2} & 0 \end{bmatrix}, \tag{5.3.5}$$

where $P_j^{1/2}P_j^{*/2} = P_j$, $Q_j^{1/2}Q_j^{*/2} = Q_j$, $R_j^{1/2}R_j^{*/2} = R_j$, and $R_{e,j}^{1/2}R_{e,j}^{*/2} = R_{e,j}$.

Using our earlier observation that $H^\infty$ filtering is just Kalman filtering in Krein space, allows us to speculate that needs to be done is to come up with a Krein space generalization of the above square-root array algorithm. This is the approach that we shall take here.

The first problem that occurs if one wants to extend the square-root array algorithm to the Krein space setting (of which the $H^\infty$ filtering problem is a special case) is that the matrices $R_i$, $Q_i$, $P_i$ and $R_{e.i}$ are in general indefinite and square-roots may not exist. To alleviate this problem we need the notion of an indefinite square-root, as defined below.

**Definition 5.3.1 (Indefinite Square-Roots)** *Suppose $A$ is an arbitrary Hermitian matrix. $A^{1/2}$ will be called an indefinite square-root of $A$ if, and only, if*

$$A = A^{1/2} S A^{*/2}$$

*where $S$ is a signature matrix (*i.e.* a diagonal matrix with diagonal elements either $+1$ or $-1$).*

However, in the Krein space case $R_i$, $Q_i$, $P_i$ and $R_{e.i}$ may all have arbitrary inertia, *i.e.*

$$R_i = R_i^{1/2} S_i^{(1)} R_i^{*/2}, \quad Q_i = Q_i^{1/2} S_i^{(2)} Q_i^{*/2}, \quad P_i = P_i^{1/2} S_i^{(3)} P_i^{*/2}, \quad R_{e,i} = R_{e,i}^{1/2} S_i^{(4)} R_{e,i}^{*/2}$$

for arbitrary signature matrices $S_i^{(k)}$, $k = 1, 2, 3, 4$. It is thus not obvious how to incorporate all these different time-varying signature matrices into a square-root array algorithm of the type (5.3.5). Although this can be done in the general case, by either introducing non-Hermitian factorizations of the Gramians, or by keeping track of the inertia, we shall not pursue these lines of thought here. The reason is that, as it turns out, in the $H^\infty$ problems that we have been studying, the Gramians satisfy certain inertia properties that allow us to extend the algorithm of (5.3.5) in a very natural way.

Indeed when a solution to the $H^\infty$ filtering problem exists, we know that $P_i \geq 0$, and that $R_{e,i}$ and $R_i$ have the same inertia (see *e.g.*, Lemmas 2.7.3 and 2.7.4). Moreover, $Q_i = I_m > 0$ and $R_i = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}$ have constant inertia (and thus so does $R_{e,i}$) so that we may write,

$$R_i = R_i^{1/2} J R_i^{*/2}, \quad Q_i = Q_i^{1/2} Q_i^{*/2}, \quad P_i = P_i^{1/2} P_i^{*/2}, \quad R_{e,i} = R_{e,i}^{1/2} J R_{e,i}^{*/2} \qquad (5.3.6)$$

with

$$R_i^{1/2} = \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}. \tag{5.3.7}$$

This suggests that in the $H^\infty$ filtering problem, the prearray in (5.3.5) should be replaced by,

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} & \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} & 0 \\ 0 & F_j P_j^{1/2} & G_j \end{bmatrix}. \tag{5.3.8}$$

[Recall that in $H^\infty$ estimation $H_j$ is replaced by $\begin{bmatrix} H_j \\ L_j \end{bmatrix}$.] Now in the $H^2$ case the prearray in (5.3.5) was triangularized by a unitary transformation (or simply by a rotation). Since the $H^2$ estimation problem can be formulated in a Hilbert space, whereas the $H^\infty$ estimation problem is most naturally formulated in a Krein space, it seems plausible that we should attempt to triangularize (5.3.8), not by an ordinary rotation, but by a hyperbolic rotation. To be more specific, we need to use a $J$-unitary transformation, as defined below.

**Definition 5.3.2 ($J$-unitary Matrices)** *For any signature matrix, $J$, (a diagonal matrix with $+1$ and $-1$ diagonal elements) the matrix $\Theta$ will be called $J$-unitary if*

$$\Theta J \Theta^* = J. \tag{5.3.9}$$

Recall that unitary transformations (or ordinary rotations) preserve the length (or ordinary norm) of vectors. $J$-unitary transformations, on the other hand, preserve the (indefinite) $J$-norm of vectors. Indeed, if $b = a\Theta$, with $\Theta$ $J$-unitary, then

$$bJb^* = a\Theta J \Theta^* a^* = aJa^*.$$

The above discussions suggest that we should attempt to triangularize (5.3.8) via a $J$-unitary transformation, where

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} & & \\ & I_n & \\ & & I_m \end{bmatrix}. \tag{5.3.10}$$

Now it is well known that it is always possible to triangularize arrays using unitary transformations. But is this also true of $J$-unitary transformations? To see if this is the case, consider a much simpler example where we are given the (two-element) row vector

$$\begin{bmatrix} a & b \end{bmatrix},$$

and are asked to hyperbolically rotate it to that the resulting vector lies along the direction of the $x$-axis.[4] For the time being, assume that such a transformation can be found. Then we can write,

$$\begin{bmatrix} a & b \end{bmatrix} \Theta = \begin{bmatrix} c & 0 \end{bmatrix}, \tag{5.3.11}$$

where

$$\Theta J \Theta^* = J \quad \text{and} \quad J = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{5.3.12}$$

Since $\Theta$ is $J$-unitary this implies that,

$$\begin{bmatrix} a & b \end{bmatrix} J \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} c & 0 \end{bmatrix} J \begin{bmatrix} c^* \\ 0 \end{bmatrix}, \tag{5.3.13}$$

or more explicitly,

$$|a|^2 - |b|^2 = |c|^2 \geq 0. \tag{5.3.14}$$

Thus, $\begin{bmatrix} a & b \end{bmatrix}$ must have nonnegative $J$-norm. In other words, if the given $\begin{bmatrix} a & b \end{bmatrix}$ has negative $J$-norm (*i.e.*, $|a|^2 - |b|^2 < 0$) then it is impossible to hyperbolically rotate it to lie along the $x$-axis. [This fact is shown in Fig. 5.1. As can be seen, standard rotations move the vector $\begin{bmatrix} a & b \end{bmatrix}$ along the circle, $a^2 + b^2 = \text{constant}$, whereas hyperbolic rotations move it along the hyperbola, $a^2 - b^2 = \text{constant}$. Thus while it is always possible to rotate $\begin{bmatrix} a & b \end{bmatrix}$ to lie along the $x$-axis, if $|a|^2 - |b|^2 < 0$ then it is impossible to do so with a *hyperbolical* rotation. Indeed hyperbolic rotations cannot move vectors from the positive to negative subspaces of a Krein space, or vice versa.]

Thus it is quite obvious that it is not always possible to triangularize arrays using $J$-unitary transformations. The precise condition follows.

[4] Note in standard (two-dimensional) Euclidean space this can always be done.

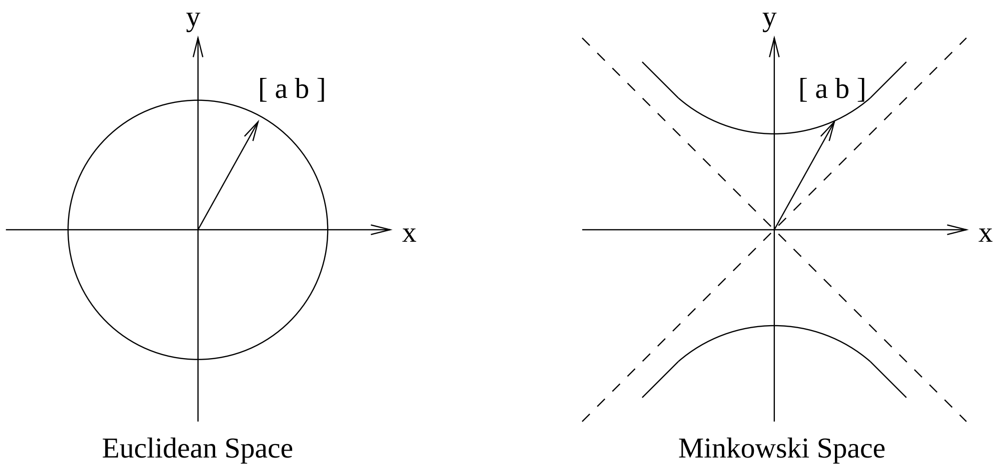


Figure 5.1: Standard rotations vs. hyperbolic rotations.

**Lemma 5.3.1 ($J$-unitary Matrices and Triangularization)** *Let $A$ and $B$ be arbitrary $n \times n$ and $n \times m$ matrices, respectively, and suppose $J = \begin{bmatrix} S_1 & \\ & S_2 \end{bmatrix}$ where $S_1$ and $S_2$ are $n \times n$ and $m \times m$ signature matrices. Then $\begin{bmatrix} A & B \end{bmatrix}$ can be triangularized by a $J$-unitary transformation $\Theta$ as*

$$\begin{bmatrix} A & B \end{bmatrix} \Theta = \begin{bmatrix} L & 0 \end{bmatrix}$$

*with $L$ lower triangular, if and only if, all leading submatrices of*

$$S_1 \quad \textit{and of} \quad AS_1A^* + BS_2B^*$$

*have the same inertia.*

**Proof:** To prove one direction suppose there exists a $J$-unitary transformation $\Theta$ that triangularizes $\begin{bmatrix} A & B \end{bmatrix}$. Consider an arbitrary partitioning of $A$, $B$ and $L$, *i.e.*

$$\begin{bmatrix} A^{(1)} & B^{(1)} \\ A^{(2)} & B^{(2)} \end{bmatrix} \quad , \quad \begin{bmatrix} L^{(1)} & 0 \\ L^{(2)} & 0 \end{bmatrix}$$

where $A^{(1)}$, $B^{(1)}$ and $L^{(1)}$ have $r$ rows. Now

$$\begin{bmatrix} A^{(1)} & B^{(1)} \\ A^{(2)} & B^{(2)} \end{bmatrix} \underbrace{\Theta J \Theta^*}_{=J} \begin{bmatrix} A^{(1)*} & A^{(2)*} \\ B^{(1)*} & B^{(2)*} \end{bmatrix} = \begin{bmatrix} L^{(1)} & 0 \\ L^{(2)} & 0 \end{bmatrix} J \begin{bmatrix} L^{(1)*} & L^{(2)*} \\ 0 & 0 \end{bmatrix}$$

so that

$$\begin{bmatrix} A^{(1)}S_1A^{(1)*}+B^{(1)}S_2B^{(1)*} & \times \\ \times & \times \end{bmatrix} = \begin{bmatrix} L^{(1)}S_1L^{(1)*} & \times \\ \times & \times \end{bmatrix} \tag{5.3.15}$$

where $\times$ denotes irrelevant entries. Moreover since $L$ is lower triangular we have $L^{(1)} = \begin{bmatrix} L^{(11)} & 0 \end{bmatrix}$, where $L^{(11)}$ is lower triangular and $r \times r$. Thus if we denote by $S_1^{(1)}$ the leading $r \times r$ submatrix of $S_1$, equating the $(1,1)$ block entries in (5.3.15) yields

$$A^{(1)}S_1A^{(1)*}+B^{(1)}S_2B^{(1)*} = L^{(11)}S_1^{(1)}L^{(11)*} \tag{5.3.16}$$

The LHS of the above equation is the leading $r \times r$ submatrix of $AS_1A^* + BS_2B^*$. Thus (5.3.16) shows that the leading $r \times r$ submatrices of $AS_1A^*+BS_2B^*$ and $S_1$ have the same inertia. Since $r$ was arbitrary the same is true for all leading submatrices.

To prove the other direction we assume that all leading submatrices of $AS_1A^* + BS_2B^*$ and $S_1$ have the same inertia. In particular the leading $(1,1)$ submatrices, so that

$$aS_1a^* + bS_2b^* = l_{11}^2 s \tag{5.3.17}$$

where $a$ and $b$ are the leading rows of $A$ and $B$, $s$ is the leading diagonal of $S_1$ and $l_{11}$ is a scalar. Now define the vector

$$v = \begin{bmatrix} a & b \end{bmatrix} + l_{11}e_1$$

where $e_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$ is the first unit *row* vector. Consider the matrix

$$\Theta_1 = I - 2\frac{Jv^*v}{vJv^*}.$$

A straightforward calculation shows that $\Theta_1 J \Theta_1^* = J$ so that $\Theta_1$ is $J$-unitary. Moreover another direct calculation shows that

$$\begin{bmatrix} a & b \end{bmatrix}\Theta_1 = l_{11}e_1 = \begin{bmatrix} l_{11} & 0 & \dots & 0 \end{bmatrix}.$$

[$\Theta_1$ may be recognized as an elementary Householder reflection in the $J$-metric.] We thus far have

$$\begin{bmatrix} A & B \end{bmatrix}\Theta_1 = \begin{bmatrix} \begin{bmatrix} l_{11} & 0 \end{bmatrix} & 0 \\ \begin{bmatrix} A_{21} & A_2 \end{bmatrix} & B_2 \end{bmatrix}. \tag{5.3.18}$$

Now if all leading submatrices of two given matrices have the same inertia, then their $(1,1)$ entries should have the same inertia and all leading submatrices of the *Schur complement* of their $(1,1)$ entries should have the same inertia. Now partition $S_1$ as

$$S_1 = \begin{bmatrix} s & \\ & S^{(1)} \end{bmatrix}$$

so that $S^{(1)}$ is the Schur complement of $s$ in $S_1$. Likewise the Schur complement of the $(1,1)$ entry of $AS_1A^* + BS_2B^*$ is $A_2S^{(1)}A_2^* + B_2S_2B_2^*$ where $A_2$ and $B_2$ are defined in (5.3.18). Therefore all leading submatrices of $A_2S^{(1)}A_2^* + B_2S_2B_2^*$ and $S^{(1)}$ have the same inertia.We may now proceed as before and find a $J$-unitary matrix $\Theta_2$ that rotates the first row of $\begin{bmatrix} A_{21} & A_2 & B_2 \end{bmatrix}$ to lie along the *second* unit vector. Continuing in a similar fashion will result in a $J$-unitary matrix $\Theta = \Theta_1\Theta_2\ldots\Theta_{n-1}$ that triangularizes $\begin{bmatrix} A & B \end{bmatrix}$.

∎

Let us now apply the result of Lemma 5.3.1 to the triangularization of the array (5.3.8) using a $J$-unitary rotation with $J$ given by (5.3.10). In fact, we need only consider the condition for the triangularization of the first block row (since setting the block $(2,3)$ entry of the post array to be zero can always be done via a standard unitary transformation). Thus we need only consider triangularizing

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} & \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} \end{bmatrix}, \tag{5.3.19}$$

using a $J$-unitary transformation with

$$J = \begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} & \\ & I_n \end{bmatrix}. \tag{5.3.20}$$

From Lemma 5.3.1, the condition obviously is that all leading submatrices of $J$ and

$$\underbrace{\begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix}}_{R_j} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} P_j^{*/2} \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} = R_{e,j}, \tag{5.3.21}$$

have the same inertia. But this is precisely the condition required for the existence of an $H^\infty$ a posteriori filter! (See Lemma 3.3.3 and Corollary 3.3.1). This result is quite amazing — it states that an $H^\infty$ (a posteriori) filter exists if, and only if, the prearray can be triangularized, *i.e.*, if, and only if, the square-root algorithm can be performed and does not break down.

This observation should now reinforce our conviction that the most natural way of studying $H^\infty$ problems is through the geometry of indefinite metric spaces. As just seen, this approach allowed us to immediately generalize the form of the conventional square-root array algorithm to the $H^\infty$ setting (by introducing indefinite square-roots). Most importantly, once this was done, the conditions for triangularizing arrays, as forced upon us by the geometry of Krein spaces, directly led us to the conditions for the existence of $H^\infty$ filters.[5] Thus in square-root implementations of $H^\infty$ filters the existence conditions are built into the algorithms themselves, so that there is no need to check for them separately.

Now that we have settled the existence question, let us return to triangularizing the prearray (5.3.8), so that we can write,

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} & \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} & 0 \\ 0 & F_j P_j^{1/2} & G_j \end{bmatrix} \Theta_j = \begin{bmatrix} A & 0 & 0 \\ B & C & 0 \end{bmatrix}, \tag{5.3.22}$$

where $A$ and $C$ are lower triangular, and where $\Theta_j$ is $J$-unitary with $J$ as in (5.3.10). The array on the left hand side of (5.3.22) is referred to as the *pre-array* and the array on the right hand side as the *post-array*. To identify the elements $A$, $B$ and $C$ in the post array let us square both sides of (5.3.22) and use the fact that $\Theta_j$ is $J$-unitary. Therefore

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} & \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} & 0 \\ 0 & F_j P_j^{1/2} & G_j \end{bmatrix} \underbrace{\Theta_j J \Theta_j^*}_{=J} \begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} & 0 \\ P_j^{*/2} \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} & P_j^{*/2} F_j^* \\ 0 & G_j^* \end{bmatrix}$$

[5]This phenomenon will occur again and again in this thesis. We shall shortly encounter it once more when we study the (fast) Chandrasekhar recursions.

$$= \begin{bmatrix} A & 0 & 0 \\ B & C & 0 \end{bmatrix} J \begin{bmatrix} A^* & B^* \\ 0 & C^* \\ 0 & 0 \end{bmatrix}. \tag{5.3.23}$$

Equating the $(1,1)$ blocks on the left hand and right hand sides of (5.3.23) yields

$$\begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} = A \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} A^*.$$

The left hand side of the above relation is simply $R_{e,j}$. Therefore $A$ is an indefinite square-root of the $R_{e,j}$, and we can write

$$A = R_{e,j}^{1/2} \qquad , \qquad R_{e,j}^{1/2} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} R_{e,j}^{*/2} = R_{e,j}. \tag{5.3.24}$$

Equating the $(2,1)$ blocks on the left hand and right hand sides of (5.3.23) yields

$$F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} = B \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} A^*.$$

Therefore

$$B = F_j P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-*/2} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix},$$

so that we can write

$$B = \bar{K}_{p,j} = K_{p,j} R_{e,j}^{1/2}. \tag{5.3.25}$$

Equating the $(2,2)$ blocks on the left hand and right hand sides of (5.3.23) yields

$$\begin{aligned} F_j P_j F_j^* + G_j G_j^* &= B \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} B^* + CC^* \\ &= K_{p,j} R_{e,j}^{1/2} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} R_{e,j}^{*/2} K_{p,j}^* + CC^* \\ &= K_{p,j} R_{e,j} K_{p,j}^* + CC^*. \end{aligned}$$

Therefore

$$CC^* = F_j P_j F_j^* + G_j G_j^* - K_{p,j} R_{e,j} K_{p,j}^* = P_{j+1},$$

so that we may write

$$C = P_{j+1}^{1/2} \quad , \quad P_{j+1}^{1/2} P_{j+1}^{*/2} = P_{j+1}. \tag{5.3.26}$$

We can now summarize our results as follows.

**Theorem 5.3.1** (**$H^\infty$ A Posteriori Square-Root Algorithm**) *The $H^\infty$ a posteriori filtering problem with level $\gamma$ has a solution if, and only if, for all $j = 0, \ldots, i$ there exist $J$-unitary matrices (with $J$ given by (5.3.10)), $\Theta_j$, such that*

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} & \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} & 0 \\ 0 & F_j P_j^{1/2} & G_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j}^{1/2} & 0 & 0 \\ K_{p,j} R_{e,j}^{1/2} & P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{5.3.27}$$

*where the algorithm is initialized with, $P_0 = \Pi_0$. If this is the case, then all possible $H^\infty$ a posteriori filters, $\check{s}_{j|j} = \mathcal{F}_{f,j}(y_0, \ldots y_j)$, are given by any choices that yield,*

$$\sum_{j=0}^{k} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix} \geq 0, \quad 0 \leq k \leq i$$

*where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix}, \qquad \hat{x}_0 = 0.$$

In the $H^\infty$ a priori filtering problem we need to begin with the pre-array

$$\begin{bmatrix} \begin{bmatrix} \gamma I_q & 0 \\ 0 & I_p \end{bmatrix} & \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j^{1/2} & 0 \\ 0 & F_j P_j^{1/2} & G_j \end{bmatrix}, \tag{5.3.28}$$

and with

$$J = \begin{bmatrix} \begin{bmatrix} -I_q & 0 \\ 0 & I_p \end{bmatrix} & & \\ & I_n & \\ & & I_m \end{bmatrix}. \tag{5.3.29}$$

[Note the reversal of the order of the $\{H_j, L_j\}$ as compared to the a posteriori case.]

Using similar arguments we may prove the following result.

**Theorem 5.3.2 ($H^\infty$ A Priori Square-Root Algorithm)** *The $H^\infty$ a priori filtering problem with level $\gamma$ has a solution if, and only if, for all $j = 0, \ldots, i$ there exist $J$-unitary matrices (with $J$ given by (5.3.29)), $\Theta_j$, such that*

$$\begin{bmatrix} \begin{bmatrix} \gamma I_q & 0 \\ 0 & I_p \end{bmatrix} & \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j^{1/2} & 0 \\ 0 & F_j P_j^{1/2} & G_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j}^{1/2} & 0 & 0 \\ K_{p,j} R_{e,j}^{1/2} & P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{5.3.30}$$

*where the algorithm is initialized with, $P_0 = \Pi_0$. If this is the case, then all possible $H^\infty$ a priori filters, $\check{s}_j = \mathcal{F}_{f,j}(y_0, \ldots y_{j-1})$, are given by any choices that yield,*

$$\sum_{j=0}^{k} \begin{bmatrix} \check{s}_j - L_j \hat{x}_j \\ y_j - H_j \hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} \check{s}_j - L_j \hat{x}_j \\ y_j - H_j \hat{x}_j \end{bmatrix} \geq 0, \quad 0 \leq k \leq i$$

*where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j} \begin{bmatrix} \check{s}_j - L_j \hat{x}_j \\ y_j - H_j \hat{x}_j \end{bmatrix}, \qquad \hat{x}_0 = 0.$$

Note that, as in the $H^2 case$, the quantities necessary to update the square-root array and to calculate the desired estimates may all be found from the triangularized post-array.

In conventional Kalman filtering, square-root arrays are preferred since the positive-definiteness of the matrices is guaranteed, and since the $\Theta_j$ are unitary, which improves the numerical stability of the algorithm. In the $H^\infty$ setting, the square-root arrays guarantee that the various matrices have their appropriate inertia; however, the $\Theta_j$ are no longer unitary but $J$-unitary. Therefore the numerical aspects need further investigation.

An interesting aspect of Theorems 5.3.1 and 5.3.2 is that there is no need to explicitly check for the existence conditions required of $H^\infty$ filters (see Theorems 3.2.1 and 3.2.2). These conditions are built into the square-root algorithms themselves: if the algorithms can be performed an $H^\infty$ estimator of the desired level exists, and if they cannot be performed such an estimator does not exist.

### 5.3.2 The Central Filters

In the previous section we obtained a square-root version of the parametrization of all $H^\infty$ a posteriori and a priori filters. Perhaps the most important filters in these classes are the so-called central filters, which, as we have seen earlier, possess the additional properties of being maximum-entropy and risk-sensitive-optimal filters, as well as being the solution to certain quadratic dynamic games (see Chapters 1, 3 abd 4). In this section we shall develop square-root algorithms specifically for such central filters. As expected, the observer gains for the central filters turn out to be readily obtainable from the square-root algorithms of Theorems 5.3.1 and 5.3.2.

Let us begin by recalling (from Theorem 3.2.1) the central $H^\infty$ a posteriori filter recursions,

$$\hat{x}_{j+1|j+1} = F_j\hat{x}_{j|j} + K_{s,j+1}(y_{j+1} - H_{j+1}F_j\hat{x}_{j|j}), \tag{5.3.31}$$

where the desired estimate is given by $\hat{s}_{j|j} = L_j\hat{x}_{j|j}$, and where the gain vector is given by

$$K_{s,j} = P_jH_j^*(I_p + H_jP_jH_j^*)^{-1}. \tag{5.3.32}$$

We will now show how to obtain the above gain vector from the a posteriori square-root recursions.

To this end, let us first note that we can rewrite the a posteriori square-root algorithm of Theorem 5.3.1 via the following two-step procedure,

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} & \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} \\ 0 & P_j^{1/2} \end{bmatrix} \Theta_j^{(1)} = \begin{bmatrix} R_{e,j}^{1/2} & 0 \\ K_{f,j}R_{e,j}^{1/2} & P_{j|j}^{1/2} \end{bmatrix}, \tag{5.3.33}$$

$$\begin{bmatrix} F_jP_{j|j}^{1/2} & G_j \end{bmatrix} \Theta_j^{(2)} = \begin{bmatrix} P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{5.3.34}$$

where $\Theta_j^{(1)}$ is $J$-unitary, with $J = I_p \oplus (-I_q) \oplus I_n$, and $\Theta_j^{(2)}$ is unitary. [Note that the above two-step procedure is the $H^\infty$ analog of Algorithm 5.2.2.] In the above recursions, we of course have

$$K_{f,j} = P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-1}, \tag{5.3.35}$$

with

$$R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix}. \tag{5.3.36}$$

Now in (5.3.33), $R_{e,j}^{1/2}$ can be any square-root of $R_{e,j}$. Let us study the consequences of choosing a lower triangular square-root. To do so, consider the following triangular factorization of $R_{e,j}$,

$$\begin{bmatrix} I_p & 0 \\ L_j P_j H_j^*(I_p + H_j P_j H_j^*)^{-1} & I_q \end{bmatrix} \begin{bmatrix} I_p + H_j P_j H_j^* & 0 \\ 0 & -\Delta_j \end{bmatrix} \begin{bmatrix} I_p & (I_p + H_j P_j H_j^*)^{-1} H_j P_j L_j^* \\ 0 & I_q \end{bmatrix}, \tag{5.3.37}$$

where we have defined the Schur complement,

$$-\Delta_j \triangleq -\gamma^2 I_q + L_j P_j L_j^* - L_j P_j H_j^*(I_p + H_j P_j H_j^*)^{-1} H_j P_j L_j^*.$$

Note that the inertia condition on $R_{e,j}$ implies that $\Delta_j > 0$, so that we may write

$$R_{e,j} = R_{e,j}^{1/2} S R_{e,j}^{*/2}, \tag{5.3.38}$$

with

$$\begin{aligned} R_{e,j}^{1/2} &= \begin{bmatrix} I_p & 0 \\ L_j P_j H_j^*(I_p + H_j P_j H_j^*)^{-1} & I_q \end{bmatrix} \begin{bmatrix} (I_p + H_j P_j H_j^*)^{1/2} & 0 \\ 0 & \Delta_j^{1/2} \end{bmatrix} \\ &= \begin{bmatrix} (I_p + H_j P_j H_j^*)^{1/2} & 0 \\ L_j P_j H_j^*(I_p + H_j P_j H_j^*)^{-*/2} & \Delta_j^{1/2} \end{bmatrix}, \end{aligned} \tag{5.3.39}$$

and $S = \begin{bmatrix} I_p & \\ & -I_q \end{bmatrix}$. Now the $(2,1)$ block entry in the postarray of (5.3.33) is given by

$$\begin{aligned} K_{f,j} R_{e,j}^{1/2} &= P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} R_{e,j}^{-*/2} \\ &= P_j \begin{bmatrix} H_j^* & L_j^* \end{bmatrix} \begin{bmatrix} (I_p + H_j P_j H_j^*)^{-*/2} & (I_p + H_j P_j H_j^*)^{-1} H_j P_j L_j^* \Delta_j^{-*/2} \\ 0 & \Delta_j^{-*/2} \end{bmatrix} \\ &= P_j \begin{bmatrix} H_j^*(I_p + H_j P_j H_j^*)^{-*/2} & \times \end{bmatrix}, \end{aligned} \tag{5.3.40}$$

where $\times$ denotes irrelevant entries.

Eqs. (5.3.39) and (5.3.40) now suggest how to compute the desired gain vector $K_{s,i}$. Indeed,

$$\begin{gathered}\left(\text{first block column of } K_{f,j}R_{e,j}^{1/2}\right) \cdot \left((1,1) \text{ block entry of } R_{e,j}^{1/2}\right)^{-1} = \\ P_j H_j^*(I_p + H_j P_j H_j^*)^{-*/2} \cdot (I_p + H_j P_j H_j^*)^{-1/2} = \\ K_{s,j}.\end{gathered} \tag{5.3.41}$$

We are thus led to the following result.

**Algorithm 5.3.1 ( Central $H^\infty$ A Posteriori Square-Root Algorithm)** *The $H^\infty$ a posteriori filtering problem with level $\gamma$ has a solution if, and only if, for all $j = 0, \ldots, i$ there exist $J$-unitary (with $J = I_p \oplus (-I_q) \oplus I_n$) matrices, $\Theta_j^{(1)}$, such that*

$$\begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \gamma I_q \end{bmatrix} & \begin{bmatrix} H_j \\ L_j \end{bmatrix} P_j^{1/2} \\ 0 & P_j^{1/2} \end{bmatrix} \Theta_j^{(1)} = \begin{bmatrix} R_{e,j}^{1/2} & 0 \\ K_{f,j}R_{e,j}^{1/2} & P_{j|j}^{1/2} \end{bmatrix}, \tag{5.3.42}$$

$$\begin{bmatrix} F_j P_{j|j}^{1/2} & G_j \end{bmatrix} \Theta_j^{(2)} = \begin{bmatrix} P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{5.3.43}$$

*with $R_{e,j}^{1/2}$ lower block triangular, and with $\Theta_j^{(2)}$ unitary. The gain vector $K_{s,j}$ needed to update the estimates in the central filter recursions*

$$\hat{x}_{j|j} = F_{j-1}\hat{x}_{j-1|j-1} + K_{s,j}(y_j - H_j F_{j-1}\hat{x}_{j-1|j-1}), \quad \hat{x}_{-1|-1} = 0,$$

*is equal to*

$$K_{s,j} = \bar{K}_{s,j}(I + H_j P_j H_j^*)^{-1/2},$$

*where $\bar{K}_{s,j}$ is given by the first block column of $\bar{K}_{f,j} = K_{f,j}R_{e,j}^{1/2}$, and $(I + H_j P_j H_j^*)^{1/2}$ is given by the $(1,1)$ block entry of $R_{e,j}^{1/2}$. The algorithm is initialized with $P_0 = \Pi_0$.*

We can now proceed with a similar argument to find square-root recursions for the central $H^\infty$ a priori filters. Let us first recall (from Theorem 3.2.2) that the central $H^\infty$ a priori filter recursions are

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{a,j}(y_j - H_j\hat{x}_j), \tag{5.3.44}$$

where the desired estimate is given by $\hat{s}_j = L_j\hat{x}_j$, and where the gain vector is given by

$$K_{a,j} = F_j\tilde{P}_jH_j^*(I_p + H_j\tilde{P}_jH_j^*)^{-1}, \tag{5.3.45}$$

with

$$\tilde{P}_j = P_j - P_jL_j^*(-\gamma^2 I_q + L_jP_jL_j^*)^{-1}L_jP_j. \tag{5.3.46}$$

We will now show how to obtain the above gain vector from the a priori square-root recursions,

$$\begin{bmatrix} \begin{bmatrix} \gamma I_q & 0 \\ 0 & I_p \end{bmatrix} & \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j^{1/2} & 0 \\ 0 & F_jP_j^{1/2} & G_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j}^{1/2} & 0 & 0 \\ K_{p,j}R_{e,j}^{1/2} & P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{5.3.47}$$

Note now that,

$$K_{p,j} = F_jP_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix} R_{e,j}^{-1}, \tag{5.3.48}$$

with

$$R_{e,j} = \begin{bmatrix} -\gamma^2 I_q & 0 \\ 0 & I_p \end{bmatrix} + \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix}. \tag{5.3.49}$$

As mentioned earlier, $R_{e,j}^{1/2}$ in (5.3.47) can be any indefinite square-root of $R_e$. Let us, once more, study the consequences of choosing a lower triangular square-root. To do so, consider the following block lower-diagonal-upper triangular factorization of the matrix, $R_{e,j}$,

$$\begin{bmatrix} I_q & 0 \\ -\gamma^{-2}H_j\tilde{P}_jL_j^* & I_p \end{bmatrix} \begin{bmatrix} -\gamma^2 I_q + L_jP_jL_j^* & 0 \\ 0 & I_p + H_j\tilde{P}_jH_j^* \end{bmatrix} \begin{bmatrix} I_q & -\gamma^{-2}L_j\tilde{P}_jH_j^* \\ 0 & I_p \end{bmatrix}, \tag{5.3.50}$$

where we have used the facts that,

$$H_jP_jL_j^*(-\gamma^2 I_q + L_jP_jL_j^*)^{-1} = -\gamma^{-2}H_j\tilde{P}_jL_j^*,$$

and, for the Schur complement,

$$I_p + H_jP_jH_j^* - H_jP_jL_j^*(-\gamma^2 I_q + L_jP_jL_j^*)^{-1}L_jP_jH_j^* = I_p + H_j\tilde{P}_jH_j^*.$$

Now the inertia conditions on $R_{e,j}$ require that, $-\gamma^{-2}H_j\tilde{P}_jL_j^* < 0$ and $I_p + H_j\tilde{P}_jH_j^* > 0$, so that we may write,

$$R_{e,j} = R_{e,j}^{1/2} S R_{e,j}^{*/2}, \tag{5.3.51}$$

with

$$R_{e,j}^{1/2} = \begin{bmatrix} (\gamma^2 I_q - L_j P_j L_j^*)^{1/2} & 0 \\ -\gamma^{-2}H_j\tilde{P}_jL_j^*(\gamma^2 I_q - L_jP_jL_j^*)^{1/2} & (I_p + H_j\tilde{P}_jH_j^*)^{1/2} \end{bmatrix}, \tag{5.3.52}$$

and $S = \begin{bmatrix} -I_q & \\ & I_p \end{bmatrix}$. Now the $(2,1)$ block entry in the postarray of (5.3.47) is given by

$$\begin{aligned} K_{p,j}R_{e,j}^{1/2} &= F_jP_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix} R_{e,j}^{-*/2} \\ &= F_jP_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix} \begin{bmatrix} (\gamma^2 I_q - L_jP_jL_j^*)^{-*/2} & \gamma^{-2}L_j\tilde{P}_jH_j^*(I_p + H_j\tilde{P}_jH_j^*)^{-*/2} \\ 0 & (I_p + H_j\tilde{P}_jH_j^*)^{-*/2} \end{bmatrix} \\ &= F_jP_j \begin{bmatrix} \times & (\gamma^{-2}L_j^*L_j\tilde{P}_j + I_n)H_j^*(I_p + H_j\tilde{P}_jH_j^*)^{-*/2} \end{bmatrix} \\ &= F_j \begin{bmatrix} \times & \tilde{P}_jH_j^*(I_p + H_j\tilde{P}_jH_j^*)^{-*/2} \end{bmatrix}, \end{aligned} \tag{5.3.53}$$

where in the last step we have used the (readily verified) identity,

$$P_j(\gamma^{-2}L_j^*L_j\tilde{P}_j + I_n) = \tilde{P}_j,$$

and where $\times$ denotes irrelevant entries.

Eqs. (5.3.52) and (5.3.53) now suggest how to compute the desired gain vector $K_{a,i}$. Indeed,

$$\begin{gathered} \left(\text{second block column of } K_{p,j}R_{e,j}^{1/2}\right) \cdot \left((2,2) \text{ block entry of } R_{e,j}^{1/2}\right)^{-1} = \\ F_j\tilde{P}_jH_j^*(I_p + H_j\tilde{P}_jH_j^*)^{-*/2} \cdot (I_p + H_j\tilde{P}_jH_j^*)^{-1/2} = \\ K_{a,j}. \end{gathered} \tag{5.3.54}$$

We are thus led to the following result.

**Algorithm 5.3.2 (Central $H^\infty$ A Priori Square-Root Algorithm)** *The $H^\infty$ a priori filtering problem with level $\gamma$ has a solution if, and only if, for all $j = 0, \ldots, i$*

*there exist $J$-unitary matrices (with $J = (-I_q) \oplus I_p \oplus I_n \oplus I_m$), $\Theta_j$, such that*

$$\begin{bmatrix} \begin{bmatrix} \gamma I_q & 0 \\ 0 & I_p \end{bmatrix} & \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j^{1/2} & 0 \\ 0 & F_j P_j^{1/2} & G_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j}^{1/2} & 0 & 0 \\ K_{p,j} R_{e,j}^{1/2} & P_{j+1}^{1/2} & 0 \end{bmatrix} \tag{5.3.55}$$

*with $R_{e,j}^{1/2}$ lower block triangular. The gain vector $K_{a,i}$ needed to update the estimates in*

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{a,j}(y_j - F_j \hat{x}_j), \quad \hat{x}_0 = 0,$$

*is equal to*

$$K_{a,j} = \bar{K}_{a,j}(I + H_j \tilde{P}_j H_j^*)^{-1/2},$$

*where $\bar{K}_{a,j}$ is given by the second block column of $\bar{K}_{p,j} = K_{p,j} R_{e,j}^{1/2}$, and $(I + H_j \tilde{P}_j H_j^*)^{1/2}$ is given by the $(2,2)$ block entry of $R_{e,j}^{1/2}$. The algorithm is initialized with $P_0 = \Pi_0$.*

## 5.4 $H^2$ Chandrasekhar Recursions

The conventional Kalman filter and square-root array recursions of Sec. 5.2 both require $O(n^3)$ operations per iteration (where $n$ is the number of states in the state-space model). However, when the state-space model is time-invariant (or if the time-variation is structured in a certain way) the Chandrasekhar recursions offer an algorithm that requires $O(n^2)$ operations per iteration [Kai72, MSK74, SK94a].

In what follows we shall assume a time-invariant state-space model of the form

$$\begin{cases} x_{j+1} = F x_j + G u_j, & x_0 \\ y_j = H x_j + v_j \end{cases} \tag{5.4.1}$$

where the $\{u_j, v_j\}$ are disturbances whose nature depends on the criterion being used, and where the $\{y_j\}$ are the observed outputs. In the $H^2$ case, where the $\{u_j, v_j\}$ are zero mean independent random variables. we shall also assume that the covariances of the $\{u_j, v_j\}$ are constant, *i.e.* $Q_j = Q \geq 0$ and $R_j = R > 0$, for all $j$. As before, we are interested in obtaining estimates of some linear combinations of the states, $s_j = L_j x_j$, and, in particular, the filtered estimates, $\hat{s}_{j|j} = L_j \hat{x}_{j|j}$, and predicted estimates, $\hat{s}_j = L_j \hat{x}_j$, that use the observations $\{y_k\}_{k=0}^{j}$ and $\{y_k\}_{k=0}^{j-1}$, respectively.

Under the aforementioned assumptions, it turns out that we can write

$$P_{j+1} - P_j = M_j S M_j, \qquad \forall j, \tag{5.4.2}$$

where $M_j$ is a $n \times d$ matrix and $S$ is a $d \times d$ signature matrix (*i.e.* a diagonal matrix with $+1$ and $-1$ on the diagonal). Thus, for time-invariant state-space models, $P_{j+1} - P_j$ has rank $d$ for all $j$ and in addition has constant inertia. In several important cases, $d$ can be much less than $n$. When this is true, propagating the smaller matrices $M_j$, which is equivalent to propagating the $P_j$, can offer computational reductions. This what is done by the Chandrasekhar recursions (see [Kai81], App. II).

In the conventional Chandrasekhar recursions, one begins with the pre-array

$$\begin{bmatrix} R_{e,j}^{1/2} & HM_j \\ \bar{K}_{p,j} & FM_j \end{bmatrix}, \tag{5.4.3}$$

where $R_{e,j}^{1/2} R_{e,j}^{*/2} = R_{e,j} = R + HP_jH^*$ and $\bar{K}_{p,j} = K_{p,j} R_{e,j}^{-*/2}$, and triangularizes the array using a $J$-unitary matrix $\Theta_j$ where $J$ is given by $\begin{bmatrix} I_p & \\ & S \end{bmatrix}$. The result of this triangularization gives us the various quantities of interest for propagating the Kalman filter recursions.

**Algorithm 5.4.1 (Conventional Chandrasekhar Recursions)** *The gain vector $K_{p,j} = \bar{K}_{p,j} R_{e,j}^{-1/2}$ necessary to obtain the state estimates in the conventional Kalman filter*

$$\hat{x}_{j+1} = F\hat{x}_j + K_{p,j}(y_j - H\hat{x}_j), \quad \hat{x}_0 = 0,$$

*can be computed using*

$$\begin{bmatrix} R_{e,j}^{1/2} & HM_j \\ \bar{K}_{p,j} & FM_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ \bar{K}_{p,j+1} & M_{j+1} \end{bmatrix}, \tag{5.4.4}$$

*where $\Theta_j$ is any $J$-unitary matrix (with $J = I_p \oplus S$) that triangularizes the above pre-array. The algorithm is initialized with*

$$R_{e,0} = R + H\Pi_0 H^*, \qquad \bar{K}_{p,0} = F\Pi_0 H^* R_{e,0}^{1/2},$$

*and*

$$P_1 - \Pi_0 = F\Pi_0 F^* + GQG^* - K_{p,0}R_{e,0}K_{p,0}^* - \Pi_0 = M_0 S M_0^*.$$

Thus, once more, the quantities necessary to update the arrays and to calculate the state estimates are all found from the triangularized post array.

The validity of the above algorithm can be readily verified by squaring both sides of the equation,

$$\begin{bmatrix} R_{e,j}^{1/2} & HM_j \\ \bar{K}_{p,j} & FM_j \end{bmatrix} \Theta_j = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}, \tag{5.4.5}$$

and using the $J$-unitarity of $\Theta_j$, to find the entries of the post array. This leads to,

$$\underbrace{\underbrace{R_{e,j}^{1/2} R_{e,j}^{*/2}}_{R_{e,j}} + H \underbrace{M_j S M_j^*}_{P_{i+1}-P_i} H^*}_{R_{e,j+1}} = AA^*, \tag{5.4.6}$$

from which we conclude that $A = R_{e,j+1}^{1/2}$, and

$$\underbrace{\underbrace{\bar{K}_{p,j} R_{e,j}^{*/2}}_{FP_jH^*} + F \underbrace{M_j S M_j^*}_{P_{i+1}-P_i} H^*}_{FP_{i+1}H^*} = BA^*, \tag{5.4.7}$$

from which we conclude that $B = FP_{i+1}H^* R_{e,j+1}^{-*/2} = \bar{K}_{p,j+1}$. Finally, we have

$$\begin{aligned} CC^* &= FM_j S M_j^* F^* + \bar{K}_{p,j}\bar{K}_{p,j}^* - BB^* \\ &= F(P_{i+1} - P_i)F^* + \bar{K}_{p,j}\bar{K}_{p,j}^* - \bar{K}_{p,j+1}\bar{K}_{p,j+1}^* \\ &= FP_{i+1}F^* + GQG^* - \bar{K}_{p,j+1}\bar{K}_{p,j+1}^* - \left[FP_iF^* + GQG^* - \bar{K}_{p,j}\bar{K}_{p,j}^*\right] \\ &= P_{i+2} - P_{i+1}, \end{aligned}$$

from which we infer that, $C = M_{j+1}$.

If instead of defining, $M_j S M_j^* = P_{j+1} - P_j$, we had defined,

$$N_j S_f N_j^* = P_{j|j} - P_{j-1|j-1}, \tag{5.4.8}$$

where $P_{j|j} = E\tilde{x}_{j|j}\tilde{x}_{j|j}^*$ is the filtered state error variance, which satisfies the recursion,

$$P_{j|j} = FP_{j-1|j-1}F^* + GQG^* - K_{f,j}R_{e,j}K_{f,j}^*, \tag{5.4.9}$$

with $K_{f,j} = P_j H R_{e,j}^{-1}$, then it is also possible to obtain the following (so-called) filtered form of the Chandrasekhar recursions.[6]

**Algorithm 5.4.2 (Conventional Chandrasekhar Recursions- Filtered Form)** *The gain vector $K_{f,j} = \bar{K}_{f,j}R_{e,j}^{-1/2}$ necessary to obtain the state estimates in the filtered form of the conventional Kalman filter*

$$\hat{x}_{j|j} = F\hat{x}_{j-1|j-1} + K_{f,j}(y_j - HF\hat{x}_{j-1|j-1}), \qquad \hat{x}_{-1|-1} = 0,$$

*can be computed using*

$$\begin{bmatrix} R_{e,j}^{1/2} & HFN_j \\ \bar{K}_{f,j} & FN_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ \bar{K}_{f,j+1} & N_{j+1} \end{bmatrix}, \tag{5.4.10}$$

*where $\Theta_j$ is any $J$-unitary matrix (with $J = I_p \oplus S_f$) that triangularizes the above pre-array. The algorithm is initialized with*

$$R_{e,0} = R + H\Pi_0 H^*, \qquad \bar{K}_{f,0} = \Pi_0 H^* R_{e,0}^{1/2},$$

*and*

$$F^{-1}(P_1 - \Pi_0)F^{-*} = \Pi_0 + F^{-1}GQG^*F^{-*} - K_{f,0}R_{e,0}K_{f,0}^* - F^{-1}\Pi_0 F^{-*} = N_0 S_f N_0^*.$$

Note that compared to the square-root formulas, the size of the pre-array in the Chandrasekhar recursions has been reduced from $(p+n)\times(p+n+m)$ to $(p+n)\times(p+d)$ where $m$ and $p$ are the dimensions of the driving disturbance and output, respectively, and where $n$ is the number of the states. Thus the number of operations for each iteration has been reduced from $O(n^3)$ to $O(n^2 d)$, with $d$ typically much less than $n$.

[6] Note that in this case it can also be shown that $P_{j+1|j+1} - P_{j|j}$ has constant inertia, given by the inertia of the signature matrix, $S_f$, for all $j$. This follows from the fact that $P_{j+1} = FP_{j|j}F^* + GQG^*$, so that $P_{j+1} - P_j = F(P_{j|j} - P_{j-1|j-1})F^*$.

## 5.5 $H^\infty$ Chandrasekhar Recursions

In this section we shall derive the $H^\infty$ counterparts of the (fast) $H^2$ Chandrasekhar recursions of the previous section. We shall essentially see that, when the underlying state-space model is time-invariant, all the arguments necessary for the development of these algorithms go through, provided that we consider the geometry of indefinite spaces. We first give the general recursions, and then specialize them to obtain the central filters.

### 5.5.1 The General Case

Recall from Sec. 5.4 that the Chandrasekhar recursions apply to time-invariant state-space models. Therefore the Krein state-space models whose Kalman filters yield the $H^\infty$ estimators must be time-invariant as well. Indeed for the $H^\infty$ a posteriori filtering problem, we need to assume the following Krein state-space model

$$\left\{\begin{array}{rcl} \mathbf{x}_{i+1} & = & F\mathbf{x}_i + G\mathbf{u}_i \\ \begin{bmatrix} \mathbf{y}_i \\ \check{\mathbf{s}}_{i|i} \end{bmatrix} & = & \begin{bmatrix} H \\ L \end{bmatrix} \mathbf{x}_i + \mathbf{v}_i \end{array}\right. \qquad i \geq 0 \tag{5.5.1}$$

with

$$\left\langle \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \\ \mathbf{x}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \\ \mathbf{x}_0 \end{bmatrix} \right\rangle = \begin{bmatrix} I_m\delta_{ij} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix}\delta_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Pi_0 \end{bmatrix}. \tag{5.5.2}$$

Suppose that the matrix $\Pi_0$ can be chosen such that $P_1 - \Pi_0$ has low rank. In other words,

$$P_1 - \Pi_0 = F\Pi_0F^* + GG^* - K_{p,0}R_{e,0}K_{p,0}^* - \Pi_0 = M_0SM_0^*, \tag{5.5.3}$$

where $M_0$ is a $n \times d$ matrix (typically $d \ll n$) and $S$ is a $d \times d$ signature matrix, and, of course,

$$K_{p,j} = FP_j \begin{bmatrix} H^* & L^* \end{bmatrix} R_{e,j}^{-1} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} P_j \begin{bmatrix} H^* & L^* \end{bmatrix}. \tag{5.5.4}$$

We shall presently show by induction that under the assumptions of a time-invariant state-space model, if the a posteriori $H^\infty$ filtering problem has a solution for all $j$, then $P_{j+1}-P_j$ has rank $d$ for all $j$ and that we may actually write $P_{j+1}-P_j = M_j S M_j^*$.

Consider the following pre-array

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} H \\ L \end{bmatrix} M_j \\ \bar{K}_{p,j} & F M_j \end{bmatrix}, \tag{5.5.5}$$

which is the extension of the pre-array in (5.4.4) to the Krein state-space model (5.5.1) of the $H^\infty$ a posteriori filtering problem. Now the $H^\infty$ a posteriori filtering problem will have a solution if, and only if, all leading submatrices of $R$ and $R_{e,j}$ (or $R_{e,j+1}$, for that matter) have the same inertia. In view of Lemma 5.3.1 this implies that the $H^\infty$ a posteriori filtering problem with level $\gamma$ will have a solution if, and only if, there exists a $J$-unitary matrix $\Theta_j$ that triangularizes (5.5.5) where

$$J = \begin{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} & \\ & S \end{bmatrix}. \tag{5.5.6}$$

Therefore we can write

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} H \\ L \end{bmatrix} M_j \\ \bar{K}_{p,j} & F M_j \end{bmatrix} \Theta_j = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}. \tag{5.5.7}$$

To identify the elements $A$, $B$ and $C$ in the post-array we square both sides of (5.5.7) and use the fact that $\Theta_j$ is $J$-unitary. Therefore

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} H \\ L \end{bmatrix} M_j \\ \bar{K}_{p,j} & F M_j \end{bmatrix} \underbrace{\Theta_j J \Theta_j^*}_{=J} \begin{bmatrix} R_{e,j}^{*/2} & \bar{K}_{p,j}^* \\ M_j^* \begin{bmatrix} H^* & L^* \end{bmatrix} & M_j F^* \end{bmatrix} = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} J \begin{bmatrix} A^* & B^* \\ 0 & C^* \end{bmatrix}. \tag{5.5.8}$$

Equating the $(1,1)$ blocks in (5.5.8) yields,

$$A \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} A^* \;=\; R_{e,j}^{1/2} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} R_{e,j}^{*/2} + \begin{bmatrix} H \\ L \end{bmatrix} M_j S M_j^* \begin{bmatrix} H^* & L^* \end{bmatrix}$$

$$
\begin{aligned}
&= R_{e,j} + \begin{bmatrix} H \\ L \end{bmatrix} (P_{j+1} - P_j) \begin{bmatrix} H^* & L^* \end{bmatrix} \\
&= R_j + \begin{bmatrix} H \\ L \end{bmatrix} P_{j+1} \begin{bmatrix} H^* & L^* \end{bmatrix} \\
&= R_{e,j+1}.
\end{aligned}
$$

Therefore $A$ is the indefinite square-root of $R_{e,j+1}$,

$$A = R_{e,j+1}^{1/2}. \tag{5.5.9}$$

Equating the $(2,1)$ blocks in (5.5.8) yields,

$$
\begin{aligned}
B \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} A^* &= \bar{K}_{p,j} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} R_{e,j}^{*/2} + FM_jSM_j^* \begin{bmatrix} H^* & L^* \end{bmatrix} \\
&= K_{p,j}R_{e,j} + F(P_{j+1} - P_j) \begin{bmatrix} H^* & L^* \end{bmatrix} \\
&= FP_j \begin{bmatrix} H^* & L^* \end{bmatrix} + F(P_{j+1} - P_j) \begin{bmatrix} H^* & L^* \end{bmatrix} \\
&= FP_{j+1} \begin{bmatrix} H^* & L^* \end{bmatrix}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
B &= FP_{j+1} \begin{bmatrix} H^* & L^* \end{bmatrix} A^{-*} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \\
&= FP_{j+1} \begin{bmatrix} H^* & L^* \end{bmatrix} R_{e,j+1}^{-*/2} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} = \bar{K}_{p,j+1}.
\end{aligned} \tag{5.5.10}
$$

Equating the $(2,2)$ blocks in (5.5.8) yields

$$CSC^* + B \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} B^* = \bar{K}_{p,j} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} \bar{K}_{p,j}^* + FM_jSM_j^*F^*.$$

Therefore

$$CSC^* + K_{p,j+1}R_{e,j+1}K_{p,j+1}^* = K_{p,j}R_{e,j}K_{p,j}^* + F(P_{j+1} - P_j)F^*.$$

We can now write

$$
\begin{aligned}
CSC^* &= FP_{j+1}F^* - K_{p,j+1}R_{e,j+1}K_{p,j+1}^* + GG^* - (FP_jF^* - K_{p,j}R_{e,j}K_{p,j}^* + GG^*) \\
&= P_{j+2} - P_{j+1},
\end{aligned}
$$

and finally

$$C = M_{j+1}. \tag{5.5.11}$$

Note that our derivation of $C$ also shows that if $P_1 - P_0 = M_0 S M_0^*$ then $P_{i+1} - P_i = M_i S M_i^*$ for all $i$.

We have thus established

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} H \\ L \end{bmatrix} M_j \\ \bar{K}_{p,j} & F M_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ \bar{K}_{p,j+1} & M_{j+1} \end{bmatrix},$$

from which we can now give the following (fast) Chandrasekhar version of the parametrization of all $H^\infty$ a posteriori filters.

**Theorem 5.5.1 ($H^\infty$ A Posteriori Chandrasekhar Recursions)** *The $H^\infty$ a posteriori filtering problem with level $\gamma$ has a solution if, and only if, all leading submatrices of*

$$R = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \quad \text{and} \quad R_{e,0} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} \Pi_0 \begin{bmatrix} H^* & L^* \end{bmatrix}$$

*have the same inertia, and if for all $j = 0, \ldots, i$ there exist $J$-unitary matrices (with $J = I_p \oplus (-I_q) \oplus S$), $\Theta_j$, such that*

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} H \\ L \end{bmatrix} M_j \\ K_{p,j} R_{e,j}^{1/2} & F M_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ K_{p,j+1} R_{e,j+1}^{1/2} & M_{j+1} \end{bmatrix} \tag{5.5.12}$$

*where the algorithm is initialized with, $R_{e,0}$, $K_{p,0} = F\Pi_0 \begin{bmatrix} H^* & L^* \end{bmatrix} R_{e,0}^{-1}$, and*

$$P_1 - \Pi_0 = F\Pi_0 F^* + GQG^* - K_{p,0} R_{e,0} K_{p,0}^* - \Pi_0 = M_0 S M_0^*. \tag{5.5.13}$$

*If this is the case, then all possible $H^\infty$ a posteriori filters, $\check{s}_{j|j} = \mathcal{F}_{f,j}(y_0, \ldots y_j)$, are given by any choices that yield,*

$$\sum_{j=0}^{k} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix} \geq 0, \quad 0 \leq k \leq i$$

*where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j} \begin{bmatrix} y_j - H_j \hat{x}_j \\ \check{s}_{j|j} - L_j \hat{x}_j \end{bmatrix}, \qquad \hat{x}_0 = 0.$$

In the $H^\infty$ a priori filtering problem, we need, instead, to start with the Krein state-space model,

$$\left\{ \begin{array}{rcl} \mathbf{x}_{i+1} & = & F\mathbf{x}_i + G\mathbf{u}_i \\ \begin{bmatrix} \check{\mathbf{s}}_i \\ \mathbf{y}_i \end{bmatrix} & = & \begin{bmatrix} L \\ H \end{bmatrix} \mathbf{x}_i + \mathbf{v}_i \end{array} \right. \qquad i \geq 0 \tag{5.5.14}$$

with

$$\left\langle \begin{bmatrix} \mathbf{u}_i \\ \mathbf{v}_i \\ \mathbf{x}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j \\ \mathbf{v}_j \\ \mathbf{x}_0 \end{bmatrix} \right\rangle = \begin{bmatrix} I_m \delta_{ij} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \begin{bmatrix} -\gamma^2 I_q & 0 \\ 0 & I_p \end{bmatrix} \delta_{ij} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Pi_0 \end{bmatrix}, \tag{5.5.15}$$

and with the prearray,

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} L \\ H \end{bmatrix} M_j \\ K_{p,j} R_{e,j}^{1/2} & F M_j \end{bmatrix}, \tag{5.5.16}$$

where now

$$K_{p,j} = F P_j \begin{bmatrix} L^* & H^* \end{bmatrix} R_{e,j}^{-1} \quad \text{and} \quad R_{e,j} = \begin{bmatrix} -\gamma^2 I_q & 0 \\ 0 & I_p \end{bmatrix} + \begin{bmatrix} L \\ H \end{bmatrix} P_j \begin{bmatrix} L^* & H^* \end{bmatrix}. \tag{5.5.17}$$

[Note that the only difference with the a posteriori case is in the order of the matrices $\{H, L\}$.]

Proceeding with an argument similar to what was done in the a posteriori case, we can show the following result.

**Theorem 5.5.2 ($H^\infty$ A Priori Chandrasekhar Recursions)** *The $H^\infty$ a priori filtering problem with level $\gamma$ has a solution if, and only if, all leading submatrices of*

$$R = \begin{bmatrix} -\gamma^2 I_q & 0 \\ 0 & I_p \end{bmatrix} \quad \textit{and} \quad R_{e,0} = \begin{bmatrix} -\gamma^2 I_q & 0 \\ 0 & I_p \end{bmatrix} + \begin{bmatrix} L \\ H \end{bmatrix} \Pi_0 \begin{bmatrix} L^* & H^* \end{bmatrix}$$

*have the same inertia, and if for all $j = 0, \ldots, i$ there exist $J$-unitary matrices (with $J = (-I_q) \oplus I_p \oplus S$), $\Theta_j$, such that*

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} L \\ H \end{bmatrix} M_j \\ K_{p,j} R_{e,j}^{1/2} & F M_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ K_{p,j+1} R_{e,j+1}^{1/2} & M_{j+1} \end{bmatrix} \tag{5.5.18}$$

*where the algorithm is initialized with,* $R_{e,0}$, $K_{p,0} = F\Pi_0 \begin{bmatrix} L^* & H^* \end{bmatrix} R_{e,0}^{-1}$, *and*

$$P_1 - \Pi_0 = F\Pi_0 F^* + GQG^* - K_{p,0} R_{e,0} K_{p,0}^* - \Pi_0 = M_0 S M_0^*. \tag{5.5.19}$$

*If this is the case, then all possible $H^\infty$ a priori filters, $\check{s}_j = \mathcal{F}_{f,j}(y_0, \ldots y_{j-1})$, are given by any choices that yield,*

$$\sum_{j=0}^{k} \begin{bmatrix} \check{s}_j - L_j \hat{x}_j \\ y_j - H_j \hat{x}_j \end{bmatrix}^* R_{e,j}^{-1} \begin{bmatrix} \check{s}_j - L_j \hat{x}_j \\ y_j - H_j \hat{x}_j \end{bmatrix} \geq 0, \quad 0 \leq k \leq i$$

*where $\hat{x}_j$ satisfies the recursion,*

$$\hat{x}_{j+1} = F_j \hat{x}_j + K_{p,j} \begin{bmatrix} \check{s}_j - L_j \hat{x}_j \\ y_j - H_j \hat{x}_j \end{bmatrix}, \qquad \hat{x}_0 = 0.$$

Note that compared to the $H^\infty$ square-root formulas, the size of the pre-array in the $H^\infty$ Chandrasekhar recursions has been reduced from $(p+q+n) \times (p+q+n+m)$ to $(p+q+n) \times (p+q+d)$ where $m$, $p$ and $q$ are the dimensions of the driving disturbance, output and states to be estimated, respectively, and where $n$ is the number of the states. Thus the number of operations for each iteration has been reduced from $O(n^3)$ to $O(n^2 d)$ with $d$ typically much less than $n$.

As in the square-root case, the Chandrasekhar recursions do not require explicitly checking the positivity conditions of Theorems 3.2.1 and 3.2.2 — if the recursions can be carried out an $H^\infty$ estimator of the desired level exists, and if not, such an estimator does not exist.

### 5.5.2 The Central Filters

The preceding section gave (fast) Chandrasekhar versions of all possible $H^\infty$ a posteriori and a priori filters. Here we shall specialize these recursions to the central a posteriori and a priori filters. We shall show that the observer gains for these filters can be readily obtained from the post-arrays of Theorems 5.5.1 and 5.5.2, provided we insist on (block) lower triangular square-roots for $R_{e,j}^{1/2}$. The development closely follows that of Sec. 5.3.2 and uses the important facts (established in Sec. 5.3.2) that if $R_{e,j}^{1/2}$ is lower triangular, then $K_{s,j}$, the gain vector for the central a posteriori filter, is given by,

$$K_{s,j} = \left(\text{first block column of } K_{f,j}R_{e,j}^{1/2}\right) \cdot \left((1,1) \text{ block entry of } R_{e,j}^{1/2}\right)^{-1}, \quad (5.5.20)$$

and $K_{a,j}$, the gain vector for the central a priori filter, is given by,

$$K_{a,j} = \left(\text{second block column of } K_{p,j}R_{e,j}^{1/2}\right) \cdot \left((2,2) \text{ block entry of } R_{e,j}^{1/2}\right)^{-1}. \quad (5.5.21)$$

The following result are now readily establishable. (The proofs are straightforward and will be omitted for brevity.)

**Algorithm 5.5.1 (Central $H^\infty$ A Posteriori Chandrasekhar Recursions)** *The $H^\infty$ a posteriori filtering problem with level $\gamma$ has a solution if, and only if, all leading submatrices of*

$$R = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} \quad \text{and} \quad R_{e,0} = \begin{bmatrix} I_p & 0 \\ 0 & -\gamma^2 I_q \end{bmatrix} + \begin{bmatrix} H \\ L \end{bmatrix} \Pi_0 \begin{bmatrix} H^* & L^* \end{bmatrix}$$

*have the same inertia, and if for all $j = 0, \ldots, i$ there exist $J$-unitary matrices, $\Theta_j$, (where $J = I_p \oplus (-I_q) \oplus S_f$) such that*

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} H \\ L \end{bmatrix} FN_j \\ K_{f,j}R_{e,j}^{1/2} & FN_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ K_{f,j}R_{e,j+1}^{1/2} & N_{j+1} \end{bmatrix} \quad (5.5.22)$$

*with $R_{e,j}^{1/2}$ and $R_{e,j+1}^{1/2}$ block lower triangular. The algorithm is initialized with, $R_{e,0}$, $K_{f,0} = \Pi_0 \begin{bmatrix} H^* & L^* \end{bmatrix} R_{e,0}^{-1}$, and*

$$F^{-1}(P_1 - \Pi_0)F^{-*} = \Pi_0 + F^{-1}GQG^*F^{-*} - K_{f,0}R_{e,0}K_{f,0}^* - F^{-1}\Pi_0 F^{-*} = N_0 S_f N_0^*.$$

*The gain vector* $K_{s,j}$ *needed to update the estimates in the central filter recursions*

$$\hat{x}_{j|j} = F_{j-1}\hat{x}_{j-1|j-1} + K_{s,j}(y_j - H_jF_{j-1}\hat{x}_{j-1|j-1}), \quad \hat{x}_{-1|-1} = 0,$$

*is equal to*

$$K_{s,j} = \bar{K}_{s,j}(I + H_jP_jH_j^*)^{-1/2},$$

*where* $\bar{K}_{s,j}$ *is given by the first block column of* $\bar{K}_{f,j} = K_{f,j}R_{e,j}^{1/2}$, *and* $(I + H_jP_jH_j^*)^{1/2}$ *is given by the* $(1,1)$ *block entry of* $R_{e,j}^{1/2}$.

**Algorithm 5.5.2 (Central $H^\infty$ A Priori Chandrasekhar Recursions)** *The* $H^\infty$ *a priori filtering problem with level* $\gamma$ *has a solution if, and only if, all leading submatrices of*

$$R = \begin{bmatrix} -\gamma^2 I_q & 0 \\ 0 & I_p \end{bmatrix} \quad \text{and} \quad R_{e,0} = \begin{bmatrix} -\gamma^2 I_q & 0 \\ 0 & I_p \end{bmatrix} + \begin{bmatrix} L \\ H \end{bmatrix} \Pi_0 \begin{bmatrix} L^* & H^* \end{bmatrix}$$

*have the same inertia, and if for all* $j = 0, \ldots, i$ *there exist* $J$*-unitary matrices,* $\Theta_j$, *(where* $J = (-I_q) \oplus I_p \oplus S$*) such that*

$$\begin{bmatrix} R_{e,j}^{1/2} & \begin{bmatrix} L \\ H \end{bmatrix} M_j \\ K_{p,j}R_{e,j}^{1/2} & FM_j \end{bmatrix} \Theta_j = \begin{bmatrix} R_{e,j+1}^{1/2} & 0 \\ K_{p,j+1}R_{e,j+1}^{1/2} & M_{j+1} \end{bmatrix} \tag{5.5.23}$$

*with* $R_{e,j}^{1/2}$ *and* $R_{e,j+1}^{1/2}$ *block lower triangular. The algorithm is initialized with,* $R_{e,0}$, $K_{p,0} = F\Pi_0 \begin{bmatrix} L^* & H^* \end{bmatrix} R_{e,0}^{-1}$ *and*

$$P_1 - \Pi_0 = F\Pi_0F^* + GQG^* - K_{p,0}R_{e,0}K_{p,0}^* - \Pi_0 = M_0SM_0^*.$$

*The gain vector* $K_{a,i}$ *needed to update the estimates in*

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{a,j}(y_j - F_j\hat{x}_j), \quad \hat{x}_0 = 0,$$

*is equal to*

$$K_{a,j} = \bar{K}_{a,j}(I + H_j\tilde{P}_jH_j^*)^{-1/2},$$

*where* $\bar{K}_{a,j}$ *is given by the second block column of* $\bar{K}_{p,j} = K_{p,j}R_{e,j}^{1/2}$, *and* $(I+H_j\tilde{P}_jH_j^*)^{1/2}$ *is given by the* $(2,2)$ *block entry of* $R_{e,j}^{1/2}$.

## 5.6 Conclusion

In this chapter, we obtained square-root array algorithms and Chandrasekhar recursions for the $H^\infty$ a posteriori and a priori filtering problems. These algorithms involve propagating the indefinite square-roots of the quantities of interest, and have the interesting property that the appropriate inertia of these quantities is preserved. Moreover, the conditions for the existence of the $H^\infty$ filters are built into the algorithms, so that filter solutions will exist if,and only if, the algorithms can be executed.

The conventional square-root arrays and Chandrasekhar recursions are preferred because of their numerical stability (in the case of square-root arrays) and their reduced computational complexity (in the case of the Chandrasekhar recursions). Since the $H^\infty$ square-root arrays and Chandrasekhar recursions are the direct analogs of their conventional counterparts, they may be more attractive for numerical implementation of $H^\infty$ filters. However, since $J$-unitary rather than unitary operations are involved, further numerical investigation is needed.

Our derivation of the $H^\infty$ square-root arrays and Chandrasekhar recursions demonstrates a virtue of the Krein space approach to $H^\infty$ estimation and control; the results appear to be more difficult to conceive and prove in the traditional $H^\infty$ approaches. We should also mention that there are many variations of the conventional square-root array and Chandrasekhar recursions, *e.g.* for control problems, and the methods given here are directly applicable to extending these variations to the $H^\infty$ setting as well. Finally, the algorithms presented here are equally applicable to risk-sensitive estimation and control problems, and to quadratic dynamic games.

# Chapter 6

# Duality and Control

This chapter deals with duality in (definite and indefinite metric) linear spaces. Duality here is introduced through the geometrical notion of dual bases for linear spaces spanned by a set of nonorthogonal basis vectors. We shall see that, apart from having conceptual value, duality is a useful tool in the study of various problems of interest. In particular, it allows for a dual approach to (definite and indefinite) quadratic problems. A large part of this chapter is therefore devoted to the use of duality in ($H^2$, $H^\infty$, game-theoretic and risk-sensitive) control, where (within the framework presented here) the notion of duality is indispensable for obtaining the solution.

## 6.1 Introduction

In this chapter we return to the geometric viewpoint of Chapter 2. We have already seen the power of such a geometrical approach in unifying the treatment of (stochastic) $H^2$ estimation and (deterministic) $H^\infty$ estimation problems, (where the solutions are obtained by projecting, respectively in a Hilbert and Krein space, one vector onto the linear span of another set of vectors), in the introduction of the innovations process (via Gram-Schmidt orthogonalization) for performing the canonical factorization of (possibly indefinite) Gramian matrices, and in the extension of conventional square-root arrays and Chandrasekhar recursions to the $H^\infty$ setting. Here we shall introduce a further natural consequence of adopting a geometric point of view; more specifically, the study of dual bases and dual linear models. The concept of a dual basis is one that has considerable (albeit, often conceptual) value in studying linear spaces spanned by a set of nonorthogonal basis vectors. It also has several ramifications; in particular, it shows how deterministic and stochastic least-squares problems can be solved using a dual approach, it has certain implications to smoothing problems, and, most importantly, it is very useful in the study of control problems (where the duality between estimation and control has long been recognized).[1] We shall, in fact, spend considerable time on a general linear-quadratic-regulator (LQR) control problem with indefinite weighting matrices, and its special cases in full information and measurement feedback $H^\infty$ control.

The remainder of this chapter is organized as follows. In Sec. 6.2 we begin by considering a different deterministic quadratic form to the one that had been studied in Sec. 2.5.2, and whose stationary point was given by a Krein space projection. These two deterministic quadratic forms are related by the fact that their coefficient matrices are inverses of one another. In Sec. 6.3, by considering the concept of a dual basis, we obtain a geometric interpretation for the stationary point of the quadratic form of Sec. 6.2, and in particular, show that it is given by a dual Krein space projection. Moreover, we obtain various useful relationships (via certain dualities and equivalences) that shed further light on the interplay between deterministic and stochastic least-squares problems (see Table 6.4.1).[2] In particular, in Sec. 6.4, we show that, given any quadratic optimization problem, it can be solved in either one of two ways; by constructing an equivalent Krein space model, for which the solution is given by a Krein space projection, or, by constructing a dual Krein space model, for which the solution is given by the negative conjugate transpose of the operator that yields the dual Krein space projection. Which one of these approaches to use depends upon the application; for estimation and adaptive filtering problems it is more natural to use the equivalent Krein space model, whereas for control problems it more natural to use the dual Krein space model.

Now the dual bases span the orthogonal complement space, or the space of the

[1]See also the comments at the end of Sec. 1.5.1.

[2]For different approaches to duality consult [Wal89] and [Lue92].

smoothing errors, of the original bases. In Sec. 6.5, we show that when the original basis has state-space structure, then the dual basis also has state-space structure. [The dual state-space model is often referred to as the adjoint or complementary state-space model.] In particular, we develop backwards-time, forwards-time and mixed dual state-space models, and in Sec. 6.6 use the mixed dual state-space model to obtain the so-called two filter formulae for smoothing.

Sec. 6.7 considers the LQR (linear-quadratic-regulator) control problem with (possibly) indefinite weighting matrices. This problem is of interest since it subsumes, as special cases, the problems of $H^2$, $H^\infty$, game-theoretic and risk-sensitive control. We solve the LQR problem using duality, *i.e.*, we first construct the dual Krein space model associated with the indefinite LQR quadratic form (which is a Krein space backwards-time dual state-space model) and then use the (negative conjugate transpose of the operator that performs the) dual Krein space projection to find its stationary point (and hence its solution). The dual projection is performed by the Krein space Kalman filter associated with the backwards-time dual state-space model, and the condition for the stationary point to be a minimum is given by certain inertia requirements on Gramians readily available from the Krein space Kalman filter. The results of Sec. 6.7 are then used in Secs. 6.8 and 6.9 to solve the full information and measurement feedback $H^\infty$ control problems. The solutions for both causal and strictly causal controllers are given, and the measurement feedback problem's solution requires a separation principle that was earlier introduced in Sec. 1.7.2.

## 6.2 An Alternative Scalar Quadratic Form

In Sec. 2.5.2 we saw that if we form a scalar quadratic form

$$J(z,y) \equiv \begin{bmatrix} z^* & y^* \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \begin{bmatrix} z \\ y \end{bmatrix}$$

where $\begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}$ may be identified as the Gramian of the the Krein space variables $\{\mathbf{z}, \mathbf{y}\}$, then the stationary point of $J(z,y)$ over $z$ can be obtained via computing the projection of $\mathbf{z}$ onto $\mathcal{L}(\mathbf{y})$. However, there is another scalar quadratic form that can be constructed from the Gramian of $\{\mathbf{z}, \mathbf{y}\}$ and that we have not yet considered. This is what is done in the following Lemma.

**Lemma 6.2.1 (Matrix to Scalar Second Order Form)** *The stationary point $y_o^d$ (over $z^d$) of*

$$I(z^d, y^d) = \begin{bmatrix} z^{d*} & y^{d*} \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} \begin{bmatrix} z^d \\ y^d \end{bmatrix}$$

*is given by*

$$y_o^d = -R_y^{-1} R_{yz} z^d,$$

*and*

$$I(z^d, y_o^d) = z^{d*}(R_z - R_{zy} R_y^{-1} R_{yz}) z^d.$$

*Moreover, this stationary point is a minimum if, and only if,*

$$R_y > 0.$$

**Proof:** The proof involves the (lower-upper) triangular factorization of $\begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}$ and is straightforward.

■

**Remark:** Note that the coefficient matrix appearing in the solution to the deterministic problem of Lemma 6.2.1 is given by the negative conjugate transpose of the coefficient matrix appearing in the projection of $\mathbf{z}$ onto $\mathbf{y}$. However, the condition for a minimum, $R_y > 0$, is the exact same as that of the stochastic problem of Theorem 2.5.1.

In view of the above result, it is now natural to speculate whether the stationary point of Lemma 6.2.1 is related to a stochastic problem in the same way that the the stationary point of Theorem 2.5.2 was related to the stochastic problem of Theorem 2.5.1. In the next section we shall see that this is indeed the case. The development involves the introduction of so-called dual bases, and will bring forth various dualities that will be useful for the study of both estimation and control problems in sections 6.6 and 6.7.

## 6.3 Dual Bases

Recall from Chapter 2 the definition of a Krein space $\mathcal{K}$ over a given ring of scalars (such as the ring of complex numbers), $\mathcal{S}$. Consider a set of linearly independent vectors $\{\mathbf{z}_0, \ldots, \mathbf{z}_m, \mathbf{y}_0, \ldots \mathbf{y}_n\}$, (with $\mathbf{z}_i \in \mathcal{K}$ $\mathbf{y}_j \in \mathcal{K}$, $i = 0, \ldots m$, $j = 0, \ldots n$), which we shall denote by $\{\mathbf{z}, \mathbf{y}\}$ where[3]

$$\mathbf{z} = \text{col}\{\mathbf{z}_0, \ldots, \mathbf{z}_m\} \quad \text{and} \quad \mathbf{y} = \text{col}\{\mathbf{y}_0, \ldots, \mathbf{y}_n\}.$$

The corresponding Gramian for this set of vectors is denoted by

$$\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \rangle = \begin{bmatrix} \langle \mathbf{z}, \mathbf{z} \rangle & \langle \mathbf{z}, \mathbf{y} \rangle \\ \langle \mathbf{y}, \mathbf{z} \rangle & \langle \mathbf{y}, \mathbf{y} \rangle \end{bmatrix} = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}, \tag{6.3.1}$$

which due to the linear independence is nonsingular. Moreover, the linear independence implies that $\{\mathbf{z}, \mathbf{y}\}$ forms a *basis* for $\mathcal{L}\{\mathbf{z}, \mathbf{y}\}$ (*i.e.*, the linear space of all vectors $a_0\mathbf{z}_0 + \ldots + a_m\mathbf{z}_m + b_0\mathbf{y}_0 + \ldots + b_n\mathbf{y}_n$, for $a_i, b_j \in \mathcal{S}$).

**Definition 6.3.1 (Dual Basis)** *Given a basis, $\{\mathbf{z}, \mathbf{y}\}$, the dual basis is defined as the pair $\{\mathbf{z}^d, \mathbf{y}^d\}$ with the properties,*

$$\mathcal{L}\{\mathbf{z}^d, \mathbf{y}^d\} = \mathcal{L}\{\mathbf{z}, \mathbf{y}\}, \tag{6.3.2}$$

*and*

$$\begin{aligned} \langle \mathbf{z}^d, \mathbf{z} \rangle = I \quad &, \quad \langle \mathbf{z}^d, \mathbf{y} \rangle = 0 \;, \\ \langle \mathbf{y}^d, \mathbf{z} \rangle = 0 \quad &, \quad \langle \mathbf{y}^d, \mathbf{y} \rangle = I \;. \end{aligned} \tag{6.3.3}$$

Note that if $\{\mathbf{z}, \mathbf{y}\}$ were an *orthonormal* basis then the dual basis would simply coincide with our original basis. In general, however, the dual basis will be different and (6.3.3) is referred to as the *bi-orthogonality condition.*

We now describe two ways - algebraic and geometric - of finding the dual basis.

[3] As a matter of fact we do not need to partition the set of independent vectors into a set of $\{\mathbf{z}_i\}$ and $\{\mathbf{y}_i\}$ in order to introduce the concept of dual bases. However, since our major interest is estimation, and the set $\{\mathbf{y}_i\}$ will typically designate the observations, while the set $\{\mathbf{z}_i\}$ will designate the quantities we want to estimate, we shall find this partitioning quite convenient.

### 6.3.1 Algebraic Specification

Clearly since $\{\mathbf{z}, \mathbf{y}\}$ and $\{\mathbf{z}^d, \mathbf{y}^d\}$ span the same space we must have

$$\begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \tag{6.3.4}$$

for some nonsingular block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Now the defining requirements (6.3.3) are equivalent to

$$\left\langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \right\rangle = \begin{bmatrix} \langle \mathbf{z}^d, \mathbf{z} \rangle & \langle \mathbf{z}^d, \mathbf{y} \rangle \\ \langle \mathbf{y}^d, \mathbf{z} \rangle & \langle \mathbf{y}^d, \mathbf{y} \rangle \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{6.3.5}$$

But

$$\left\langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \right\rangle = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \left\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \right\rangle = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}. \tag{6.3.6}$$

Combining the above two relations we have

$$\begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}. \tag{6.3.7}$$

Moreover, the Gramian of $\{\mathbf{z}^d, \mathbf{y}^d\}$ is seen to be

$$\begin{aligned} \left\langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \right\rangle &= \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \left\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \right\rangle \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-*} \\ &= \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-*} = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1}, \end{aligned}$$

since Gramians are Hermitian.

**Lemma 6.3.1 (Dual Basis)** *Consider the pair $\{\mathbf{z}, \mathbf{y}\}$ with nonsingular Gramian given by (6.3.1). Then the the dual basis $\{\mathbf{z}^d, \mathbf{y}^d\}$ satisfying the bi-orthogonality condition (6.3.3) is given by*

$$\begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}.$$

*Moreover, the Gramian of the dual basis* $\{\mathbf{z}^d, \mathbf{y}^d\}$ *is*

$$\left\langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \right\rangle = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1}.$$

The above arguments and result are quite specific, but may not be very intuitive. Let us now consider the geometric description.

### 6.3.2 Geometric Specification

Let us first introduce the notations

$$\hat{\mathbf{z}}_{|\mathbf{y}} = \text{the projection of } \mathbf{z} \text{ onto } \mathcal{L}\{\mathbf{y}\}\ , \tag{6.3.8}$$

and the error

$$\tilde{\mathbf{z}} \triangleq \tilde{\mathbf{z}}_{|\mathbf{y}} = \mathbf{z} - \hat{\mathbf{z}}_{|\mathbf{y}}. \tag{6.3.9}$$

From the orthogonality principle we have $\langle \tilde{\mathbf{z}}, \mathbf{y} \rangle = 0$. On the other hand, due to the property that $\langle \mathbf{z}^d, \mathbf{y} \rangle = 0$, and since $\{\tilde{\mathbf{z}}, \mathbf{y}\}$ and $\{\mathbf{z}^d, \mathbf{y}\}$ span the same space, we clearly see that $\{\tilde{\mathbf{z}}\}$ and $\{\mathbf{z}^d\}$ must span the same space as well. Thus we must have

$$\mathbf{z}^d = M\tilde{\mathbf{z}}, \tag{6.3.10}$$

for some nonsingular $M$. However, note that

$$I = \langle \mathbf{z}^d, \mathbf{z} \rangle = M\langle \tilde{\mathbf{z}}, \mathbf{z} \rangle = M\langle \tilde{\mathbf{z}}, \tilde{\mathbf{z}} + \hat{\mathbf{z}} \rangle = M\langle \tilde{\mathbf{z}}, \tilde{\mathbf{z}} \rangle = MR_{\tilde{z}}\ , \tag{6.3.11}$$

so that

$$\mathbf{z}^d = R_{\tilde{z}}^{-1}\tilde{\mathbf{z}}_{|\mathbf{y}}. \tag{6.3.12}$$

Similarly, of course we have

$$\mathbf{y}^d = R_{\tilde{y}}^{-1}\tilde{\mathbf{y}}_{|\mathbf{z}}. \tag{6.3.13}$$

The reader may want to verify that the invertibility of the matrices $R_x, R_y$ and the Gramian (6.3.1) guarantees the invertibility of $R_{\tilde{z}}$ and $R_{\tilde{y}}$.

**Lemma 6.3.2 (Dual Basis - Revisited)** *Consider the pair $\{\mathbf{z}, \mathbf{y}\}$ with nonsingular Gramian given by (6.3.1). Then the the dual basis $\{\mathbf{z}^d, \mathbf{y}^d\}$ satisfying the bi-orthogonality condition (6.3.3) is given by*

$$\mathbf{z}^d = R_{\tilde{z}}^{-1}\tilde{\mathbf{z}}_{|\mathbf{y}} \quad \textit{and} \quad \mathbf{y}^d = R_{\tilde{y}}^{-1}\tilde{\mathbf{y}}_{|\mathbf{z}} \ ,$$

*where $\tilde{\mathbf{z}}_{|\mathbf{y}}$ and $\tilde{\mathbf{y}}_{|\mathbf{z}}$ are the errors in projecting $\mathbf{z}$ and $\mathbf{y}$ onto $\mathcal{L}\{\mathbf{y}\}$ and $\mathcal{L}\{\mathbf{z}\}$, respectively, and $R_{\tilde{z}}$ and $R_{\tilde{y}}$ are their corresponding Gramians.*

The vectors $\{\mathbf{z}^d, \mathbf{y}^d\}$ can be readily sketched as in Fig. 6.1. [In simple cases, determining the dual basis from the geometry can be simpler than inverting the Gramian matrix.]

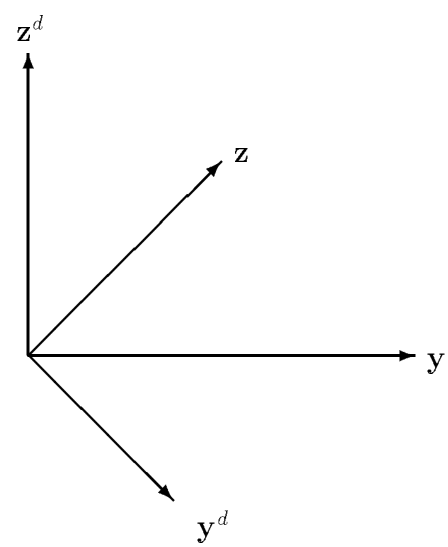


Figure 6.1: Dual bases.

But why introduce dual bases? There are several reasons. One is the fact that to compute the projection of a vector $\mathbf{x}$ onto $\mathcal{L}\{\mathbf{y}, \mathbf{z}\}$, say

$$\hat{\mathbf{x}}_{|\mathbf{y},\mathbf{z}} = A\mathbf{z} + B\mathbf{y} \ , \tag{6.3.14}$$

requires solving a system of linear equations to determine the coefficient matrices $\{A, B\}$, except, of course, when $\mathbf{y}$ and $\mathbf{z}$ are orthogonal. In the orthogonal case, the coefficients are obtained by projecting $\mathbf{x}$ separately on $\mathbf{z}$ and $\mathbf{y}$, so that

$$\hat{\mathbf{x}}_{|\mathbf{y},\mathbf{z}} = \langle \mathbf{x}, \mathbf{z} \rangle \|\mathbf{z}\|^{-2}\mathbf{z} + \langle \mathbf{x}, \mathbf{y} \rangle \|\mathbf{y}\|^{-2}\mathbf{y}, \quad \text{when } \langle \mathbf{y}, \mathbf{z} \rangle = 0. \tag{6.3.15}$$

In the general case, we can obtain a somewhat similar expression by using the dual basis.

**Lemma 6.3.3 (Projection via the Dual Basis)** $\hat{\mathbf{x}}_{|\mathbf{y},\mathbf{z}}$, *the projection of* $\mathbf{x}$ *onto* $\mathcal{L}\{\mathbf{y}, \mathbf{z}\}$, *can be written as*

$$\hat{\mathbf{x}}_{|\mathbf{y},\mathbf{z}} = \langle \mathbf{x}, \mathbf{z}^d \rangle \mathbf{z} + \langle \mathbf{x}, \mathbf{y}^d \rangle \mathbf{y}, \tag{6.3.16}$$

*where* $\{\mathbf{y}^d, \mathbf{z}^d\}$ *is the dual basis.*

**Proof:** We need to find $\{A, B\}$ such that

$$\mathbf{x} - A\mathbf{z} - B\mathbf{y} \ \perp \ \mathcal{L}\{\mathbf{y}, \mathbf{z}\} = \mathcal{L}\{\mathbf{y}^d, \mathbf{z}^d\}.$$

But

$$0 = \langle \mathbf{x} - A\mathbf{z} - B\mathbf{y}, \mathbf{z}^d \rangle = \langle \mathbf{x}, \mathbf{z}^d \rangle - A\langle \mathbf{z}, \mathbf{z}^d \rangle - B\langle \mathbf{y}, \mathbf{z}^d \rangle = \langle \mathbf{x}, \mathbf{z}^d \rangle - A.$$

Similarly,

$$0 = \langle \mathbf{x}, \mathbf{y}^d \rangle - B.$$

■

Of course, the apparent simplicity of the formula (6.3.16) is, in general, only conceptual rather than computational. The algebraic equations for $\{A, B\}$ are

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} = \begin{bmatrix} \langle \mathbf{x}, \mathbf{z} \rangle & \langle \mathbf{x}, \mathbf{y} \rangle \end{bmatrix}, \tag{6.3.17}$$

and inverting the (Gramian) coefficient matrix is equivalent to determining the dual basis $\{\mathbf{z}^d, \mathbf{y}^d\}$! Nevertheless, the conceptual simplification makes the introduction of dual bases, and their geometric interpretation, quite useful.

We can also obtain an interesting result by combining the algebraic and geometric characterizations just given. Recall that the geometric characterization was

$$\begin{bmatrix} \tilde{\mathbf{z}}_{|\mathbf{y}} \\ \tilde{\mathbf{y}}_{|\mathbf{z}} \end{bmatrix} = \begin{bmatrix} R_{\tilde{z}} & 0 \\ 0 & R_{\tilde{y}} \end{bmatrix} \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}. \tag{6.3.18}$$

Therefore we have

$$\begin{aligned} \left\langle \begin{bmatrix} \tilde{\mathbf{z}}_{|\mathbf{y}} \\ \tilde{\mathbf{y}}_{|\mathbf{z}} \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \right\rangle &= \begin{bmatrix} R_{\tilde{z}} & 0 \\ 0 & R_{\tilde{y}} \end{bmatrix} \left\langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \right\rangle \\ &= \begin{bmatrix} R_{\tilde{z}} & 0 \\ 0 & R_{\tilde{y}} \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1}. \end{aligned} \tag{6.3.19}$$

But note that

$$\langle \tilde{\mathbf{z}}_{|\mathbf{y}}, \mathbf{z}^d \rangle = \langle \mathbf{z} - R_{zy} R_y^{-1} \mathbf{y}, \mathbf{z}^d \rangle = I - 0 = I \ ,$$

while

$$\langle \tilde{\mathbf{z}}_{|\mathbf{y}}, \mathbf{y}^d \rangle = \langle \mathbf{z} - R_{zy} R_y^{-1} \mathbf{y}, \mathbf{y}^d \rangle = 0 - R_{zy} R_y^{-1} I.$$

With similar results for $\langle \tilde{\mathbf{y}}_{|\mathbf{z}}, \mathbf{z}^d \rangle$ and $\langle \tilde{\mathbf{y}}_{|\mathbf{z}}, \mathbf{y}^d \rangle$, we see that

$$\left\langle \begin{bmatrix} \tilde{\mathbf{z}}_{|\mathbf{y}} \\ \tilde{\mathbf{y}}_{|\mathbf{z}} \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \right\rangle = \begin{bmatrix} I & -R_{zy} R_y^{-1} \\ -R_{yz} R_z^{-1} & I \end{bmatrix}.$$

But this leads immediately to the identity

$$\begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} = \begin{bmatrix} R_{\tilde{z}}^{-1} & 0 \\ 0 & R_{\tilde{y}}^{-1} \end{bmatrix} \begin{bmatrix} I & -R_{zy} R_y^{-1} \\ -R_{yz} R_z^{-1} & I \end{bmatrix}, \tag{6.3.20}$$

whose origin is otherwise not so evident. We can also rewrite (6.3.20) as

$$\begin{bmatrix} R_{z^d} & R_{z^d y^d} \\ R_{y^d z^d} & R_{y^d} \end{bmatrix} = \begin{bmatrix} R_{\tilde{z}}^{-1} & -R_{\tilde{z}}^{-1} R_{zy} R_y^{-1} \\ -R_{\tilde{y}}^{-1} R_{yz} R_z^{-1} & R_{\tilde{y}}^{-1} \end{bmatrix}. \tag{6.3.21}$$

In particular, this implies that

$$R_{z^d} = R_{\tilde{z}}^{-1} \quad , \quad R_{y^d} = R_{\tilde{y}}^{-1} \tag{6.3.22}$$

and

$$R_{zy} R_y^{-1} = -R_{\tilde{z}} R_{z^d y^d} = -R_{z^d}^{-1} R_{z^d y^d} = -(R_{y^d z^d} R_{z^d}^{-1})^*. \tag{6.3.23}$$

But recalling that

$$\hat{\mathbf{z}}_{|\mathbf{y}} = R_{zy} R_y^{-1} \mathbf{y} \quad \text{and} \quad \hat{\mathbf{y}}^d_{|\mathbf{z}^d} = R_{y^d z^d} R_{z^d}^{-1} \mathbf{z}^d \tag{6.3.24}$$

we see that the gain matrix for estimating $\mathbf{z}$ from $\mathbf{y}$ is the negative conjugate transpose of the gain matrix for estimating $\mathbf{y}^d$ from $\mathbf{z}^d$!

This is an interesting result, and we therefore collect the above identities into the following Lemma.

**Lemma 6.3.4 (Dual Computation of Projection)** *The projection of* $\mathbf{z}$ *onto* $\mathcal{L}(\mathbf{y})$*, say* $\hat{\mathbf{z}}_{|\mathbf{y}}$*, can be computed as*

$$\hat{\mathbf{z}}_{|\mathbf{y}} = -R_{z^d}^{-1} R_{z^d y^d} \mathbf{y} \ ,$$

*where* $\{R_{z^d}, R_{z^d y^d}\}$ *are the Gramians and cross-Gramians of the dual basis vectors* $\{\mathbf{z}^d, \mathbf{y}^d\}$*. In other words, we have the identity*

$$R_{zy} R_y^{-1} = -R_{z^d}^{-1} R_{z^d y^d}.$$

*Also, the minimum mean-square-error covariance matrix is given by*

$$R_{\tilde{z}} = R_{z^d}^{-1}.$$

The above lemma captures some of the duality between $\{\mathbf{z}, \mathbf{y}\}$ and the dual basis $\{\mathbf{z}^d, \mathbf{y}^d\}$. However, before proceeding with this duality, we shall study the consequence of the above results to the case where the $\{\mathbf{z}, \mathbf{y}\}$ are related in a linear fashion.

### 6.3.3 Linear Models

It is interesting to apply the above results to the simple linear model,[4]

$$\mathbf{y} = H\mathbf{z} + \mathbf{v}, \tag{6.3.25}$$

[4] It can, in fact, be shown that there is no loss of generality in assuming such a linear model (rather than assuming the general case of arbitrarily related Krein space variables, $\mathbf{z}$ and $\mathbf{y}$). However, this representation is useful since it has (and will) frequently occur in this thesis.

where the Gramian matrix of $\mathbf{z}$ and $\mathbf{v}$ is block-diagonal,

$$\left\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{v} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{v} \end{bmatrix} \right\rangle = \begin{bmatrix} R_z & 0 \\ 0 & R_v \end{bmatrix}. \tag{6.3.26}$$

Moreover, assume that $R_z$ and $R_v$ are invertible. Then

$$\begin{aligned} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} = \left\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \right\rangle &= \begin{bmatrix} I & 0 \\ H & I \end{bmatrix} \left\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{v} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{v} \end{bmatrix} \right\rangle \begin{bmatrix} I & H^* \\ 0 & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ H & I \end{bmatrix} \begin{bmatrix} R_z & 0 \\ 0 & R_v \end{bmatrix} \begin{bmatrix} I & H^* \\ 0 & I \end{bmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} &= \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \\ &= \left\{ \begin{bmatrix} I & 0 \\ H & I \end{bmatrix} \begin{bmatrix} R_z & 0 \\ 0 & R_v \end{bmatrix} \begin{bmatrix} I & H^* \\ 0 & I \end{bmatrix} \right\}^{-1} \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \\ &= \begin{bmatrix} I & -H^* \\ 0 & I \end{bmatrix} \begin{bmatrix} R_z^{-1} & 0 \\ 0 & R_v^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -H & I \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \\ &= \begin{bmatrix} I & -H^* \\ 0 & I \end{bmatrix} \begin{bmatrix} R_z^{-1} & 0 \\ 0 & R_v^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{v} \end{bmatrix}, \\ &= \begin{bmatrix} R_z^{-1}\mathbf{z} - H^* R_v^{-1}\mathbf{v} \\ R_v^{-1}\mathbf{v} \end{bmatrix}. \end{aligned}$$

We thus see that

$$\mathbf{y}^d = R_v^{-1}\mathbf{v}, \tag{6.3.27}$$

and that $\mathbf{z}^d$ arises from a *dual* linear model,

$$\mathbf{z}^d = -H^* R_v^{-1}\mathbf{v} + R_z^{-1}\mathbf{z} = -H^*\mathbf{y}^d + R_z^{-1}\mathbf{z} = -H^*\mathbf{y}^d + \mathbf{v}^c, \tag{6.3.28}$$

where we have introduced $\mathbf{v}^c = R_z^{-1}\mathbf{z}$ and, hence,

$$\left\langle \begin{bmatrix} \mathbf{y}^d \\ \mathbf{v}^c \end{bmatrix}, \begin{bmatrix} \mathbf{y}^d \\ \mathbf{v}^c \end{bmatrix} \right\rangle = \begin{bmatrix} R_v^{-1} & 0 \\ 0 & R_z^{-1} \end{bmatrix}.$$

The Gramian matrix of the dual basis is readily seen to be

$$\left\langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \right\rangle = \begin{bmatrix} R_{z^d d} & R_{z^d d y^d} \\ R_{y^d z^d d} & R_{y^d} \end{bmatrix} = \begin{bmatrix} R_z^{-1} + H^* R_v^{-1} H & -H^* R_v^{-1} \\ -R_v^{-1} H & R_v^{-1} \end{bmatrix}. \quad (6.3.29)$$

We can therefore compute the projection of $\mathbf{y}^d$ onto $\mathbf{z}^d$ as follows

$$\hat{\mathbf{y}}^d_{|\mathbf{z}^d} = R_{y^d z^d d} R_{y^d}^{-1} \mathbf{z}^d = -R_v^{-1} H (R_z^{-1} + H^* R_v^{-1} H)^{-1} \mathbf{z}^d. \quad (6.3.30)$$

On the other hand,

$$\hat{\mathbf{z}}_{|\mathbf{y}} = R_{zy} R_y^{-1} = R_z H^* (R_v + H R_z H^*)^{-1} \mathbf{y}. \quad (6.3.31)$$

But as claimed in Lemma 6.3.4, the coefficient matrices for these two problems must be the negative conjugate transpose of each other, *i.e.*, we must have

$$R_z H^* (R_v + H R_z H^*)^{-1} = -(-R_v^{-1} H (R_z^{-1} + H^* R_v^{-1} H)^{-1})^* = (R_z^{-1} + H^* R_v^{-1} H)^{-1} H^* R_v^{-1} \quad (6.3.32)$$

which is an identity that can also be verified algebraically.

We note that these so-called "information form" expressions for the estimates actually correspond to an estimation problem for certain dual variables. To summarize the results, we have the following Lemma.

**Lemma 6.3.5 (Linear Models and Dual Bases)** *Suppose* $\{\mathbf{z}, \mathbf{y}\}$ *satisfy the linear model*

$$\mathbf{y} = H\mathbf{z} + \mathbf{v},$$

*where*

$$\left\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{v} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{v} \end{bmatrix} \right\rangle = \begin{bmatrix} R_z & 0 \\ 0 & R_v \end{bmatrix},$$

*and both* $R_z$ *and* $R_v$ *are assumed nonsingular. Then the dual basis* $\{\mathbf{z}^d, \mathbf{y}^d\}$ *will also satisfy a linear model, namely,*

$$\mathbf{z}^d = -H^* \mathbf{y}^d + \mathbf{v}^c,$$

*where*

$$\mathbf{y}^d = R_v^{-1} \mathbf{v} \quad \textit{and} \quad \mathbf{v}^c = R_z^{-1} \mathbf{z}.$$

*We therefore have the identities*

$$R_{zy}R_y^{-1} = R_zH^*(R_v + HR_zH^*)^{-1} = (R_z^{-1} + H^*R_v^{-1}H)^{-1}H^*R_v^{-1} = -R_{z^dd}R_{z^ddy^d}\ ,$$

*and*

$$\langle \tilde{\mathbf{z}}_{|y}, \tilde{\mathbf{z}}_{|y}\rangle = R_{\tilde{z}} = R_{z^d}^{-1} = (R_z^{-1} + H^*R_v^{-1}H)^{-1}.$$

## 6.4 A Pair of Duality and Equivalence Relationships

In the previous section we considered the problem of projecting the dual vector $\mathbf{y}^d$ onto the dual vector $\mathbf{z}^d$, and saw that the problem was dual to the problem (considered so far) of projecting $\mathbf{z}$ onto $\mathbf{y}$. In this section we shall study the consequences of this observation to the solution of deterministic least-squares problems. The main conclusion is, that given a deterministic quadratic minimization problem, one can solve it in two-ways: either by constructing an equivalent Krein space model (which is what has been done so far, say in the $H^\infty$ filtering problems of Chapter 3), where the solution is the same as the original problem, or by constructing a dual Krein space model where the solution is given by the negative transpose of the solution of the original problem. Although, not done here, it is possible to solve the general estimation problem using this dual approach to obtain the (so-called) Information-form Kalman filter [AM79, Kai81]. However, the major benefit of using this dual approach is in the solution of LQR (linear quadratic regulator) control problems that will be studied in Sec. 6.7.

### 6.4.1 General Equivalence and Duality Relationships

Lemma 6.3.4 captures the duality between the original basis $\{\mathbf{z}, \mathbf{y}\}$ and the dual basis $\{\mathbf{z}^d, \mathbf{y}^d\}$. Most noteworthy it points out that the inverse of the matrix that appears in $I(z^d, y^d)$ of Lemma 6.2.1, is the Gramian of the dual basis $\{\mathbf{z}^d, \mathbf{y}^d\}$. We shall presently continue to pursue this connection.

Note now that the scalar quadratic form appearing in Lemma 6.2.1 can be written as

$$I(z,y) = \begin{bmatrix} z^{d*} & y^{d*} \end{bmatrix} \langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \rangle^{-1} \begin{bmatrix} z^d \\ y^d \end{bmatrix}. \tag{6.4.1}$$

Using Theorems 2.5.1 and 2.5.2 this implies that the stationary point of $I(z^d, y^d)$ over $y^d$ is given by the same coefficient matrix that computes the projection of $\mathbf{y}^d$ onto $\mathbf{z}^d$. Thus

$$\hat{\mathbf{y}}^d_{|z^d} = R_{y^d z^d} R_{z^d}^{-1} \mathbf{z}^d \quad \text{and} \quad y^d_o = R_{y^d z^d} R_{z^d}^{-1} z^d. \tag{6.4.2}$$

Now from Lemma 6.3.4 we have $R_{y^d z^d} R_{z^d}^{-1} = -R_y^{-1} R_{yz}$, so that

$$y^d_o = -R_y^{-1} R_{yz} z, \tag{6.4.3}$$

as in Lemma 6.2.1. The condition for a minimum is (using Theorem 2.5.2 and Lemma 6.3.4)

$$R_{y^d}^{-1} - R_{y^d z^d} R_{z^d}^{-1} R_{z^d y^d} = R_y^{-1} > 0, \tag{6.4.4}$$

which is the same condition as in Lemma 6.2.1. Thus the deterministic problem of Lemma 6.2.1 is related to the stochastic problem of projecting $\mathbf{y}^d$ onto $\mathbf{z}^d$.

The results of this section are summarized in Table 6.4.1. It collects the relationships between the two dual stochastic problems and their corresponding deterministic quadratic forms. Vertical transitions from (i) to (iii) and from (ii) to (iv) correspond to going from the original bases to the dual bases, so that the solutions are the dual (negative conjugate transpose) of each other and the conditions for a minimum are different. Horizontal transitions from (i) to (ii) and (iii) to (iv) correspond to going from matrix-valued to scalar-valued quadratic forms so that these solutions are also the dual of one another, however, now the conditions for a minimum are the same. Diagonal transitions from (i) to (iv) and (ii) to (iii) relate problems with the same solution, as in Theorems 2.5.1 and 2.5.2, but with different conditions for a minimum.

It is important to keep the picture of these four relationships in mind. As we have seen so far, and as we shall see in the remainder of this thesis, with appropriate choice of the variables $\mathbf{z}$ and $\mathbf{y}$, we are able to solve a number of interesting problems in estimation and control, and to study their properties. All these results are special

| | Stochastic Problems | Deterministic Problems |
|---|---|---|
| Model | (i) $\left\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \right\rangle = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}$ | $(ii)$ $\{z^d, y^d\}$ |
| Problem | $\min_{\hat{\mathbf{z}} \in \mathcal{L}\{\mathbf{y}\}} \langle \mathbf{z} - \hat{\mathbf{z}}, \mathbf{z} - \hat{\mathbf{z}} \rangle$ | $\min_{y^d} \begin{bmatrix} y^{d*} & z^{d*} \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix} \begin{bmatrix} y^d \\ z^d \end{bmatrix}$ |
| Solution | $\hat{\mathbf{z}} = K_o \mathbf{y}$ <br> $K_o = R_{zy} R_y^{-1}$ | $y_o^d = K_o^d z^d$ <br> $K_o^d = -R_y^{-1} R_{yz}$ |
| Cond. for Min. | $R_y > 0$ | $R_y > 0$ |
| Value at Min. | $R_z - R_{zy} R_y^{-1} R_{yz} = R_{z^d}^{-1}$ | $z^{d*}(R_z - R_{zy} R_y^{-1} R_{yz}) z^d = z^{d*} R_{z^d}^{-1} z^d$ |
| Model | (iii) $\left\langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \right\rangle = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1}$ | $(iv)$ $\{z, y\}$ |
| Problem | $\min_{\hat{\mathbf{y}}^d \in \mathcal{L}\{\mathbf{z}^d\}} \langle \mathbf{y}^d - \hat{\mathbf{y}}^d, \mathbf{y}^d - \hat{\mathbf{y}}^d \rangle$ | $\min_z \begin{bmatrix} z^* & y^* \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \begin{bmatrix} z \\ y \end{bmatrix}$ |
| Solution | $\hat{\mathbf{y}}^d = K_o^d \mathbf{z}^d$ <br> $K_o^d = R_{y^d z^d} R_{z^d}^{-1} = -R_y^{-1} R_{yz}$ | $z_o = K_o y$ <br> $K_o = R_{zy} R_y^{-1}$ |
| Cond. for Min. | $R_y^d = (R_y - R_{zy} R_y^{-1} R_{yz})^{-1} > 0$ | $R_z - R_{zy} R_y^{-1} R_{yz} > 0$ |
| Value at Min. | $R_{y^d}^{-1} - R_{y^d z^d} R_{z^d}^{-1} R_{z^d y^d} = R_y^{-1}$ | $y^* R_y^{-1} y$ |

Table 6.1: General equivalences and dualities.

cases of these four problems, when we specialize to a particular choice and structure for $\mathbf{z}$ and $\mathbf{y}$.

We can now explain how these four relations may be used to solve various problems. Suppose we are given an optimization problem of the form (iv), *i.e.* ,

$$\min z \begin{bmatrix} z^* & y^* \end{bmatrix} \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1} \begin{bmatrix} z \\ y \end{bmatrix}. \tag{6.4.5}$$

Then to obtain the solution we may proceed in either one of two ways. We can construct an *equivalent* Krein space model of the form

$$\{\mathbf{z}, \mathbf{y}\}, \tag{6.4.6}$$

with

$$\langle \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} \rangle = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}, \tag{6.4.7}$$

where $R_z$, $R_{zy} = R_{yz}^*$ and $R_y$ are given in (iv). Then the projection of $\mathbf{z}$ onto $\mathbf{y}$ provides the solution to (iv).

Or, we can construct the *dual* Krein space model

$$\{\mathbf{z}^d, \mathbf{y}^d\}, \tag{6.4.8}$$

with

$$\langle \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix}, \begin{bmatrix} \mathbf{z}^d \\ \mathbf{y}^d \end{bmatrix} \rangle = \begin{bmatrix} R_z & R_{zy} \\ R_{yz} & R_y \end{bmatrix}^{-1}, \tag{6.4.9}$$

where, once more, $R_z$, $R_{zy} = R_{yz}^*$ and $R_y$ are given in (iv). Now if we find the projection of $\mathbf{y}^d$ onto $\mathbf{z}^d$, we can use the negative conjugate transpose of the matrix that performs this projection to solve problem (iv).

Which one of the above two approaches to use depends upon the application at hand. It so turns out that for estimation and adaptive filtering problems it is more natural to use the equivalent Krein space model, whereas for control problems it more natural to use the dual Krein space model.

# 6.5 Dual State-Space Models

In this section we shall study the structure of the dual bases when the observations $\{\mathbf{y}_i\}$ have an underlying state-space structure. As demonstrated in the previous section, the dual bases $\mathbf{z}^d$ and $\mathbf{y}^d$ span the same space as the *smoothed* estimation errors $\tilde{\mathbf{z}}_{|y}$ and $\tilde{\mathbf{y}}_{|z}$, respectively. Thus, in a certain sense, we will also be obtaining state-space models for the smoothed estimation errors as well. The state-space models obtained here will be useful in the study of smoothing problems, in obtaining the information-form Kalman filter, and especially in the study of LQR control problems.

## 6.5.1 The Backwards Dual Model

Consider once more the time-varying state-space model

$$\left\{\begin{array}{rcl} \mathbf{x}_{i+1} & = & F_i\mathbf{x}_i + \mathbf{u}_i, \quad 0 \le i \le N \\ \mathbf{y}_i & = & H_i\mathbf{x}_i + \mathbf{v}_i \end{array}\right. \tag{6.5.1}$$

with

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \right\rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q_i\delta_{ij} & 0 \\ 0 & 0 & R_i\delta_{ij} \end{bmatrix}.$$

Recall the definitions of the observability map,

$$\mathcal{O} = \begin{bmatrix} H_0 \\ H_1F_0 \\ \vdots \\ H_NF_{N-1}\dots F_0 \end{bmatrix},$$

and the impulse response matrix,

$$\Gamma = \begin{bmatrix} 0 & & & \\ H_1G_0 & 0 & & \\ H_2F_1G_0 & H_2G_1 & & \\ \vdots & \vdots & \ddots & \\ H_NF_{N-1}\dots F_1G_0 & H_NF_{N-1}\dots F_2G_1 & \dots & H_NG_{N-1} \end{bmatrix},$$

so that we may write

$$\mathbf{y} = \mathcal{O}\mathbf{x}_0 + \Gamma\mathbf{u} + \mathbf{v} = \begin{bmatrix} \mathcal{O} & \Gamma & I \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathcal{O} & \Gamma \end{bmatrix} & I \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{v} \end{bmatrix},$$

where we have defined, $\mathbf{z} \triangleq \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \end{bmatrix}$. We are now interested in $\{\mathbf{z}^d, \mathbf{y}^d\}$, the dual basis to $\{\mathbf{z}, \mathbf{y}\}$. To this end note that

$$\langle \begin{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \end{bmatrix} \\ \mathbf{v} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \end{bmatrix} \\ \mathbf{v} \end{bmatrix} \rangle = \begin{bmatrix} \begin{bmatrix} \Pi_0 & 0 \\ 0 & Q \end{bmatrix} & 0 \\ 0 & R \end{bmatrix}$$

where

$$Q \triangleq Q_0 \oplus \ldots \oplus Q_N \quad \text{and} \quad R \triangleq R_0 \oplus \ldots \oplus R_N.$$

We now have the following interesting result.

**Lemma 6.5.1 (Dual Model for State-Space Structure)** *Consider the linear state-space model*

$$\mathbf{y} = \begin{bmatrix} \mathcal{O} & \Gamma \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \end{bmatrix} + \mathbf{v},$$

*with*

$$\langle \begin{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \end{bmatrix} \\ \mathbf{v} \end{bmatrix}, \begin{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \end{bmatrix} \\ \mathbf{v} \end{bmatrix} \rangle = \begin{bmatrix} \begin{bmatrix} \Pi_0 & 0 \\ 0 & Q \end{bmatrix} & 0 \\ 0 & R \end{bmatrix},$$

*and $\Pi_0$, $Q$ and $R$ all invertible. Then choosing $\mathbf{z} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \end{bmatrix}$, the dual basis $\{\mathbf{z}^d, \mathbf{y}^d\}$ is given by*

$$\mathbf{z}^d = -\begin{bmatrix} \mathcal{O}^* \\ \Gamma^* \end{bmatrix} \mathbf{y}^d + \mathbf{v}^c, \tag{6.5.2}$$

*where*

$$\mathbf{y}^d = R^{-1}\mathbf{v} \quad \textit{and} \quad \mathbf{v}^c = \begin{bmatrix} \Pi_0^{-1}\mathbf{x}_0 \\ Q^{-1}\mathbf{u} \end{bmatrix}. \tag{6.5.3}$$

*In addition we may write*

$$\mathbf{z}^d = \begin{bmatrix} -\mathcal{O}^* R^{-1}\mathbf{v} + \Pi_0^{-1}\mathbf{x}_0 \\ -\Gamma^* R^{-1}\mathbf{v} + Q^{-1}\mathbf{u} \end{bmatrix} = \begin{bmatrix} R_{\tilde{x}_{0|y}} & R_{\tilde{x}_{0|y}\tilde{u}_{|y}} \\ R_{\tilde{u}_{|y}\tilde{x}_{0|y}} & R_{\tilde{u}_{|y}} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{x}}_{0|y} \\ \tilde{\mathbf{u}}_{|y} \end{bmatrix} = \begin{bmatrix} R^{-1}_{\tilde{x}_{0|y,u}} \tilde{\mathbf{x}}_{0|y,u} \\ R^{-1}_{\tilde{u}_{|y,x_0}} \tilde{\mathbf{u}}_{|y,x_0} \end{bmatrix}. \tag{6.5.4}$$

**Proof:** Eqs. (6.5.2) and (6.5.3) follow immediately from the dual bases formulae for linear models given in Lemma 6.3.5. The first of the equalities in (6.5.4), *i.e.*,

$$\mathbf{z}^d = \begin{bmatrix} -\mathcal{O}^* R^{-1}\mathbf{v} + \Pi_0^{-1}\mathbf{x}_0 \\ -\Gamma^* R^{-1}\mathbf{v} + Q^{-1}\mathbf{u} \end{bmatrix},$$

follows from (6.5.2-6.5.3), the second equality, *i.e.*,

$$\mathbf{z}^d = \begin{bmatrix} R_{\tilde{x}_{0|y}} & R_{\tilde{x}_{0|y}\tilde{u}_{|y}} \\ R_{\tilde{u}_{|y}\tilde{x}_{0|y}} & R_{\tilde{u}_{|y}} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{x}}_{0|y} \\ \tilde{\mathbf{u}}_{|y} \end{bmatrix},$$

follows from the equation $\mathbf{z}^d = R^{-1}_{\tilde{z}_{|y}} \tilde{\mathbf{z}}_{|y}$, given in Lemma 6.3.2, and the third equality, from the fact that

$$\begin{bmatrix} R_{\tilde{x}_{0|y}} & R_{\tilde{x}_{0|y}\tilde{u}_{|y}} \\ R_{\tilde{u}_{|y}\tilde{x}_{0|y}} & R_{\tilde{u}_{|y}} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{x}}_{0|y} \\ \tilde{\mathbf{u}}_{|y} \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}^d_{0|y} \\ \tilde{\mathbf{u}}^d_{|y} \end{bmatrix} = \begin{bmatrix} R^{-1}_{\tilde{x}_{0|y,u}} \tilde{\mathbf{x}}_{0|y,u} \\ R^{-1}_{\tilde{u}_{|y,x_0}} \tilde{\mathbf{u}}_{|y,x_0} \end{bmatrix}.$$

■

Note that the above lemma gives a dual linear model (viz. (6.5.2-6.5.3)) that describes the dual bases $\{\mathbf{z}^d, \mathbf{y}^d\}$ and also gives a physical interpretation for $\mathbf{z}^d$ in terms of various smoothing errors. It turns out, however, that since we have state-space structure, we can give an even more explicit representation of the dual basis, $\mathbf{z}^d$, and in fact show that it also possesses state-space structure.

**Lemma 6.5.2 (Backwards Dual State-Space Model)** *The vector* $\mathbf{z}^d$ *in the dual basis* $\{\mathbf{z}^d, \mathbf{y}^d\}$ *of Lemma 6.5.1 is given by*

$$\mathbf{z}^d = \begin{bmatrix} \xi_0^d + \Pi_0^{-1}\mathbf{x}_0 \\ \eta^d \end{bmatrix}, \tag{6.5.5}$$

*where* $\eta^d = col\{\eta_0^d, \ldots, \eta_N^d\}$ *has the following backwards-time state-space model*

$$\left\{\begin{array}{lcl} \xi_i^d & = & F_i^* \xi_{i+1}^d - H_i^* R_i^{-1} \mathbf{v}_i \quad \xi_{N+1}^d = 0 \\ \eta_i^d & = & G_i^* \xi_{i+1}^d + Q_i^{-1} \mathbf{u}_i \end{array}\right. , \quad i = N, N-1, \ldots 0. \tag{6.5.6}$$

*and where* $\xi_0^d$ *is its final state.*

*Moreover, the (backwards) estimation errors of the states,* $\tilde{\xi}_{i|i}^d \triangleq \xi_i^d - \hat{\xi}_{i|i}^d$*, given the (future) observations* $\{\eta_i^d, \ldots, \eta_N^d\}$ *in (6.5.6) have the following interpretation*

$$\tilde{\xi}_{i|i}^d = P_{i|i}^{-b} \tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1} \mathbf{x}_i, \tag{6.5.7}$$

*where* $\tilde{\mathbf{x}}_{i|i}^b \triangleq \mathbf{x}_i - \hat{\mathbf{x}}_{i|i}^b$ *is the (backwards) error in estimating* $\mathbf{x}_i$ *using the (future) observations* $\{\mathbf{y}_i, \ldots, \mathbf{y}_N\}$*, where* $P_{i|i}^b$ *is the corresponding error Gramian, and where* $\Pi_i$ *is the Gramian of* $\mathbf{x}_i$*.*

**Proof:** Note from Lemma 6.5.1 that we need to show

$$\left[\begin{array}{c} -\mathcal{O}^* R^{-1} \mathbf{v} + \Pi_0^{-1} \mathbf{x}_0 \\ -\Gamma^* R^{-1} \mathbf{v} + Q^{-1} \mathbf{u} \end{array}\right] = \left[\begin{array}{c} \xi_0^d + \Pi_0^{-1} \mathbf{x}_0 \\ \eta^d \end{array}\right].$$

Now the state space model (6.5.6) can be written in matrix form as

$$\left[\begin{array}{c} \eta_0^d \\ \eta_1^d \\ \vdots \\ \eta_N^d \end{array}\right] = \mathcal{C}^* \xi_{N+1}^d - \Gamma^* R^{-1} \mathbf{v} + Q^{-1} \mathbf{u}$$

where

$$\mathcal{C} = \left[\begin{array}{cccc} F_N F_{N-1} \ldots F_1 G_0 & F_N F_{N-1} \ldots F_2 G_1 & \ldots & G_N \end{array}\right],$$

is the controllability map of the original state space model. But when $\xi_{N+1}^d = 0$ this is the matrix relation defining $\eta^d$.

To prove that the first entry of the dual basis is $\xi_0^d + \Pi_0^{-1} \mathbf{x}_0$, we use the state-space recursion to write

$$\xi_0^d = \Phi_F^* \xi_{N+1}^d - \mathcal{O} R^{-1} \mathbf{v} = -\mathcal{O} R^{-1} \mathbf{v},$$

where $\Phi_F^* = F_0^* \ldots F_N^*$ is the state transition matrix corresponding to (6.5.6) and since $\xi_{N+1}^d = 0$. But if we recall that the first entry of $\mathbf{z}^d$ is

$$-\mathcal{O}^* R^{-1}\mathbf{v} + \Pi_0^{-1}\mathbf{x}_0,$$

we obtain the desired result.

Finally, to prove (6.5.7) let us first recall from (6.5.4) that

$$\mathbf{z}^d = \begin{bmatrix} \xi_0^d + \Pi_0^{-1}\mathbf{x}_0 \\ \eta^d \end{bmatrix} = \begin{bmatrix} R_{\tilde{x}_{0|y}} & R_{\tilde{x}_{0|y}\tilde{u}_{|y}} \\ R_{\tilde{u}_{|y}\tilde{x}_{0|y}} & R_{\tilde{u}_{|y}} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\mathbf{x}}_{0|y} \\ \tilde{\mathbf{u}}_{|y} \end{bmatrix} \tag{6.5.8}$$

and that therefore the Gramian of $\mathbf{z}^d$ is

$$\begin{bmatrix} R_{\tilde{x}_{0|y}} & R_{\tilde{x}_{0|y}\tilde{u}_{|y}} \\ R_{\tilde{u}_{|y}\tilde{x}_{0|y}} & R_{\tilde{u}_{|y}} \end{bmatrix}^{-1}.$$

Let us now consider $\{\xi_0^d + \Pi_0^{-1}\mathbf{x}_0, \eta^d\}$ and try to construct its dual basis $\{(\xi_0^d + \Pi_0^{-1}\mathbf{x}_0)^d, (\eta^d)^d\}$. This is straightforward to find, since

$$\begin{bmatrix} (\xi_0^d + \Pi_0^{-1}\mathbf{x}_0)^d \\ (\eta^d)^d \end{bmatrix} = \begin{bmatrix} R_{\tilde{x}_{0|y}} & R_{\tilde{x}_{0|y}\tilde{u}_{|y}} \\ R_{\tilde{u}_{|y}\tilde{x}_{0|y}} & R_{\tilde{u}_{|y}} \end{bmatrix} \begin{bmatrix} \xi_0^d + \Pi_0^{-1}\mathbf{x}_0 \\ \eta^d \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{x}}_{0|y} \\ \tilde{\mathbf{u}}_{|y} \end{bmatrix},$$

where in the last equality we have used (6.5.8). Therefore we conclude

$$(\xi_0^d + \Pi_0^{-1}\mathbf{x}_0)^d = \tilde{\mathbf{x}}_{0|y}.$$

But on the other hand

$$(\xi_0^d + \Pi_0^{-1}\mathbf{x}_0)^d = M^{-1}(\tilde{\xi}_{0|\eta}^d + \Pi_0^{-1}\tilde{\mathbf{x}}_{0|\eta}),$$

where $M$ is the Gramian of $\tilde{\xi}_{0|\eta}^d + \Pi_0^{-1}\tilde{\mathbf{x}}_{0|\eta}$. But since $\eta$ is a function of only the $\{\mathbf{u}_i, \mathbf{v}_i\}$, we conclude that $\tilde{\mathbf{x}}_{0|\eta} = \mathbf{x}_0$ , and therefore

$$M^{-1}(\tilde{\xi}_{0|\eta}^d + \Pi_0^{-1}\mathbf{x}_0) = \tilde{\mathbf{x}}_{0|y}.$$

Taking the Gramian of both sides of the above equation we conclude that $M = R_{x_{0|y}}^{-1}$. Therefore we may write

$$\tilde{\xi}_{0|\eta}^d = R_{x_{0|y}}^{-1}\tilde{\mathbf{x}}_{0|y} - \Pi_0^{-1}\mathbf{x}_0.$$

But noting that $\tilde{\mathbf{x}}_{0|y} = \tilde{\mathbf{x}}^b_{0|0}$ and $R_{x_{0|y}} = P^b_{0|0}$, we have

$$\tilde{\xi}^d_{0|0} = P^{-b}_{0|0}\tilde{\mathbf{x}}^b_{0|0} - \Pi_0^{-1}\mathbf{x}_0.$$

Since, in principle, we could have taken any other time instant (instead of $i = 0$) as the initial point, we also have

$$\tilde{\xi}^d_{i|i} = P^{-b}_{i|i}\tilde{\mathbf{x}}^b_{i|i} - \Pi_i^{-1}\mathbf{x}_i,$$

which is the desired result.

■

The backwards-time state-space model

$$\left\{ \begin{array}{lcl} \xi^d_i & = & F^*_i\xi^d_{i+1} - H^*_iR^{-1}_i\mathbf{v}_i \quad \xi^d_{N+1} = 0 \\ \eta^d_i & = & G^*_i\xi^d_{i+1} + Q^{-1}_i\mathbf{u}_i \end{array} \right. , \quad i = N, N-1, \ldots 0. \tag{6.5.9}$$

is well-known in the system theory and is referred to as the *adjoint* state-space model, or as the backwards-time *complementary* model (since its output spans the orthogonal complement space of the output of the original model) [WD81, AK89]. However, we shall continue to call it the backwards dual model since its output describes the dual basis.

The duality with the original model is quite explicit. Indeed, we see that if we perform the following transformations;

$$F_i \to F^*_i \ , \ G_i \to H^*_i \ , \ H_i \to G^*_i \ \ , \ Q_i \to R^{-1}_i \ , \ R_i \to Q^{-1}_i$$

and reverse forward time to backward time we can obtain the dual state-space model from its original.

One further remark is that, as we shall see in Sec. 6.7 the above backwards dual state-space model will be very useful in the study of LQR control problems.

We close this section with one interesting observation regarding the output of the backwards dual state-space model (6.5.6).

**Corollary 6.5.1 (An Orthogonality Result)** *Consider the backwards dual model (6.5.6) and the original state-space model (6.5.1). Then we have*

$$\mathbf{x}_i \ \perp \ \eta^d_j \quad j = i, i+1, \ldots, N. \tag{6.5.10}$$

**Proof:** Follows readily from the orthogonality of the $\{\mathbf{x}_0, \{\mathbf{u}_i\}, \{\mathbf{v}_i\}\}$ and the facts that

$$\mathbf{x}_i \in \mathcal{L}\{\mathbf{u}_0, \mathbf{v}_0, \ldots \mathbf{u}_{i-1}, \mathbf{v}_{i-1}\} \quad \text{and} \quad \eta_j \in \mathcal{L}\{\mathbf{u}_j, \mathbf{v}_j \ldots \mathbf{u}_N, \mathbf{v}_N\}.$$

■

### 6.5.2 The Forwards Dual Model

In the previous section we obtained the dual basis to $\{\mathbf{z}, \mathbf{y}\}$ where $\mathbf{z} = \text{col}\{\mathbf{x}_0, \mathbf{u}\}$ and $\mathbf{y}$ were related via the standard state-space model (6.5.1). This led us to the basis $\{\mathbf{z}^d, \mathbf{y}^d\}$ of Lemma 6.5.1, where one component of $\mathbf{z}^d$ satisfied the backwards-time state-space model (6.5.6).

As is obvious from the defining relation (6.3.3), the dual basis depends on our choice of the original basis. In the state-space context, $\{\mathbf{u}, \mathbf{x}_0, \mathbf{y}\}$ is just one choice of basis for the underlying space $\mathcal{L}\{\mathbf{u}, \mathbf{x}_0, \mathbf{y}\}$, and alternative dual bases are found when one chooses different bases for $\mathcal{L}\{\mathbf{u}, \mathbf{x}_0, \mathbf{y}\}$. In this and the next section, we shall study the consequences of choosing such different bases.

When the matrices $\{F_i\}_{i=0}^N$ are nonsingular, $\{\mathbf{u}, \mathbf{x}_{N+1}, \mathbf{y}\}$ is a basis for $\mathcal{L}\{\mathbf{u}, \mathbf{x}_0, \mathbf{y}\}$, since the matrix obtained in the transformation

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{x}_{N+1} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ \mathcal{C} & \Phi_F & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_0 \\ \mathbf{y} \end{bmatrix}, \tag{6.5.11}$$

is nonsingular. (Recall that $\mathcal{C}$ and $\Phi_F = F_N \ldots F_0$ are the controllability and state-transition matrices for the state-space model (6.5.1), and that $\mathbf{x}_{N+1} = \Phi_F \mathbf{x}_0 + \mathcal{C}\mathbf{u}$.)

Defining $\mathbf{z}' \triangleq \text{col}\{\mathbf{u}, \mathbf{x}_{N+1}\}$, we can now consider $\{\mathbf{z}', \mathbf{y}\}$ as a basis for $\mathcal{L}\{\mathbf{u}, \mathbf{x}_0, \mathbf{y}\}$. Our goal is to find the dual basis $\{\mathbf{z}'^d, \mathbf{y}^d\}$.

Note that with the above transformation we may write the linear relationship between $\{\mathbf{u}, \mathbf{x}_{N+1}, \mathbf{v}\}$ as follows

$$\begin{aligned} \mathbf{y} &= \mathcal{O}\mathbf{x}_0 + \Gamma\mathbf{u} + \mathbf{v} \\ &= \mathcal{O}(\Phi^{-1}\mathbf{x}_{N+1} - \Phi^{-1}\mathcal{C}) + \Gamma\mathbf{u} + \mathbf{v} \\ &= \mathcal{O}\Phi^{-1}\mathbf{x}_{N+1} + (\Gamma - \mathcal{O}\Phi^{-1}\mathcal{C})\mathbf{u} + \mathbf{v} \end{aligned}$$

$$= \begin{bmatrix} \Gamma - \mathcal{O}\Phi^{-1}\mathcal{C} & \mathcal{O}\Phi^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_{N+1} \end{bmatrix} + \mathbf{v}.$$

The above linear model for $\mathbf{y}$, allows us to apply the results of Lemma 6.3.5 in order to obtain the dual basis $\{\mathbf{z}'^d, \mathbf{y}^d\}$. Thus

$$\begin{aligned}
\mathbf{z}'^d &= -\begin{bmatrix} \Gamma^* - \mathcal{C}^*\Phi^{-*}\mathcal{O}^* \\ \Phi^{-*}\mathcal{O}^* \end{bmatrix} R^{-1}\mathbf{v} + \langle \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_{N+1} \end{bmatrix}, \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_{N+1} \end{bmatrix} \rangle^{-1} \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_{N+1} \end{bmatrix} \\
&= -\begin{bmatrix} \Gamma^* - \mathcal{C}^*\Phi^{-*}\mathcal{O}^* \\ \Phi^{-*}\mathcal{O}^* \end{bmatrix} R^{-1}\mathbf{v} + \langle \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_0 \end{bmatrix}, \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_{N+1} \end{bmatrix} \rangle^{-1} \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_0 \end{bmatrix} \\
&= -\begin{bmatrix} \Gamma^* - \mathcal{C}^*\Phi^{-*}\mathcal{O}^* \\ \Phi^{-*}\mathcal{O}^* \end{bmatrix} R^{-1}\mathbf{v} + \begin{bmatrix} Q & Q\mathcal{C}^* \\ 0 & \Pi_0\Phi^* \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_0 \end{bmatrix} \\
&= -\begin{bmatrix} \Gamma^* - \mathcal{C}^*\Phi^{-*}\mathcal{O}^* \\ \Phi^{-*}\mathcal{O}^* \end{bmatrix} R^{-1}\mathbf{v} + \begin{bmatrix} Q^{-1} & -\mathcal{C}^*\Phi^{-*}\Pi_0^{-1} \\ 0 & \Phi^{-*}\Pi_0^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{x}_0 \end{bmatrix}
\end{aligned}$$

from which we finally conclude

$$\mathbf{z}'^d = -\begin{bmatrix} -\mathcal{C}^*\Phi^{-*}\Pi_0^{-1}\mathbf{x}_0 - (\Gamma^* - \mathcal{C}^*\Phi^{-*}\mathcal{O}^*)R^{-1}\mathbf{v} + Q^{-1}\mathbf{u} \\ \Phi^{-*}\Pi_0^{-1}\mathbf{x}_0 - \Phi^{-*}\mathcal{O}^*R^{-1}\mathbf{v} \end{bmatrix}. \tag{6.5.12}$$

The above expression can be used to obtain a state-space model for the dual basis $\mathbf{z}'^d$. Indeed we have the following result.

**Lemma 6.5.3 (Forwards Dual State-Space Model)** *Consider the standard state-space model (6.5.1). Then the vector* $\mathbf{z}'^d$ *in the dual basis* $\{\mathbf{z}'^d, \mathbf{y}'^d\}$ *of* $\{\mathbf{z}', \mathbf{y}\}$ *(with* $\mathbf{z}' \triangleq col\{\mathbf{u}, \mathbf{x}_{N+1}\}$*) is given by*

$$\mathbf{z}^d = \begin{bmatrix} \eta'^d \\ \xi'^d_{N+1} \end{bmatrix}, \tag{6.5.13}$$

*where* $\eta'^d = col\{\eta'^d_0, \ldots, \eta'^d_N\}$ *has the following forwards-time state-space model*

$$\left\{ \begin{array}{rcl} \xi'^d_{i+1} &=& F_i^{-*}\xi'^d_i - F_i^{-*}H_i^*R_i^{-1}\mathbf{v}_i, \quad \xi'^d_0 = \Pi_0^{-1}\mathbf{x}_0 \\ \eta'^d_i &=& G_i^*F_i^{-*}\xi'^d_i - G_i^*F_i^{-*}H_i^*R_i^{-1}\mathbf{v}_i + Q_i^{-1}\mathbf{u}_i \end{array} \right. \quad i = 0, 1, \ldots N \tag{6.5.14}$$

*and where* $\xi'^d_{N+1}$ *is its final state.*

*Moreover, the prediction errors of the states,* $\tilde{\xi}_i'^d \triangleq \xi_i'^d - \hat{\xi}_i'^d$, *given the (past) observations* $\{\eta_0'^d, \ldots, \eta_{i-1}'^d\}$ *in (6.5.14) have the following interpretation*

$$\tilde{\xi}_i'^d = P_i^{-1}\tilde{\mathbf{x}}_i, \tag{6.5.15}$$

*where* $\tilde{\mathbf{x}}_i \triangleq \mathbf{x}_i - \hat{\mathbf{x}}_i$ *is the prediction error in estimating* $\mathbf{x}_i$ *using the (past) observations* $\{\mathbf{y}_0, \ldots, \mathbf{y}_{i-1}\}$, *and where* $P_i$ *is the corresponding error Gramian.*

**Proof:** Note that the state-space equations can be written in global form as

$$\eta'^d = \begin{bmatrix} \eta_0'^d \\ \eta_1'^d \\ \vdots \\ \mathbf{y}_N'^d \end{bmatrix} = \underbrace{\begin{bmatrix} G_0^* F_0^{-*} H_0^{-*} & & & \\ G_1^* F_1^{-*} F_0^{-*} H_0^{-*} & G_1^* F_1^{-*} H_1^{-*} & & \\ \vdots & \vdots & \ddots & \\ G_N^* F_N^{-*} \ldots F_0^{-*} H_0^{-*} & G_N^* F_N^{-*} \ldots F_1^{-*} H_1^{-*} & \ldots & G_N^* F_N^{-*} H_N^{-*} \end{bmatrix}}_{-\Gamma^* + \mathcal{C}^* \mathcal{F}^{-*} \mathcal{O}^*} \begin{bmatrix} R_0^{-1}\mathbf{v}_0 \\ R_1^{-1}\mathbf{v}_1 \\ \ldots \\ R_N^{-1}\mathbf{v}_N \end{bmatrix}$$

$$- \begin{bmatrix} G_0^* F_0^{-*} \\ G_1^* F_1^{-*} F_0^{-*} \\ \vdots \\ G_N^* \ldots F_0^{-*} \end{bmatrix} \Pi_0^{-1}\mathbf{x}_0 + \begin{bmatrix} Q_0^{-1}\mathbf{u}_0 \\ Q_1^{-1}\mathbf{u}_1 \\ \ldots \\ Q_N^{-1}\mathbf{u}_N \end{bmatrix}. \tag{6.5.16}$$

which is precisely the equation defining $\eta'^d$ in (6.5.12). Likewise using (6.5.14) we may write

$$\xi_{N+1}'^d = \Phi_F^{-*}\Pi_0^{-1}\mathbf{x}_0 - \Phi_F^{-*}\mathcal{O}^* R^{-1}\mathbf{v}.$$

Comparing with (6.5.12) we obtain the desired result that the econd component of $\mathbf{z}'^d$ is the end-state, $\xi_{N+1}'^d$.

The proof of (6.5.15) is analogous to the proof of (6.5.7) in Lemma 6.5.2, *i.e.*, we show that $(\xi_{N+1}'^d)^d = \tilde{\mathbf{x}}_{N+1}$ and that $(\xi_{N+1}'^d)^d = M^{-1}\tilde{\xi}_{N+1}'^d$ where $M$ is the Gramian of $\tilde{\xi}_{N+1}'^d$. These two equalities then imply that $M = P_{N+1}$, so that we may write

$$\tilde{\xi}_{N+1}'^d = P_{N+1}^{-1}\tilde{\mathbf{x}}_{N+1}.$$

Since the endpoint $i = N+1$ is arbitrary we conclude that for all $i$,

$$\tilde{\xi}_i'^d = P_i^{-1}\tilde{\mathbf{x}}_i,$$

which is the desired result.

■

The forwards-time state-space model

$$\left\{ \begin{array}{rcl} \xi'^d_{i+1} & = & F_i^{-*}\xi'^d_i - F_i^{-*}H_i^*R_i^{-1}\mathbf{v}_i, \quad \xi'^d_0 = \Pi_0^{-1}\mathbf{x}_0 \\ \eta'^d_i & = & G_i^*F_i^{-*}\xi'^d_i - G_i^*F_i^{-*}H_i^*R_i^{-1}\mathbf{v}_i + Q_i^{-1}\mathbf{u}_i \end{array} \right. \qquad i = 0, 1, \ldots N \tag{6.5.17}$$

is less well-known in the system theory and is referred to as the forwards-time complementary state-space model [AK89]. However, as before, we shall continue to call it the backwards dual model since its output describes the dual basis.

The reader at this point may note that (6.5.14) is essentially the backwards dual state-space model of Lemma 6.5.2 whose time has been reversed. Indeed this is one way of obtaining the forwards-time dual model. Another route would be to construct the backwards Markovian model for the original state-space model (6.5.1), and to find its dual using the approach of the previous section. We shall not the provide the details of these derivations here and will leave them as an exercise for the interested reader.

One further remark here is that the above forwards dual state-space model allows one to obtain the information form Kalman filter.

We close this section with one interesting observation regarding the output of the forwards dual state-space model (6.5.14).

**Corollary 6.5.2 (An Orthogonality Result)** *Consider the forwards dual model (6.5.14) and the original state-space model (6.5.1). Then we have*

$$\mathbf{x}_i \ \perp \ \eta'^d_j \quad j = 0, 1, \ldots, i-1. \tag{6.5.18}$$

**Proof:** Note that $\mathcal{L}\{\eta'^d\} = \mathcal{L}\{\tilde{\mathbf{u}}_{|y,x_{N+1}}\}$ (since it is the second component of the dual basis $\mathbf{z}'^d$). Therefore we readily see that

$$\mathbf{x}_{N+1} \ \perp \ \eta'^d_j \quad j = 0, 1, \ldots, N.$$

But since the endpoint $i = N+1$ is arbitrary, we conclude that for any value of $i$,

$$\mathbf{x}_i \ \perp \ \eta'^d_j \quad j = 0, 1, \ldots, i-1,$$

which is the desired result.

■

### 6.5.3 The Mixed Dual Model

Note that if we assume that the $\{F_j\}_{j=0}^{i}$ are nonsingular, then, for any $i = 0, \ldots N+1$, we can take $\{\mathbf{u}_0, \ldots, \mathbf{u}_{i-1}, \mathbf{x}_i, \mathbf{u}_i, \ldots, \mathbf{u}_N, \mathbf{y}\}$ as a basis for $\mathcal{L}\{\mathbf{u}, \mathbf{x}_0, \mathbf{y}\}$. For this so-called mixed basis we can obtain the following result using the arguments presented in Secs. 6.5.1 and 6.5.2.

**Lemma 6.5.4 (Mixed Dual Model)** *Consider the standard state-space model (6.5.1). Then the dual basis to* $\{\mathbf{u}_0, \ldots, \mathbf{u}_{i-1}, \mathbf{x}_i, \mathbf{u}_i, \ldots, \mathbf{u}_N, \mathbf{y}\}$ *is given by*

$$\{\eta_0'^d, \ldots, \eta_{i-1}'^d, \xi_i^d + \xi_i'^d, \eta_i^d, \ldots \eta_N^d, R^{-1}\mathbf{v}\}, \tag{6.5.19}$$

*where* $\eta_j^d$ *has the following backwards-time dual state-space model*

$$\left\{ \begin{array}{rcl} \xi_j^d & = & F_j^* \xi_{j+1}^d - H_j^* R_j^{-1} \mathbf{v}_j \quad \xi_{N+1}^d = 0 \\ \eta_j^d & = & G_j^* \xi_{j+1}^d + Q_j^{-1} \mathbf{u}_j \end{array} \right. , \quad j = N, N-1, \ldots i. \tag{6.5.20}$$

*with* $\xi_i^d$ *its final state, and where* $\eta_j'^d$ *has the following forwards-time state-space model*

$$\left\{ \begin{array}{rcl} \xi_{j+1}'^d & = & F_j^{-*} \xi_j'^d - F_j^{-*} H_j^* R_j^{-1} \mathbf{v}_j, \quad \xi_0'^d = \Pi_0^{-1} \mathbf{x}_0 \\ \eta_j'^d & = & G_j^* F_j^{-*} \xi_j'^d - G_j^* F_j^{-*} H_j^* R_j^{-1} \mathbf{v}_j + Q_j^{-1} \mathbf{u}_j \end{array} \right. \quad i = 0, 1, \ldots i-1 \tag{6.5.21}$$

*with* $\xi_i'^d$ *its final state.*

**Proof:** Instead of actually deriving the dual basis using the approach of the previous sections, we shall verify that the expressions given in the Lemma are indeed the dual basis. To show this, we must verify that

$$\langle \begin{bmatrix} \eta_0'^d \\ \vdots \\ \eta_{i-1}'^d \\ \xi_i^d + \xi_i'^d \\ \eta_i^d \\ \vdots \\ \eta_N^d \\ R^{-1}\mathbf{v} \end{bmatrix}, \begin{bmatrix} \mathbf{u}_0 \\ \\ \vdots \\ \mathbf{u}_{i-1} \\ \mathbf{x}_i \\ \mathbf{u}_i \\ \vdots \\ \mathbf{u}_N \\ \mathbf{y} \end{bmatrix} \rangle = I. \tag{6.5.22}$$

But $\langle R^{-1}\mathbf{v}, \mathbf{y}\rangle = I$ and from our construction of the backwards and forwards dual bases we already know that $\eta_k^d \perp \mathbf{y}_l$ and $\eta_k'^d \perp \mathbf{y}_l$, for all $k$ and $l$. Thus to verify the inner products with respect to $\mathbf{y}$ in (6.5.22) we need only show

$$\xi_i^d + \xi_i'^d \;\perp\; \mathbf{y}_j \quad j = 0, 1 \ldots, N.$$

But for $j < i$ we have $\xi_i'^d \perp \mathbf{y}_j$, so we need to show

$$\xi_i^d \;\perp\; \mathbf{y}_j \quad j = 0, 1 \ldots, i-1$$

which is immediate since $\xi_i^d \in \mathcal{L}\{\mathbf{v}_i, \ldots, \mathbf{v}_N\}$. On the other hand, for $j \geq i$ we have $\xi_i^d + \Pi_i^{-1}\mathbf{x}_i \perp \mathbf{y}_j$, so we need to show

$$-\Pi_i^{-1}\mathbf{x}_i + \xi_i'^d \;\perp\; \mathbf{y}_j \quad j = i, i+11 \ldots, N$$

which follows from the fact that $\xi_i'^d = \Pi_i^{-1}\mathbf{x}_i + \mu_i$ where $\mu_i \in \mathcal{L}\{\mathbf{v}_0, \ldots, \mathbf{v}_{i-1}\}$.

From Lemmas 6.5.2 and 6.5.3, we already know that

$$\langle \begin{bmatrix} \eta_0'^d \\ \vdots \\ \eta_{i-1}'^d \\ \xi_i^d + \xi_i'^d \end{bmatrix}, \begin{bmatrix} \mathbf{u}_0 \\ \\ \vdots \\ \mathbf{u}_{i-1} \end{bmatrix} \rangle = I \quad \text{and} \quad \langle \begin{bmatrix} \xi_i^d + \xi_i'^d \\ \eta_i^d \\ \vdots \\ \eta_N^d \end{bmatrix}, \begin{bmatrix} \mathbf{u}_i \\ \vdots \\ \mathbf{u}_N \end{bmatrix} \rangle = I.$$

Therefore all that remains to be shown is that

$$\left\langle \begin{bmatrix} \eta_0'^d \\ \vdots \\ \eta_{i-1}'^d \end{bmatrix}, \begin{bmatrix} \mathbf{u}_i \\ \vdots \\ \mathbf{u}_N \end{bmatrix} \right\rangle = 0 \quad , \quad \left\langle \begin{bmatrix} \eta_i^d \\ \vdots \\ \eta_N^d \end{bmatrix}, \begin{bmatrix} \mathbf{u}_0 \\ \\ \vdots \\ \\ \mathbf{u}_{i-1} \end{bmatrix} \right\rangle = 0 \;\text{ and }\; \left\langle \begin{bmatrix} \eta_0'^d \\ \vdots \\ \eta_{i-1}'^d \\ \xi_i^d + \xi_i'^d \\ \eta_i^d \\ \vdots \\ \eta_N^d \end{bmatrix}, \mathbf{x}_i \right\rangle = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

But the first two requirements are obvious since the $\{\eta_j'^d\}_{j=0}^{i-1}$ and the $\{\eta_j^d\}_{j=i}^{N}$ depend only on $\{\mathbf{x}_0, \{\mathbf{u}_j\}_{j=0}^{i-1}, \{\mathbf{v}_j\}_{j=0}^{i-1}\}$ and $\{\{\mathbf{u}_j\}_{j=i}^{N}, \{\mathbf{v}_j\}_{j=i}^{N}\}$. respectively. Also, from our Corollaries 6.5.1 and 6.5.2, $\mathbf{x}_i$ is orthogonal to the $\{\eta_j^d\}_{j=i}^{N}$ and to the $\{\eta_j'^d\}_{j=0}^{i-1}$.

Thus the only requirement to verify is

$$\langle \xi_i^d + \xi_i'^d, \mathbf{x}_i \rangle = I.$$

But

$$\begin{aligned} \langle \xi_i^d, \mathbf{x}_i \rangle &= \underbrace{\langle \tilde{\xi}_i^d + \hat{\xi}_i^d, \mathbf{x}_i \rangle = \langle \tilde{\xi}_i^d, \mathbf{x}_i \rangle}_{\text{since } \hat{\xi}_i^d \in \mathcal{L}\{\eta_j^d\}_{j=i}^{N}} \\ &= \langle P_{i|i}^{-b} \tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1} \mathbf{x}_i, \mathbf{x}_i \rangle \\ &= \langle P_{i|i}^{-b} \tilde{\mathbf{x}}_{i|i}^b, \mathbf{x}_i \rangle - I \\ &= P_{i|i}^{-b} \langle \tilde{\mathbf{x}}_{i|i}^b, \tilde{\mathbf{x}}_{i|i}^b + \hat{\mathbf{x}}_{i|i}^b \rangle - I \\ &= P_{i|i}^{-b} \langle \tilde{\mathbf{x}}_{i|i}^b, \tilde{\mathbf{x}}_{i|i}^b \rangle - I = 0. \end{aligned}$$

A similar argument shows that

$$\langle \xi_i'^d, \mathbf{x}_i \rangle = I,$$

so that

$$\langle \xi_i^d + \xi_i'^d, \mathbf{x}_i \rangle = 0 + I = I,$$

as desired.

■

As we shall see in the section, the above mixed dual state-space model will be useful for deriving the so-called two-filter smoothing formulas [May66, Fra67].

## 6.6 Application to Smoothing

The key observation that allows one to use dual bases for solving smoothing problems is the following. Suppose we are given a Krein space variable $\mathbf{x} \in \mathcal{L}\{\mathbf{z}, \mathbf{y}\}$ and would like to find, $\hat{\mathbf{x}}_{|y}$, the smoothed estimate of $\mathbf{x}$ given $\mathbf{y}$. Then since $\mathbf{z}^d$ is one possible basis for the orthogonal complement space of $\mathcal{L}\{\mathbf{y}\}$ in $\mathcal{L}\{\mathbf{z}, \mathbf{y}\}$ (in other words, $\mathcal{L}\{\mathbf{z}, \mathbf{y}\} = \mathcal{L}\{\mathbf{y}\} \oplus \mathcal{L}\{\mathbf{z}^d\}$), we may uniquely decompose $\mathbf{x}$ as

$$\mathbf{x} = \hat{\mathbf{x}}_{|y} + \hat{\mathbf{x}}_{|z^d},$$

from which it follows

$$\hat{\mathbf{x}}_{|y} = \mathbf{x} - \hat{\mathbf{x}}_{|z^d}. \tag{6.6.1}$$

Thus we can obtain the desired estimate by projecting onto the dual basis, $\mathbf{z}^d$, and subtracting the result from $\mathbf{x}$.

The above approach is a general method for obtaining smoothed estimates. Of course the resulting algorithms will depend on the choice of $\mathbf{z}$ (and hence $\mathbf{z}^d$). By projecting onto the backwards and forwards dual state-space models of Lemmas 6.5.2 and 6.5.3 it is possible to derive the standard Bryson-Frazier and Rauch-Tung-Striebel (RTS) smoothing formulae and their variants [RTS65]. The algebra involved in these derivations is (at times) rather tedious and therefore we shall not present them here. [The interested reader may refer to [AK89] for details.] Instead, we shall illustrate the general method by deriving the so-called two-filter smoothing formulae by projecting onto the mixed dual state-space model of Lemma 6.5.4.

### 6.6.1 Two-Filter Formulae

Consider the standard state-space model (6.5.1) and suppose that we would like to obtain, $\hat{\mathbf{x}}_{i|N}$, the smoothed estimate of the state $\mathbf{x}_i$ given all the observations $\{\mathbf{y}_0, \ldots, \mathbf{y}_N\}$. To do so, consider the following mixed dual basis that was introduced in Lemma 6.5.4,

$$\{\eta_0'^d, \ldots, \eta_{i-1}'^d, \xi_i^d + \xi_i'^d, \eta_i^d, \ldots \eta_N^d, R^{-1}\mathbf{v}\}. \tag{6.6.2}$$

To find $\hat{\mathbf{x}}_{i|N}$ we will make use of the formula

$$\hat{\mathbf{x}}_{i|N} \stackrel{\Delta}{=} \hat{\mathbf{x}}_{|y} = \mathbf{x}_i - \hat{\mathbf{x}}_{i|z^d}, \tag{6.6.3}$$

where $\mathbf{z}^d$ is the first part of the aforementioned mixed dual basis, *i.e.*,

$$\mathbf{z}^d = \operatorname{col}\{\eta_0'^d, \ldots, \eta_{i-1}'^d, \xi_i^d + \xi_i'^d, \eta_i^d, \ldots \eta_N^d\}. \tag{6.6.4}$$

Now due to Corollaries 6.5.1 and 6.5.2, the state $\mathbf{x}_i$ is orthogonal to both $\{\eta_j^d\}_{j=i}^N$ and $\{\eta_j'^d\}_{j=0}^{i-1}$. Therefore to find $\hat{\mathbf{x}}_{i|z^d}$ we need only project onto $\xi_i^d + \xi_i'^d$.

Moreover, due to the above orthogonalities, projecting $\mathbf{x}_i$ onto $\xi_i^d + \xi_i'^d$ is the same as projecting $\mathbf{x}_i$ onto $\tilde{\xi}_{i|i}^d + \tilde{\xi}_i'^d$, where $\tilde{\xi}_{i|i}^d \triangleq \xi_i^d - \hat{\xi}_{i|i}^d$, is the state estimation error given the (future) observations $\{\eta_i^d, \ldots, \eta_N^d\}$ and $\tilde{\xi}_i'^d \triangleq \xi_i'^d - \hat{\xi}_i'^d$, is the state estimation error given the (past) observations $\{\eta_0'^d, \ldots, \eta_{i-1}'^d\}$. Now in Lemmas 6.5.2 and 6.5.3 we have shown

$$\tilde{\xi}_{i|i}^d = P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i \quad \text{and} \quad \tilde{\xi}_i'^d = P_i^{-1}\tilde{\mathbf{x}}_i$$

where $\tilde{\mathbf{x}}_{i|i}^b \triangleq \mathbf{x}_i - \hat{\mathbf{x}}_{i|i}^b$ is the (backwards) error in estimating $\mathbf{x}_i$ using the (future) observations $\{\mathbf{y}_i, \ldots, \mathbf{y}_N\}$, with $P_{i|i}^b$ its corresponding error Gramian, and where $\tilde{\mathbf{x}}_i \triangleq \mathbf{x}_i - \hat{\mathbf{x}}_i$ is the prediction error in estimating $\mathbf{x}_i$ using the (past) observations $\{\mathbf{y}_0, \ldots, \mathbf{y}_{i-1}\}$, with $P_i$ its corresponding error Gramian.

In view of the above arguments, we have

$$\hat{\mathbf{x}}_{i|z^d} = \langle \mathbf{x}_i, P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i + P_i^{-1}\tilde{\mathbf{x}}_i \rangle \| P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i + P_i^{-1}\tilde{\mathbf{x}}_i \|^{-2} \left( P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i + P_i^{-1}\tilde{\mathbf{x}}_i \right). \tag{6.6.5}$$

But

$$\langle \mathbf{x}_i, P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i + P_i^{-1}\tilde{\mathbf{x}}_i \rangle = I - I + I = I, \tag{6.6.6}$$

and

$$\| P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i + P_i^{-1}\tilde{\mathbf{x}}_i \|^2 = \| \tilde{\xi}_{i|i}^d + \tilde{\xi}_i'^d \|^2 = \| \tilde{\xi}_{i|i}^d \|^2 + \| \tilde{\xi}_i'^d \|^2 = \| P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i \|^2 + \| P_i^{-1}\tilde{\mathbf{x}}_i \|^2, \tag{6.6.7}$$

since $\tilde{\xi}_{i|i}^d \in \mathcal{L}\{\mathbf{u}_i, \mathbf{v}_i, \ldots, \mathbf{u}_N, \mathbf{v}_N\}$ and $\tilde{\xi}_i'^d \in \mathcal{L}\{\mathbf{x}_0, \mathbf{u}_0, \mathbf{v}_0, \ldots, \mathbf{u}_{i-1}, \mathbf{v}_{i-1}\}$ (see Eqs. (6.5.6) and (6.5.14)). Moreover,

$$\| P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i \|^2 = P_{i|i}^{-b} - \Pi_i^{-1} \quad \text{and} \quad \| P_i^{-1}\tilde{\mathbf{x}}_i \|^2 = P_i^{-1}. \tag{6.6.8}$$

Therefore we may finally write

$$\mathbf{x}_i - \hat{\mathbf{x}}_{i|N} = \hat{\mathbf{x}}_{i|z^d} = I \cdot \left( P_{i|i}^{-b} - \Pi_i^{-1} + P_i^{-1} \right)^{-1} \left( P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^b - \Pi_i^{-1}\mathbf{x}_i + P_i^{-1}\tilde{\mathbf{x}}_i \right), \tag{6.6.9}$$

or

$$\left(P_{i|i}^{-b} - \Pi_i^{-1} + P_i^{-1}\right)\left(\mathbf{x}_i - \hat{\mathbf{x}}_{i|N}\right) = \left(P_{i|i}^{-b}\tilde{\mathbf{x}}_{i|i}^{b} - \Pi_i^{-1}\mathbf{x}_i + P_i^{-1}\tilde{\mathbf{x}}_i\right), \tag{6.6.10}$$

from which we conclude

$$\left(P_{i|i}^{-b} - \Pi_i^{-1} + P_i^{-1}\right)\hat{\mathbf{x}}_{i|N} = P_{i|i}^{-b}\hat{\mathbf{x}}_{i|i}^{b} + P_i^{-1}\hat{\mathbf{x}}_i. \tag{6.6.11}$$

Computing the Gramian of $\tilde{\mathbf{x}}_{i|N}$ from Eq. (6.6.9) we readily see

$$P_{i|N} = \left(P_{i|i}^{-b} - \Pi_i^{-1} + P_i^{-1}\right)^{-1}, \tag{6.6.12}$$

so that (6.6.11) becomes

$$\hat{\mathbf{x}}_{i|N} = P_{i|N}\left(P_{i|i}^{-b}\hat{\mathbf{x}}_{i|i}^{b} + P_i^{-1}\hat{\mathbf{x}}_i\right), \tag{6.6.13}$$

which is the desired two-filter smoother.

It is left as an exercise to the reader to show that if we had instead begun with the dual basis

$$\mathrm{col}\{\eta_0'^d, \ldots, \eta_i'^d, \xi_i^d + \xi_i'^d, \eta_{i+1}^d, \ldots \eta_N^d\} \tag{6.6.14}$$

then we would have obtained the following two-filter formula

$$\hat{\mathbf{x}}_{i|N} = P_{i|N}\left(P_i^{-b}\hat{\mathbf{x}}_i^{b} + P_{i|i}^{-1}\hat{\mathbf{x}}_{i|i}\right), \tag{6.6.15}$$

with

$$P_{i|N}^{-1} = \left(P_i^{-b} + P_{i|i}^{-1} - \Pi_i^{-1}\right)^{-1}, \tag{6.6.16}$$

where $\hat{\mathbf{x}}_i^b$, $\hat{\mathbf{x}}_{i|i}$, $P_i^b$ and $P_{i|i}$ have the obvious meanings.

The results are summarized in the following theorem.

**Theorem 6.6.1 (Two-Filter Smoothing Formulae)** *Consider the standard state-space model (6.5.1) with the $\{F_i\}$ assumed to be invertible. Then we can write*

$$\hat{\mathbf{x}}_{i|N} = P_{i|N}(P_i^{-1}\hat{\mathbf{x}}_i + P_{i|i}^{-b}\hat{\mathbf{x}}_{i|i}^{b}),$$

*with*

$$P_{i|N}^{-1} = \left(P_i^{-1} + P_{i|i}^{-b} - \Pi_i^{-1}\right)^{-1},$$

*or, alternatively,*

$$\hat{\mathbf{x}}_{i|N} = P_{i|N}(P_{i|i}^{-1}\hat{\mathbf{x}}_{i|i} + P_i^{-b}\hat{\mathbf{x}}_i^b),$$

*with*

$$P_{i|N}^{-1} = \left(P_{i|i}^{-1} + P_i^{-b} - \Pi_i^{-1}\right)^{-1}.$$

*Both sets of formulae require forwards and backwards sweeps over the original data: the first formulae require a forwards sweep from* $\mathbf{y}_0$ *to* $\mathbf{y}_i$ *and a backwards sweep from* $\mathbf{y}_N$ *down to* $\mathbf{y}_{i+1}$*, while the second formulae require a forwards sweep from* $\mathbf{y}_0$ *to* $\mathbf{y}_{i-1}$ *and a backwards sweep from* $\mathbf{y}_N$ *down to* $\mathbf{y}_i$*.*

## 6.7 The LQR Control Problem

In this section we shall study the dual problem to Kalman filtering, *i.e.* the linear quadratic regulator (LQR) control problem. The generalization considered here is that the LQR cost function may be indefinite (actually problems in $H^\infty$ control immediately result in such cost functions). Therefore, contrary to the conventional case where the problem always had a minimizing solution, we can at most guarantee a stationarizing solution. Further conditions must be met for the solution to be a minimum.

The usual method for solving the conventional LQR problem is through a dynamic programming argument (see *e.g.* ,[BK65, BH69]). Although the dynamic programming principle can be extended from minimizing solutions to include stationarizing solutions, our approach to solving the LQR problem is similar to the approach we used for solving $H^\infty$ estimation: we identify a Krein state-space model that corresponds to the LQR quadratic form (via simple inspection), and then invoke the Krein space Kalman filter to write down the stationarizing solution and its condition for being a minimum. We should also remark that the Krein space model that most naturally lends itself to the solution of the LQR control problem is the dual model of Sec. 6.4, which is what will be used here.

Finally, we should mention that the results of this section will be used to solve the finite horizon full information and measurement feedback control problems.

### 6.7.1 Problem Formulation

Consider the (possibly) time-varying state-space model

$$x_{i+1} = F_i x_i + G_i u_i, \qquad 0 \leq i \leq N \tag{6.7.1}$$

and some given linear combination of the states

$$s_i = L_i x_i, \tag{6.7.2}$$

where the $\{u_i, x_i\}$ are deterministic quantities. One would like to choose the control signal $u_i$ to regulate the $s_i$ in a certain sense. The conventional LQR (linear quadratic regulator) method proposes the following criterion for choosing the $u_i$

$$\min_{\{u_i\}} \quad x_{N+1}^* P_{N+1}^c x_{N+1} + \sum_{i=0}^{N} u_i^* Q_i^c u_i + \sum_{i=0}^{N} s_i^* R_i^c s_i \tag{6.7.3}$$

subject to the state-space constraints (6.7.1) and (6.7.2), where $P_{N+1}^c$, $Q_i^c$ and $R_i^c$ are positive-definite matrices that penalize the final state, the inputs and the intermediary states, respectively.

In many applications, such as the $H^\infty$ problems to be studied, one needs to consider the quadratic form in (6.7.3) when the $P_{N+1}^c$, $Q_i^c$ and $R_i^c$ are indefinite Hermitian matrices. Therefore, let us introduce the notation for the quadratic form in (6.7.3) as

$$J_N^c(x_0, u_0, \ldots, u_N) = x_{N+1}^* P_{N+1}^c x_{N+1} + \sum_{i=0}^{N} u_i^* Q_i^c u_i + \sum_{i=0}^{N} s_i^* R_i^c s_i, \tag{6.7.4}$$

where from now on we shall drop the positivity conditions on the $P_{N+1}^c$, $Q_i^c$ and $R_i^c$. Since $J_N^c$ will now be an indefinite quadratic form, it does not necessarily have a minimum over the $\{u_i\}$, and the most that we can guarantee is a stationary point.

With this in mind, we may now state the finite horizon LQR control problem as follows.

**Problem 6.7.1 (Finite Horizon LQR Control Problem)** *Find the control signals $\{\hat{u}_i\}_{i=0}^N$ that stationarize the quadratic form*

$$J_N^c(x_0, u_0, \ldots, u_N) = x_{N+1}^* P_{N+1}^c x_{N+1} + \sum_{i=0}^{N} u_i^* Q_i^c u_i + \sum_{i=0}^{N} s_i^* R_i^c s_i$$

*subject to the state-space relations (6.7.1) and (6.7.2), where by a stationary point we mean*

$$\frac{\partial J_N^c}{\partial u_i}(x_0, \hat{u}_0, \ldots, \hat{u}_N) = 0, \qquad 0 \le i \le N.$$

*Moreover, find the necessary and sufficient conditions for this stationary point to be a minimum.*

### 6.7.2 Solution Based on Duality

Let us begin by defining $u = \text{col}\{u_0, \ldots, u_N\}$ and $s = \text{col}\{s_0, \ldots, s_N\}$, so that we may use the state equation (6.7.1) and the output equation (6.7.2) to write

$$x_{N+1} = \Phi_F x_0 + \mathcal{C}u \quad \text{and} \quad y = \mathcal{O}x_0 + \Gamma \tag{6.7.5}$$

where $\Phi_F = F_N F_{N-1} \ldots F_0$ is the state transition matrix,

$$\mathcal{O} = \begin{bmatrix} L_0 \\ L_1 F_0 \\ \vdots \\ L_N F_{N-1} \ldots F_0 \end{bmatrix},$$

is the observability map,

$$\mathcal{C} = \begin{bmatrix} F_N F_{N-1} \ldots F_1 G_0 & F_N F_{N-1} \ldots F_2 G_1 & \ldots & G_0 \end{bmatrix},$$

is the controllability map, and

$$\Gamma = \begin{bmatrix} 0 & & & \\ H_1 G_0 & 0 & & \\ H_2 F_1 G_0 & H_2 G_1 & & \\ \vdots & \vdots & \ddots & \\ H_N F_{N-1} \ldots F_1 G_0 & H_N F_{N-1} \ldots F_2 G_1 & \ldots & H_N G_{N-1} \end{bmatrix},$$

is the impulse response matrix. If we further define,

$$Q^c = \text{diag}\{Q_0^c, Q_1^c, \ldots, Q_N^c\} \quad \text{and} \quad R^c = \text{diag}\{R_0^c, R_1^c, \ldots, R_N^c\},$$

then we can write the LQR cost function, $J_N^c$, as a quadratic form in terms of $x_0$ and $u$. We thus have

$$J_N^c = (x_0^*\Phi_F^* + u^*\mathcal{C}^*)P_{N+1}^c(\Phi_F x_0 + \mathcal{C}u) + (x_0^*\mathcal{O}^* + u^*\Gamma^*)R^c(\mathcal{O}x_0 + \Gamma u) + u^*Q^c u,$$

or, after gathering terms,

$$J_N^c = \begin{bmatrix} x_0^* & u^* \end{bmatrix} \begin{bmatrix} \Phi_F^* P_{N+1}^c \Phi_F + \mathcal{O}^* R^c \mathcal{O} & \Phi_F^* P_{N+1}^c \mathcal{C} + \mathcal{O}^* R^c \Gamma \\ \mathcal{C}^* P_{N+1}^c \Phi_F + \Gamma^* R^c \mathcal{O} & \mathcal{C}^* P_{N+1}^c \mathcal{C} + \Gamma^* R^c \Gamma + Q^c \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix}. \tag{6.7.6}$$

Now if we define

$$\begin{cases} \mathbf{x}_0^c &= \Phi_F^* \mathbf{x}_{N+1}^c + \mathcal{O}^* \mathbf{u}^c \\ \mathbf{y}^c &= \mathcal{C}^* \mathbf{x}_{N+1}^c + \Gamma^* \mathbf{u}^c + \mathbf{v}^c \end{cases} \tag{6.7.7}$$

where $\mathbf{x}_{N+1}^c$, $\mathbf{u}^c$ and $\mathbf{v}^c$ are Krein space variables with

$$\left\langle \begin{bmatrix} \mathbf{x}_{N+1}^c \\ \mathbf{u}^c \\ \mathbf{v}^c \end{bmatrix}, \begin{bmatrix} \mathbf{x}_{N+1}^c \\ \mathbf{u}^c \\ \mathbf{v}^c \end{bmatrix} \right\rangle = \begin{bmatrix} P_{N+1}^c & 0 & 0 \\ 0 & R^c & 0 \\ 0 & 0 & Q^c \end{bmatrix},$$

then it is straightforward to see that

$$\left\langle \begin{bmatrix} \mathbf{x}_0^c \\ \mathbf{y}^c \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0^c \\ \mathbf{y}^c \end{bmatrix} \right\rangle = \begin{bmatrix} \Phi_F^* P_{N+1}^c \Phi_F + \mathcal{O}^* R^c \mathcal{O} & \Phi_F^* P_{N+1}^c \mathcal{C} + \mathcal{O}^* R^c \Gamma \\ \mathcal{C}^* P_{N+1}^c \Phi_F + \Gamma^* R^c \mathcal{O} & \mathcal{C}^* P_{N+1}^c \mathcal{C} + \Gamma^* R^c \Gamma + Q^c \end{bmatrix}, \tag{6.7.8}$$

so that the LQR cost function in (6.7.6) can be rewritten as

$$J_N^c = \begin{bmatrix} x_0^* & u^* \end{bmatrix} \begin{bmatrix} R_{x_0^c} & R_{x_0^c y^c} \\ R_{y^c x_0^c} & R_{y^c} \end{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix}. \tag{6.7.9}$$

The minimizing solution over $u$ for $J_N^c$ is now readily seen to be

$$\hat{u} = -R_{y^c}^{-1} R_{y^c x_0^c} x_0, \tag{6.7.10}$$

*i.e.*, its gain matrix $-R_{y^c}^{-1} R_{y^c x_0^c}$ is the negative (conjugate) transpose of the gain matrix in estimating the random variable $\mathbf{x}_0^c$ from $\mathbf{y}^c$. Note that this is the dual stochastic approach to deterministic least-squares problems that was introduced in Sec. 6.4. Therefore to find our desired solution we can project $\mathbf{x}_0^c$ onto $\mathbf{y}^c$ and use the negative (conjugate) transpose of the resulting gain matrix to find $\hat{u}$.

To this end, let us first note that (6.7.7) has the following backwards dual state-space model

$$\left\{ \begin{array}{lcl} \mathbf{x}_i^c & = & F_i^* \mathbf{x}_{i+1}^c + L_i^* \mathbf{u}_i^c \quad , \quad \mathbf{x}_{N+1}^c \\ \mathbf{y}_i^c & = & G_i^* \mathbf{x}_{i+1}^c + \mathbf{v}_i^c \end{array} \right. \tag{6.7.11}$$

with

$$\left\langle \begin{bmatrix} \mathbf{u}_i^c \\ \mathbf{v}_i^c \\ \mathbf{x}_{N+1}^c \end{bmatrix}, \begin{bmatrix} \mathbf{u}_j^c \\ \mathbf{v}_j^c \\ \mathbf{x}_{N+1}^c \end{bmatrix} \right\rangle = \begin{bmatrix} R_i^c \delta_{ij} & 0 & 0 \\ 0 & Q_i^c \delta_{ij} & 0 \\ 0 & 0 & P_{N+1}^c \end{bmatrix},$$

as can be readily checked by defining $\mathbf{y}^c = \text{col}\{\mathbf{y}_0^c, \ldots, \mathbf{y}_N^c\}$, $\mathbf{u}^c = \text{col}\{\mathbf{u}_0^c, \ldots, \mathbf{u}_N^c\}$ and $\mathbf{v}^c = \text{col}\{\mathbf{v}_0^c, \ldots, \mathbf{v}_N^c\}$, noting the relations

$$\mathbf{x}_0^c = \Phi_F^* \mathbf{x}_{N+1}^c + \mathcal{O}^* \mathbf{u}^c \quad , \quad \mathbf{y}^c = \mathcal{C}^* \mathbf{x}_{N+1}^c + \Gamma^* \mathbf{u}^c + \mathbf{v}^c \tag{6.7.12}$$

and computing the respective Gramians.

To find the projection of $\mathbf{x}_0^c$ onto $\mathbf{y}^c$ we can invoke the Krein space Kalman filter corresponding to (6.7.11). Thus

$$\hat{\mathbf{x}}_i^c = F_i^* \hat{\mathbf{x}}_{i+1}^c + K_{p,i}^c \mathbf{e}_i^c, \quad \hat{\mathbf{x}}_{N+1}^c = 0 \tag{6.7.13}$$

where

$$\mathbf{e}_i^c = \mathbf{y}_i^c - G_i^* \hat{\mathbf{x}}_{i+1}^c, \tag{6.7.14}$$

is the innovations,

$$K_{p,i}^c = F_i^* P_{i+1}^c G_i (R_{e,i}^c)^{-1} \quad \text{and} \quad R_{e,i}^c = Q_i^c + G_i^* P_{i+1}^c G_i \tag{6.7.15}$$

are the Kalman gain and innovations variance, respectively, and where $P_i^c$ satisfies the backwards-time Riccati recursion

$$P_i^c = F_i^* P_{i+1}^c F_i + L_i^* R_i^c L_i - K_{p,i}^c R_{e,i}^c K_{p,i}^{c*}, \quad P_{N+1}^c. \tag{6.7.16}$$

Now (6.7.13) may be rewritten as

$$\hat{\mathbf{x}}_i^c = \Phi_i \hat{\mathbf{x}}_{i+1}^c + K_{p,i}^c \mathbf{y}_i^c, \quad \hat{\mathbf{x}}_{N+1}^c = 0 \tag{6.7.17}$$

where

$$\Phi_i = (F_i^* - K_{p,i}^c G_i^*).$$

The recursion (6.7.17) can be solved to yield

$$\hat{\mathbf{x}}_0^c = \begin{bmatrix} K_{p,0}^c & \Phi_0 K_{p,1}^c & \dots & \Phi_0 \dots \Phi_{N-1} K_{p,N}^c \end{bmatrix} \begin{bmatrix} \mathbf{y}_0^c \\ \mathbf{y}_1^c \\ \vdots \\ \mathbf{y}_N^c \end{bmatrix}. \tag{6.7.18}$$

Now the solution to the LQR problem is given by the negative of the conjugate transpose of the above solution. Thus

$$\begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_N \end{bmatrix} = - \begin{bmatrix} K_{p,0}^{c*} \\ K_{p,1}^{c*} \Phi_0^* \\ \vdots \\ K_{p,N}^{c*} \Phi_{N-1}^* \dots \Phi_0^* \end{bmatrix} x_0. \tag{6.7.19}$$

But this implies

$$\hat{u}_0 = -K_{p,0}^{c*},$$

and

$$\begin{aligned} \hat{u}_1 = -K_{p,1}^{c*} \Phi_0^* &= -K_{p,1}^{c*} (F_0 - G_0 K_{p,0}^{c*}) x_0 \\ &= -K_{p,1}^{c*} (F_0 x_0 + G_0 \hat{u}_0) = -K_{p,1}^{c*} x_1, \end{aligned}$$

and continuing in a similar fashion

$$\hat{u}_i = -K_{p,i}^{c*} x_i, \quad i = 0, \dots, N \tag{6.7.20}$$

which is the wellknown state-feedback law of LQR control.

Equations (6.7.15), (6.7.16) and (6.7.20) constitute the solution to the LQR control problem. Of course, we still need to check the condition for a minimum over, the $\{u_i\}$. But, using Lemma 6.2.1, this is readily seen to be

$$R_{y^c} = \mathcal{C}^* P_{N+1}^c \mathcal{C} + \Gamma^* R^c \Gamma + Q^c > 0. \tag{6.7.21}$$

Moreover, the Krein space Kalman filter corresponding to $R_{y^c}$ allows us to recursively check the above condition via the innovations Gramian, viz.,

$$R_{e,i}^c = Q_i^c + G_i^* P_{i+1}^c G_i > 0, \quad i = 0, \dots, N. \tag{6.7.22}$$

Now the control signal, $\hat{u} = -R_{y^c}^{-1} R_{y^c x_0^c} x_0$, is referred to as the optimal *open-loop* control, since it only depends on the initial state, $x_0$. The control signal, $\hat{u}_i = -K_{p,i}^{c*} x_i$, on the other hand, is referred to as the optimal *closed-loop* control, since it depends only on the current state, $x_i$. Note that at any given time, $i$, the optimal closed-loop control signal, $\hat{u}_i = -K_{p,i}^{c*} x_i$, coincides with the optimal open-loop signal if, and only if, all previous choices of the control signal were also optimal. (Since non-optimal choices of $\{u_j, j < i\}$ affect the value of $x_i$, and thereby the value of the optimal closed-loop control, $\hat{u}_i = -K_{p,i}^{c*} x_i$.)

The closed-loop control signal, however, has the additional property that it is the optimal control at time $i$, *irrespective of what the previous control signals were.* To be more specific, it solves the LQR problem,

$$\min_{u_i} \sum_{j=i}^{N} u_j^* Q_j^c u_j + \sum_{j=i}^{N} s_j^* R_j^c s_j + x_{N+1}^* P_{N+1}^c x_{N+1}. \tag{6.7.23}$$

The validity of the above claim is readily seen from the fact that in the above argument, which led to the optimal choice of $u_0$, *i.e.*, $\hat{u}_0 = -K_{p,0}^{c*} x_0$, the choice of the initial time, $i = 0$, was arbitrary.

The closed-loop control signals, $\hat{u}_i = -K_{p,i}^{c*} x_i$, also allow for a certain decomposition of the LQR cost, which will prove to be extremely useful. Indeed,[5]

$$J_N^c(x_0, u_0, \ldots, u_N) = x_0^* P_0^c x_0 + \sum_{j=0}^{N} (u_j - \hat{u}_j)^* R_{e,j}^c (u_j - \hat{u}_j). \tag{6.7.24}$$

We can now summarize the results obtained so far in the following Theorem, where, to conform with the results of Sec. 1.5.3, we have defined, $K_{c,i} \triangleq K_{p,i}^{c*}$.

[5]This identity can be shown in several different ways. One way is to complete the squares in the expression for $J_N^c$, (6.7.6), to write

$$J_N^c = x_0^* P_0^c x_0 + (u - R_{y^c}^{-1} R_{y^c x_0^c} x_0)^* R_{y^c} (u - R_{y^c}^{-1} R_{y^c x_0^c} x_0).$$

The triangular decomposition, $R_{y^c} = L^* R_{e^c} L$, (obtained from the Krein space Kalman filter for $R_{y^c}$) can then (after some amount of algebra) be used to show the desired result. A shorter proof is to repeatedly apply the (readily verified via completion of squares) identity,

$$u_j^* Q_j^c u_j + s_j^* R_j^c s_j + x_{j+1}^* P_{j+1}^c x_{j+1} = x_j^* P_j^c x_j + (u_j - \hat{u}_j)^* R_{e,j}^c (u_j - \hat{u}_j),$$

to the quadratic form, $J_N^c$.

**Theorem 6.7.1 (Finite Horizon LQR Solution)** *Consider the LQR cost function*

$$J_N^c(x_0, u_0, \ldots, u_N) = x_{N+1}^* P_{N+1}^c x_{N+1} + \sum_{i=0}^{N} u_i^* Q_i^c u_i + \sum_{i=0}^{N} s_i^* R_i^c s_i,$$

*with $P_{N+1}^c$, $Q_i^c$, and $R_i^c$, given Hermitian matrices, and suppose that we want to minimize $J_N^c(x_0, u_0, \ldots, u_N)$ over the variables $\{u_i\}_{i=0}^N$ subject to the state-space constraint $x_{i+1} = F_i x_i + G_i u_i$, $i = 0, \ldots, N$. Then the minimizing solution is given by the state-feedback law*

$$\hat{u}_i = -K_{c,i} x_i, \quad i = 0, \ldots, N \tag{6.7.25}$$

*where*

$$K_{c,i} = F_i^* P_{i+1}^c G_i (R_{e,i}^c)^{-1} \tag{6.7.26}$$

*and*

$$R_{e,i}^c = Q_i^c + G_i^* P_{i+1}^c G_i \tag{6.7.27}$$

*and where $P_i^c$ satisfies the backwards-time Riccati recursion*

$$P_i^c = F_i^* P_{i+1}^c F_i + H_i^* R_i^c H_i - K_{c,i} R_{e,i}^c K_{c,i}^*, \qquad P_{N+1}^c. \tag{6.7.28}$$

*The condition for the above solution to be a minimum is that*

$$R_{e,i}^c > 0, \quad i = 0, \ldots, N. \tag{6.7.29}$$

*Moreover, we have the identity,*

$$J_N^c(x_0, u_0, \ldots, u_N) = x_0^* P_0^c x_0 + \sum_{j=0}^{N} (u_j - \hat{u}_j)^* R_{e,j}^c (u_j - \hat{u}_j). \tag{6.7.30}$$

**Remark:** Note that we obtained the solution to the LQR problem by appealing to the dual approach for solving deterministic least-squares problems. The solution was obtained by constructing the Kalman filter for the backwards-time dual state-space model. This, therefore, directly suggests the wellknown duality between Kalman filtering and LQR control. To see this duality more explicitly, note that if we apply the transformations given below to the solution of Theorem 6.7.1,

$$\begin{array}{lll} F_i^* \to F_i & P_{i+1}^c \to P_i & K_{c.i} \to K_{p,i}^* \\ H_i^* \to G_i & R_i^c \to Q_i & \text{backward time} \to \text{forward time} \\ G_i^* \to H_i & Q_i^c \to R_i & \end{array}$$

then we recover the Kalman filter solution corresponding to the standard forward-time state-space model.

## 6.8 Full Information $H^\infty$ Control

In this section we shall study the full information $H^\infty$ control problem. We first introduced, and in fact solved, this problem in Sec. 1.7.1 using an approach based on the canonical factorization of a certain indefinite transfer operator. Here we shall take a slightly different approach, closer to the general spirit of Chapters 2 and 3, that is based upon associating an indefinite quadratic form, rather than an indefinite transfer operator, with the problem.[6] The resulting indefinite quadratic form is a special case of the LQR cost function studied in the previous section and Theorem 6.7.1 will be used to stationarize the quadratic form (thereby obtaining the solution) and to check the conditions for a minimum (thereby yielding the existence conditions for the controller).

For the early motivation of full information $H^\infty$ control and alternative approaches to its solution the reader is referred to [Zam81, Fra87, FD87, BC87, Kim87, DGKF89, GD89, Tad90, Kwa91, GLD+91, IPJ91, LAKG92, BB95, GL95, ZDG96] and the references therein.

[6]For comparison of the two methods, see the remarks at the beginning of Sec. 3.3.

### 6.8.1 Problem Formulation

Consider the (possibly) time-variant state-space model

$$\begin{aligned} x_{i+1} &= F_i x_i + G_{1,i} w_i + G_{2,i} w_i \\ &= F_i x_i + \begin{bmatrix} G_{1,i} & G_{2,i} \end{bmatrix} \begin{bmatrix} w_i \\ u_i \end{bmatrix} \qquad i = 0, \dots N, \end{aligned} \tag{6.8.1}$$

where the $\{u_i\}$ are the control inputs, and $x_0$ and the $\{w_i\}$ are deterministic disturbances. The $\{w_i\}$ may be interpreted as process noise or driving disturbance. Consider a given linear combination of the states,

$$s_i = L_i x_i, \tag{6.8.2}$$

that we intend to regulate using the control signal $u_i$. In the full information problem, it is assumed that that the control signal, $u_i$, has access to current and past values of the disturbances, $\{x_0, w_0, \dots, w_i\}$. Let

$$\check{u}_i = \mathcal{F}_i(x_0, w_0, \dots, w_i), \tag{6.8.3}$$

denote a such a full information control strategy.

Using this control strategy, let $\mathcal{T}(\mathcal{F})$ denote the transfer operator that maps the disturbances

$$\left\{ \Pi_0^{-\frac{1}{2}} x_0, \{(Q_i^w)^{\frac{1}{2}} w_i\}_{i=0}^N \right\},$$

to the variables

$$\left\{ (P_{N+1}^c)^{\frac{1}{2}} x_{N+1}, \{(Q_i^c)^{\frac{1}{2}} \check{u}_i\}_{i=0}^N, \{(R_i^c)^{\frac{1}{2}} s_i\}_{i=0}^N \right\},$$

where $\Pi_0$, $P_{N+1}^c$, $Q_i^w$, $Q_i^c$ and $R_i^c$ are positive (semi)definite weighting matrices. The finite horizon full information $H^\infty$ control problem can now be stated as follows.

**Problem 6.8.1 (Optimal Full Information $H^\infty$ Control Problem)** *Find a finite horizon full information $H^\infty$-optimal control strategy $\check{u}_i = \mathcal{F}_i(x_0, w_0, \dots, w_i)$ that minimizes $\mathcal{T}(\mathcal{F})$ and obtain the resulting*

$$\gamma_{opt}^2 = \inf_{\mathcal{F}} \| T(\mathcal{F}) \|_\infty^2 = \inf_{\mathcal{F}} \sup_{x_0, w \in h_2} \frac{x_{N+1}^* P_{N+1}^c x_{N+1} + \| (Q_i^c)^{\frac{1}{2}} \check{u}_i \|^2 + \| (R_i^c)^{\frac{1}{2}} s_i \|^2}{x_0 \Pi_0^{-1} x_0 + \| (Q_i^w)^{\frac{1}{2}} w_i \|^2}. \tag{6.8.4}$$

Note that the numerator in (6.8.4) is simply the LQR cost function. Thus the above problem formulation guarantees the smallest LQR cost over all possible disturbances of fixed energy. $H^\infty$ controllers must therefore accommodate for all conceivable disturbances, which reflects in better robust behaviour with respect to disturbance variation.

A closed form solution to the optimal full information $H^\infty$ control problem is available only for some special cases, and a simpler problem results if one relaxes the minimization condition and settles for a suboptimal solution.

**Problem 6.8.2 (Suboptimal $H^\infty$ Problem)** *Find a suboptimal full information $H^\infty$ control strategy $\check{u}_i = \mathcal{F}_i(x_0, w_0, \ldots, w_i)$ that achieves $\| \mathcal{T}(\mathcal{F}) \|_\infty < \gamma$. This clearly requires checking whether $\gamma \geq \gamma_{opt}$.*

### 6.8.2 Solution

Before giving the solution to the full information $H^\infty$ control problem, it will be useful to study the structure of the problem in slightly more detail. In view of Prob. 6.8.2, we must find a control strategy $\check{u}_i = \mathcal{F}_i(x_0, w_0, \ldots w_i)$ such that

$$\sup_{x_0, w \in h_2} \frac{x_{N+1}^* P_{N+1}^c x_{N+1} + \| (Q_i^c)^{\frac{1}{2}} \check{u}_i \|^2 + \| (R_i^c)^{\frac{1}{2}} s_i \|^2}{x_0 \Pi_0^{-1} x_0 + \| (Q_i^w)^{\frac{1}{2}} w_i \|^2} < \gamma^2,$$

or, in other words, a control strategy, such that for all nonzero $x_0$ and $\{w_i\}_{i=0}^N$,

$$\frac{x_{N+1}^* P_{N+1}^c x_{N+1} + \sum_{i=0}^N \check{u}_i^* Q_i^c \check{u}_i + \sum_{i=0}^N s_i^* R_i^c z_i}{x_0 \Pi_0^{-1} x_0 + \sum_{i=0}^N w_i^* Q_i^w w_i} < \gamma^2. \tag{6.8.5}$$

Multiplying (6.8.5) by the nonzero denominator, we obtain the following result.

**Lemma 6.8.1 (Full Information $H^\infty$ Control and Quadratic Forms)** *Given a scalar $\gamma > 0$, there exists a full information controller that achieves $\| \mathcal{T}(\mathcal{F}) \|_\infty < \gamma$ if, and only if, there exists $\check{u}_i = \mathcal{F}_i(x_0, w_0, \ldots w_i)$ (for all $i = 0, \ldots, N$) such that for all nonzero complex vectors $x_0$ and all nonzero sequences $\{w_i\}_{i=0}^N$, the scalar quadratic form*

$$J_N^c = x_0^* \Pi_0^{-1} x_0 - \gamma^{-2} (x_{N+1}^* P_{N+1}^c x_{N+1} + \sum_{i=0}^N \check{u}_i^* Q_i^c \check{u}_i - \gamma^2 \sum_{i=0}^N w_i^* Q_i^w w_i + \sum_{i=0}^N s_i^* R_i^c s_i)$$

$$= \quad x_0^* \Pi_0^{-1} x_0 - \gamma^{-2} (x_{N+1}^* P_{N+1}^c x_{N+1} + \sum_{i=0}^{N} \begin{bmatrix} \check{u}_i^* & w_i^* \end{bmatrix} \begin{bmatrix} Q_i^c & 0 \\ 0 & -\gamma^2 Q_i^w \end{bmatrix} \begin{bmatrix} \check{u}_i \\ w_i \end{bmatrix} + \sum_{i=0}^{N} s_i^* R_i^c s_i)$$

*satisfies*

$$J_N^c > 0.$$

The quadratic form appearing in the parentheses of $J_N^c$ is simply the LQR cost function associated with the state-space model (6.8.1), and with weighting matrices $P_{N+1}$, $\begin{bmatrix} Q_i^c & 0 \\ 0 & -\gamma^2 Q_i^w \end{bmatrix}$, and $R_i^c$, respectively. Note that one of the weighting matrices is indefinite, which is why we considered the Krein space setting in Sec. 6.7. We may thus use Eq. (6.7.30) of Theorem 6.7.1 to rewrite $J_N^c$ as

$$J_N^c = x_0^* \Pi_0^{-1} x_0 - \gamma^{-2} \left( x_0^* P_0^c x_0 + \right.$$

$$\left. \sum_{i=0}^{N} \begin{bmatrix} \check{u}_i - \hat{u}_i \\ w_i - \hat{w}_i \end{bmatrix}^* \begin{bmatrix} Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} & G_{2,i}^* P_{i+1}^c G_{1,i} \\ G_{1,i}^* P_{i+1}^c G_{2,i} & -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} \end{bmatrix} \begin{bmatrix} \check{u}_i - \hat{u}_i \\ w_i - \hat{w}_i \end{bmatrix} \right). \tag{6.8.6}$$

where $\hat{u}_i$ and $\hat{w}_i$ are given by

$$\begin{bmatrix} \hat{u}_i \\ \hat{w}_i \end{bmatrix} = -K_{c,i} x_i, \tag{6.8.7}$$

with

$$K_{c,i} = (R_{e,i}^c)^{-1} \begin{bmatrix} G_{2,i}^* \\ G_{1,i}^* \end{bmatrix} P_{i+1}^c F_i, \tag{6.8.8}$$

where

$$R_{e,i}^c = \begin{bmatrix} Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} & G_{2,i}^* P_{i+1}^c G_{1,i} \\ G_{1,i}^* P_{i+1}^c G_{2,i} & -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} \end{bmatrix}, \tag{6.8.9}$$

and where $P_i^c$ satisfies the backwards Riccati recursion

$$P_i^c = F_i^* P_{i+1}^c F_i + L_i^* R_i^c L_i - K_{c,i}^* R_{e,i}^c K_{c,i}, \quad P_{N+1}^c. \tag{6.8.10}$$

In order to solve the full information problem, let us introduce the following block LDU factorization of $R_{e,i}^c$,

$$\begin{bmatrix} I_{m_2} & 0 \\ G_{1,i}^* P_{i+1}^c G_{2,i} R_{G^c,i}^{-1} & I_{m_1} \end{bmatrix} \begin{bmatrix} R_{G^c,i} & 0 \\ 0 & \Delta_i \end{bmatrix} \begin{bmatrix} I_{m_1} & R_{G^c,i}^{-1} G_{2,i}^* P_{i+1}^c G_{1,i} \\ 0 & I_{m_1} \end{bmatrix}, \tag{6.8.11}$$

where we have defined

$$R_{G^c,i} \triangleq Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i}, \tag{6.8.12}$$

and the Schur complement,

$$\Delta_i \triangleq -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} - G_{1,i}^* P_{i+1}^c G_{2,i} R_{G^c,i}^{-1} G_{2,i}^* P_{i+1}^c G_{1,i}. \tag{6.8.13}$$

The factorization in (6.8.11) allows us to write $J_N^c$ as

$$J_N^c = x_0^*(\Pi_0^{-1} - \gamma^{-2} P_0^c)x_0 - \gamma^{-2}\sum_{i=0}^{N}(\check{u}_i - \bar{u}_i)^* R_{G^c,i}(\check{u}_i - \bar{u}_i) - \gamma^{-2}\sum_{i=0}^{N}(w_i - \hat{w}_i)^*\Delta_i(w_i - \hat{w}_i), \tag{6.8.14}$$

where we have defined $\bar{u}_i$ via

$$\begin{bmatrix} \check{u}_i - \bar{u}_i \\ w_i - \hat{w}_i \end{bmatrix} = \begin{bmatrix} I_{m_2} & 0 \\ G_{1,i}^* P_{i+1}^c G_{2,i} R_{G^c,i}^{-1} & I_{m_1} \end{bmatrix} \begin{bmatrix} \check{u}_i - \hat{u}_i \\ w_i - \hat{w}_i \end{bmatrix}, \tag{6.8.15}$$

so that, after some algebraic simplification, we obtain

$$\bar{u}_i = -R_{G^c,i}^{-1} G_{2,i}^* P_{i+1}^c F_i x_i - R_{G^c,i}^{-1} G_{2,i}^* P_{i+1}^c G_{1,i} w_i. \tag{6.8.16}$$

Note that $\bar{u}_i$ is a function of $\{x_0, w_0, \ldots, w_i\}$, whereas $\hat{u}_i$ and $\hat{w}_i$ are functions of $\{x_0, w_0, \ldots, w_{i-1}\}$.

Referring back to Eq. (6.8.14) we see that since $\check{u}_i$ is only allowed to be a function of $\{x_0, w_0, \ldots, w_i\}$ it cannot influence the first and third terms in the summation for $J_N^c$.[7] Thus a necessary condition for $J_N^c$ to be positive for all nonzero $\{x_0, w_0, \ldots w_N\}$ is that the first and third terms be positive for all such disturbances. In other words, if

$$\Pi_0^{-1} - \gamma^{-2} P_0^c > 0, \tag{6.8.17}$$

and

$$\Delta_i < 0, \quad i = 0, \ldots N. \tag{6.8.18}$$

The above conditions are also sufficient, since we can always choose the control signal to be,

$$\check{u}_i = \bar{u}_i, \tag{6.8.19}$$

[7] To see why, note that $x_0^*(\Pi_0^{-1} - \gamma^{-2} P_0^c)x_0$ is independent of $\check{u}_i$, and, moreover, that the controller cannot influence the value of $(w_i - \hat{w}_i)^*\Delta_i(w_i - \hat{w}_i)$ since $\hat{w}_i$ depends only on previous controls which have no knowledge of $w_i$.

which yields, incidentally, the central controller.

We have thus established the following result.

**Theorem 6.8.1 (Finite-Horizon Full Information $H^\infty$ Control)** *Consider the state-space model (6.8.1) and define $\mathcal{T}(\mathcal{F})$ as the transfer operator that maps the disturbances $\{\Pi_0^{-\frac{1}{2}}x_0, \{(Q_i^w)^{\frac{1}{2}}w_i\}_{i=0}^N\}$ to the variables $\{(P_{N+1}^c)^{\frac{1}{2}}x_{N+1}, \{(Q_i^c)^{\frac{1}{2}}\check{u}_i, (R_i^c)^{\frac{1}{2}}s_i\}_{i=0}^N\}$, where $\Pi_0$, $P_{N+1}^c$, $Q_i^w$, $Q_i^c$ and $R_i^c$ are positive (semi)definite weighting matrices. Then for any given $\gamma > 0$, a full information $H^\infty$ control strategy, $\check{u}_i = \mathcal{F}_i(x_0, w_0, \ldots, w_i)$, that yields*

$$\| \mathcal{T}(\mathcal{F}) \|_\infty < \gamma,$$

*exists if, and only if,*

*(i)* $\Pi_0^{-1} - \gamma^{-2}P_0^c > 0$

*(ii)* $\Delta_i = -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} - G_{1,i}^* P_{i+1}^c G_{2,i} R_{G^c,i}^{-1} G_{2,i}^* P_{i+1}^c G_{1,i} < 0 \quad i = 0, \ldots, N$

*where $R_{G^c,i} = Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i}$, and $P_{i+1}^c$ satisfies the backwards Riccati recursion*

$$P_i^c = F_i^* P_{i+1}^c F_i + L_i^* R_i^c L_i - Kc, i^* R_{e,i}^c K_{c,i}, \quad P_{N+1}^c$$

*with*

$$K_{c,i} = (R_{e,i}^c)^{-1} \begin{bmatrix} G_{2,i}^* \\ G_{1,i}^* \end{bmatrix} P_{i+1}^c F_i,$$

*and*

$$R_{e,i}^c = \begin{bmatrix} Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} & G_{2,i}^* P_{i+1}^c G_{1,i} \\ G_{1,i}^* P_{i+1}^c G_{2,i} & -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} \end{bmatrix}.$$

*If this is the case, then all possible full information $H^\infty$ control strategies, $\check{u}_i = \mathcal{F}_i(x_0, w_0, \ldots, w_i)$, are given by those that satisfy,*

$$x_0^*(\Pi_0^{-1} - \gamma^{-2}P_0^c)x_0 - \gamma^{-2}\sum_{i=0}^N (\check{u}_i - \bar{u}_i)^* R_{G^c,i}(\check{u}_i - \bar{u}_i) - \gamma^{-2}\sum_{i=0}^N (w_i - \hat{w}_i)^* \Delta_i (w_i - \hat{w}_i) > 0,$$

*where $\hat{w}_i$ is defined via,*

$$\begin{bmatrix} \hat{u}_i \\ \hat{w}_i \end{bmatrix} = -K_{c,i} x_i,$$

*and* $\bar{u}_i$, *via*

$$\bar{u}_i = -R_{G^c,i}^{-1} G_{2,i}^* P_{i+1}^c F_i x_i - R_{G^c,i}^{-1} G_{2,i}^* P_{i+1}^c G_{1,i} w_i.$$

*Finally, the so-called central controller is given by,*

$$\check{u}_i = \bar{u}_i. \tag{6.8.20}$$

**Remarks:**

(i) Note that the above Theorem is simply the finite-horizon time-variant counterpart of Theorem 1.7.3. (See also the comments at the end of Sec. 1.7.3.)

(ii) The solution to the $H^\infty$ control problem with perfect state measurement, as given by Theorem 6.8.1, is a special case of the LQR control problem studied in Sec. 6.7. Note that we needed the Krein space extension of the LQR theory (with indefinite weighting matrices) since the $H^\infty$ control problem leads to such types of cost functions.

(iii) There are two conditions for the existence of $H^\infty$ controllers: one is a certain positivity check for the solution of the backwards Riccati at each time instant, and one is a coupling condition between the end-point solution of the Riccati and the weighting on the initial state $x_0$. This last condition is reminiscent of the separation principle and coupling condition that occurs in measurement feedback $H^\infty$ control, which we shall study in the next section.

(iv) The central full information controllers given by Eq. (6.8.20) have the same basic properties that the central filters of $H^\infty$ estimation had. In other words, they are risk-sensitive optimal with risk-sensitivity parameter, $\theta = -\gamma^{-2}$, they are the solution to certain quadratic dynamic games, and they are maximum entropy controllers.

(v) We should mention that the central control law (6.8.1) is not exactly a state feedback law (as happens in continuous-time full information $H^\infty$ control) since the second term involves $w_i$. The reason is, of course, that we have allowed the control signal to be a causal function of the disturbances, *i.e.*, a function of $\{x_0, w_0, \ldots w_i\}$. If we restrict the control signal to be a strictly causal function of the disturbances, *i.e.*, a function of $\{x_0, w_0, \ldots w_{i-1}\}$, then, as shown below, we do obtain a state feedback law for the central controller. (Compare with the remarks following Theorem 1.7.3.)

When the control signal is restricted to be a strictly causal function of the disturbances, *i.e.*, $\check{u}_i = \mathcal{F}_i(x_0, w_0, \ldots w_{i-1})$, then the only difference in the arguments preceding Theorem 6.8.1 is that we need to replace the block LDU factorization (6.8.11) of $R^c_{e,i}$ with its block upper-diagonal-lower (UDL) factorization,

$$\begin{bmatrix} I_{m_2} & G^*_{2,i}P^c_{i+1}G_{1,i}(R'_{G^c,i})^{-1} \\ 0 & I_{m_1} \end{bmatrix} \begin{bmatrix} \Delta'_i & 0 \\ 0 & R'_{G^c,i} \end{bmatrix} \begin{bmatrix} I_{m_1} & 0 \\ (R'_{G^c,i})^{-1}G^*_{1,i}P^c_{i+1}G_{2,i} & I_{m_1} \end{bmatrix}, \tag{6.8.21}$$

where we have defined

$$R'_{G^c,i} \triangleq -\gamma^2 Q^w_i + G^*_{1,i}P^c_{i+1}G_{1,i}, \tag{6.8.22}$$

and the Schur complement,

$$\Delta'_i \triangleq Q^c_i + G^*_{2,i}P^c_{i+1}G_{2,i} - G^*_{2,i}P^c_{i+1}G_{1,i}(R'_{G^c,i})^{-1}G^*_{1,i}P^c_{i+1}G_{2,i}. \tag{6.8.23}$$

The factorization in (6.8.21) allows us to write $J^c_N$ as

$$J^c_N = x_0^*(\Pi_0^{-1} - \gamma^{-2}P^c_0)x_0 - \gamma^{-2}\sum_{i=0}^{N}(\check{u}_i - \hat{u}_i)^*\Delta'_i(\check{u}_i - \hat{u}_i) - \gamma^{-2}\sum_{i=0}^{N}(w_i - \bar{w}_i)^*R'_{G^c,i}(w_i - \bar{w}_i), \tag{6.8.24}$$

where $\bar{w}_i$ is defined via,

$$\begin{bmatrix} \check{u}_i - \hat{u}_i \\ w_i - \bar{w}_i \end{bmatrix} = \begin{bmatrix} I_{m_2} & G^*_{2,i}P^c_{i+1}G_{1,i}(R'_{G^c,i})^{-1} \\ 0 & I_{m_1} \end{bmatrix} \begin{bmatrix} \check{u}_i - \hat{u}_i \\ w_i - \hat{w}_i \end{bmatrix}. \tag{6.8.25}$$

Note that since $u_i$ is only a function of $(x_0, w_0, \ldots w_{i-1})$, it can only effectively influence the second term in the summation for $J^c_N$ in Eq. (6.8.24). Therefore proceeding with an argument similar to the one that led to the proof of Theorem 6.8.1, we obtain the following result.

**Theorem 6.8.2 (Strictly Causal Full Information $H^\infty$ Control)** *Consider the state-space model (6.8.1) and define $\mathcal{T}(\mathcal{F})$ as the transfer operator that maps the disturbances $\{\Pi_0^{-\frac{1}{2}}x_0, \{(Q_i^w)^{\frac{1}{2}}w_i\}_{i=0}^N\}$ to the variables $\{(P_{N+1}^c)^{\frac{1}{2}}x_{N+1}, \{(Q_i^c)^{\frac{1}{2}}\check{u}_i, (R_i^c)^{\frac{1}{2}}s_i\}_{i=0}^N\}$, where $\Pi_0$, $P_{N+1}^c$, $Q_i^w$, $Q_i^c$ and $R_i^c$ are positive (semi)definite weighting matrices. Then for any given $\gamma > 0$, a strictly causal full information $H^\infty$ control strategy, $\check{u}_i = \mathcal{F}_i(x_0, w_0, \ldots, w_{i-1})$, that yields*

$$\| \mathcal{T}(\mathcal{F}) \|_\infty < \gamma,$$

*exists if, and only if,*

*(i)* $\Pi_0^{-1} - \gamma^{-2}P_0^c > 0$

*(ii)* $R'_{G^c,i} = -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} < 0 \quad i = 0, \ldots, N$

*where $P_{i+1}^c$ is the same as in Theorem 6.8.1. If this is the case, then all possible strictly causal full information $H^\infty$ control strategies, $\check{u}_i = \mathcal{F}_i(x_0, w_0, \ldots, w_{i-1})$, are given by those that satisfy,*

$$x_0^*(\Pi_0^{-1} - \gamma^{-2}P_0^c)x_0 - \gamma^{-2}\sum_{i=0}^N(\check{u}_i - \hat{u}_i)^*\Delta_i'(\check{u}_i - \hat{u}_i) - \gamma^{-2}\sum_{i=0}^N(w_i - \bar{w}_i)^* R'_{G^c,i}(w_i - \bar{w}_i) > 0,$$

*where $\Delta_i' = Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} - G_{2,i}^* P_{i+1}^c G_{1,i}(R'_{G^c,i})^{-1} G_{1,i}^* P_{i+1}^c G_{2,i}$, and where*

$$\bar{w}_i = -(R'_{G^c,i})^{-1} G_{1,i}^* P_{i+1}^c F_i x_i - (R'_{G^c,i})^{-1} G_{1,i}^* P_{i+1}^c G_{2,i}\check{u}_i, \tag{6.8.26}$$

*and*

$$\hat{u}_i = -(Q_i^c + G_{2,i}^* \tilde{P}_{i+1}^c G_{2,i})^{-1} G_{2,i}^* \tilde{P}_{i+1}^c F_i x_i, \tag{6.8.27}$$

*with*

$$\tilde{P}_{i+1}^c = P_{i+1}^c - P_{i+1}^c G_{1,i}(R'_{G^c,i})^{-1} G_{1,i}^* P_{i+1}^c. \tag{6.8.28}$$

*Finally, the so-called central controller is given by,*

$$\check{u}_i = \hat{u}_i. \tag{6.8.29}$$

## 6.9 Measurement Feedback $H^\infty$ Control

In this section we shall study the finite horizon $H^\infty$ control problem with measurement feedback. We first introduced, and in fact solved, this problem in Sec. 1.7.2. The approach used there was based on a certain separation principle that reduced the measurement feedback problem to two coupled problems: one a full information control problem, and one an estimation problem (essentially for estimating the unobservable full information control signals using the available measurements). The approach taken here is the same, and is a direct application of our earlier results on $H^\infty$ filtering (from Chapter 3) and full information $H^\infty$ control (from Sec. 6.8), coupled with the aforementioned separation principle. We should mention that the separation principle presented here is slightly different from the separation principles first presented in $H^\infty$ control [DGKF89] and risk-sensitive control [Whi90], and is closer to the separation principle given in [GL95]. We should also mention, however, as pointed out earlier in Remark (i) following Theorem 1.7.4, that the two separation principles are closely related, although we believe that the one presented here is more natural since it does not need to appeal to state-space models.

### 6.9.1 Problem Formulation

Consider the (possibly) time-variant state-space model

$$\begin{cases} x_{i+1} &= F_i x_i + G_{1,i} w_i + G_{2,i} u_i \\ y_i &= H_i x_i + v_i \\ s_i &= L_i x_i \end{cases} \qquad i = 0, \ldots N, \tag{6.9.1}$$

where the $\{u_i\}$ are the control inputs, the $\{x_0, \{w_i\}, \{v_i\}\}$ are unknown disturbances, the $\{y_i\}$ are the observed outputs, and the $\{s_i\}$ are the signals we intend to regulate. The $\{w_i\}$ may be interpreted as process noise and the $\{v_i\}$ as measurement noise.

Moreover let $\check{u}_i$ be a control strategy that uses current and past observations, *i.e.*

$$\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_i).$$

With this control strategy, let $\mathcal{T}(\mathcal{F})$ denote the transfer operator that maps the unknown disturbances

$$\left\{\Pi_0^{-\frac{*}{2}}x_0, \{(Q_i^w)^{\frac{1}{2}}w_i\}_{i=0}^N, \{(R_i^v)^{-\frac{*}{2}}v_i\}_{i=0}^N\right\},$$

to the variables

$$\left\{(P_{N+1}^c)^{\frac{1}{2}}x_{N+1}, \{(R_i^c)^{\frac{1}{2}}s_i\}_{i=0}^N, \{(Q_i^c)^{\frac{1}{2}}\check{u}_i\}_{i=0}^N\right\},$$

where $\Pi_0$, $P_{N+1}^c$, $Q_i^w$, $Q_i^c$ $R_i^v$ and $R_i^c$ are positive (semi)definite weighting matrices. The $H^\infty$ control problem with output feedback can now be stated as follows. (See figure 6.2.)

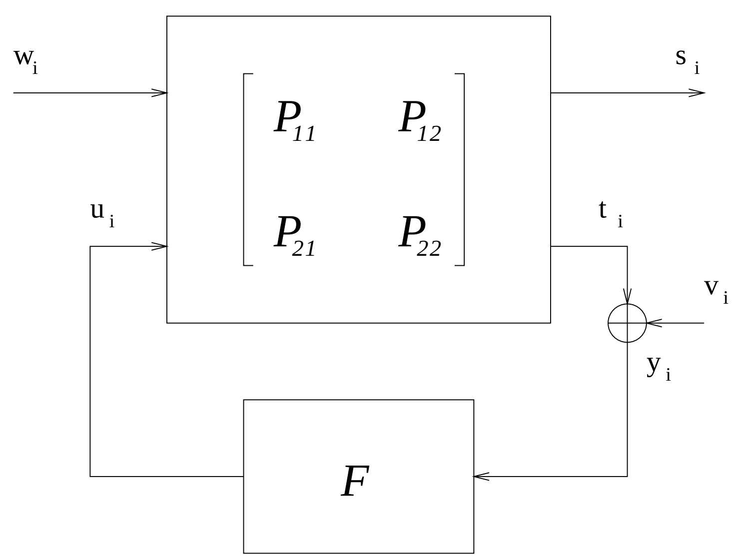


Figure 6.2: $H^\infty$ control with measurement feedback.

**Problem 6.9.1 (Optimal Measurement Feedback $H^\infty$ Control Problem)** *Find a finite horizon $H^\infty$-optimal measurement feedback control strategy $\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_i)$ that minimizes $\mathcal{T}(\mathcal{F})$ and obtain the resulting*

$$\gamma_{opt}^2 = \inf_{\mathcal{F}} \| \mathcal{T}(\mathcal{F}) \|_\infty^2 = \inf_{\mathcal{F}} \sup_{x_0, w\in h_2, v\in h_2} \frac{x_{N+1}^* P_{N+1}^c x_{N+1} + \| (Q_i^c)^{\frac{1}{2}}\check{u}_i \|^2 + \| (R_i^c)^{\frac{1}{2}} L_i x_i \|^2}{x_0 \Pi_0^{-1} x_0 + \| (Q_i^w)^{\frac{1}{2}} w_i \|^2 + \| (R_i^v)^{-\frac{*}{2}} v_i \|^2}. \tag{6.9.2}$$

Note that the numerator in (6.9.2) is simply the LQR cost function. Thus, as before, the above problem formulation guarantees the smallest LQR cost over all possible disturbances of fixed energy. The resulting $H^\infty$ controllers are thus conservative, which reflects in a better robust behaviour with respect to disturbance variation.

As we have seen earlier, closed form solutions to the optimal $H^\infty$ problem are available for only some special cases. We therefore obtain a simpler problem if we relax the minimization condition and settle for a suboptimal solution.

**Problem 6.9.2 (Suboptimal $H^\infty$ Problem)** *Find a suboptimal measurement feedback $H^\infty$ control strategy $\check{u}_i = \mathcal{F}_i(y_0, \ldots y_i)$ that achieves $\| \mathcal{T}(\mathcal{F}) \|_\infty < \gamma$. This clearly requires checking whether $\gamma \geq \gamma_{opt}$.*

## 6.9.2 Solution

We begin by examining the structure of the problem in slightly more detail. In view of Prob. 6.9.2, we must find a control strategy $\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_i)$ such that

$$\sup_{x_0, w \in h_2, v \in h_2} \frac{x_{N+1}^* P_{N+1}^c x_{N+1} + \| (Q_i^c)^{\frac{1}{2}} \check{u}_i \|^2 + \| (R_i^c)^{\frac{1}{2}} L_i x_i \|^2}{x_0 \Pi_0^{-1} x_0 + \| (Q_i^w)^{\frac{1}{2}} w_i \|^2 + \| (R_i^v)^{-\frac{*}{2}} v_i \|^2} < \gamma^2,$$

or, in other words, a control strategy, such that for all nonzero $x_0$, $\{v_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$,

$$\frac{x_{N+1}^* P_{N+1}^c x_{N+1} + \sum_{i=0}^N \check{u}_i^* Q_i^c \check{u}_i + \sum_{i=0}^N x_i^* L_i * R_i^c L_i x_i}{x_0 \Pi_0^{-1} x_0 + \sum_{i=0}^N w_i^* Q_i^w w_i + \sum_{i=0}^N v_i^* (R_i^v)^{-1} v_i} < \gamma^2. \tag{6.9.3}$$

Multiplying (6.9.3) by the nonzero denominator, we obtain the following result.

**Lemma 6.9.1 (Measurement Feedback $H^\infty$ Control and Quadratic Forms)** *Given a scalar $\gamma > 0$, there exists a measurement feedback controller that achieves $\| \mathcal{T}(\mathcal{F}) \|_\infty < \gamma$ if, and only if, there exists $\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_i)$ (for all $i = 0, \ldots, N$) such that for all nonzero complex vectors $x_0$, and all nonzero sequences $\{v_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$, the scalar quadratic form*

$$\begin{aligned} J_N^m \;\; = \;\; & x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^N w_i^* Q_i^w w_i + \sum_{i=0}^N (y_i - H_i x_i)^* (R_i^v)^{-1} (y_i - H_i x_i) - \\ & \gamma^{-2} (\sum_{i=0}^N x_i^* L_i^* R_i^c L_i x_i + \sum_{i=0}^N \check{u}_i^* Q_i^c \check{u}_i + x_{N+1}^* P_{N+1}^c x_{N+1}), \end{aligned}$$

*satisfies*

$$J_N^m > 0.$$

*In other words if, and only if,*

(i) $J_N^m(x_0, w_0, \ldots, w_N, \check{u}_0, \ldots, \check{u}_N)$ *has a minimum over* $\{x_0, w_0, \ldots, w_N\}$.

(ii) *The* $\{\check{u}_0, \ldots, \check{u}_N\}$ *can be chosen such that the value of* $J_N^m$ *at this minimum be positive.*

The above Lemma requires us to minimize $J_N^m$ over the variables, $\{x_0, w_0, \ldots, w_N\}$ and to compute its value at this minimum. Since the expression for $J_N^m$ is quite complicated, this appears to be a formidable task. To somewhat alleviate the problem, let us reorganize $J_N^m$ in the following fashion,

$$\begin{aligned} J_N^m &= x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N}(y_i - H_i x_i)^*(R_i^v)^{-1}(y_i - H_i x_i) \\ &\quad -\gamma^{-2}\left[\sum_{i=0}^{N} x_i^* L_i^* R_i^c L_i x_i + \sum_{i=0}^{N}\begin{bmatrix} w_i \\ \check{u}_i \end{bmatrix}^* \begin{bmatrix} -\gamma^2 Q_i^w & 0 \\ 0 & Q_i^c \end{bmatrix}\begin{bmatrix} w_i \\ \check{u}_i \end{bmatrix} + x_{N+1}^* P_{N+1}^c x_{N+1}\right]. \end{aligned}$$

Note that the expression inside the brackets is simply the indefinite quadratic form associated with the full information problem. (See Lemma 6.8.1). Therefore we can use Eq. (6.8.6) to write it as

$$x_0^* P_0^c x_0 + \sum_{i=0}^{N}\begin{bmatrix} w_i - \hat{w}_i \\ \check{u}_i - \hat{u}_i \end{bmatrix}^* (R_{e,i}^c)^{-1}\begin{bmatrix} w_i - \hat{w}_i \\ \check{u}_i - \hat{u}_i \end{bmatrix}, \tag{6.9.4}$$

where $\hat{w}_i$ and $\hat{u}_i$ are given by

$$\begin{bmatrix} \hat{w}_i \\ \hat{u}_i \end{bmatrix} = -K_{c,i}x_i \triangleq -\begin{bmatrix} K_{w,i} \\ K_{u,i} \end{bmatrix} x_i, \tag{6.9.5}$$

with

$$K_{c,i} = (R_{e,i}^c)^{-1}\begin{bmatrix} G_{1,i}^* \\ G_{2,i}^* \end{bmatrix} P_{i+1}^c F_i, \tag{6.9.6}$$

where

$$R_{e,i}^c = \begin{bmatrix} -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} & G_{1,i}^* P_{i+1}^c G_{2,i} \\ G_{2,i}^* P_{i+1}^c G_{1,i} & Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} \end{bmatrix}, \tag{6.9.7}$$

and where $P_i^c$ satisfies the backwards Riccati recursion

$$P_i^c = F_i^* P_{i+1}^c F_i + L_i^* R_i^c L_i - K_{c,i}^* R_{e,i}^c K_{c,i}, \quad P_{N+1}^c. \tag{6.9.8}$$

Now Eq. (6.9.4) allows us to write $J_N^m$ as follows,

$$\begin{aligned} J_N^m &= x_0^*(\Pi_0^{-1} - \gamma^{-2} P_0^c)x_0 + \\ &\sum_{i=0}^N \begin{bmatrix} w_i - \hat{w}_i \\ \check{u}_i - \hat{u}_i \\ y_i - H_i x_i \end{bmatrix}^* \begin{bmatrix} -\gamma^{-2} \begin{bmatrix} R_{e,i}^c(1,1) & R_{e,i}^c(1,2) \\ R_{e,i}^c(2,1) & R_{e,i}^c(2,2) \end{bmatrix} & 0 \\ 0 & (R_i^v)^{-1} \end{bmatrix} \begin{bmatrix} w_i - \hat{w}_i \\ \check{u}_i - \hat{u}_i \\ y_i - H_i x_i \end{bmatrix}, \end{aligned}$$

where the $R_{e,i}^c(i,j)$ denote the (obvious) block entries of $R_{e,i}^c$. Now if we define,

$$\begin{bmatrix} \Delta_i^{-1} & \bar{S}_i \\ \bar{S}_i^* & (\Delta_i')^{-1} \end{bmatrix} \triangleq \begin{bmatrix} R_{e,i}^c(1,1) & R_{e,i}^c(1,2) \\ R_{e,i}^c(2,1) & R_{e,i}^c(2,2) \end{bmatrix}, \tag{6.9.9}$$

then $J_N^m$ can be written as,

$$J_N^m = x_0^*(\Pi_0^{-1} - \gamma^{-2} P_0^c)x_0 + \sum_{i=0}^N \begin{bmatrix} w_i - \hat{w}_i \\ \begin{bmatrix} \check{u}_i \\ y_i \end{bmatrix} - \begin{bmatrix} K_{u,i} \\ H_i \end{bmatrix} \end{bmatrix}^* \begin{bmatrix} Q_i & S_i \\ S_i^* & R_i \end{bmatrix}^{-1} \begin{bmatrix} w_i - \hat{w}_i \\ \begin{bmatrix} \check{u}_i \\ y_i \end{bmatrix} - \begin{bmatrix} K_{u,i} \\ H_i \end{bmatrix} \end{bmatrix}, \tag{6.9.10}$$

where we have defined,

$$Q_i \triangleq -\gamma^2 \Delta_i^{-1} \quad , \quad S_i \triangleq -\gamma^2 \begin{bmatrix} \bar{S}_i & 0 \end{bmatrix} \quad , \quad R_i \triangleq \begin{bmatrix} \gamma^2 (\Delta_i')^{-1} & 0 \\ 0 & R_i^v \end{bmatrix}. \tag{6.9.11}$$

But note that $J_N^m$, as given by (6.9.10), is now the standard estimation quadratic form of Sec. 2.6 (see Eq. (2.6.2)) for the Krein space state-space model,[8]

$$\begin{cases} \mathbf{x}_{i+1} = (F_i - G_{1,i}K_{w,i})\mathbf{x}_i + G_{1,i}(\mathbf{w}_i - \hat{\mathbf{w}}_i) + G_{2,1}\check{u}_i \\ \begin{bmatrix} \check{\mathbf{u}}_i \\ \mathbf{y}_i \end{bmatrix} = \begin{bmatrix} -K_{u,i} \\ H_i \end{bmatrix} \mathbf{x}_i + \mathbf{v}_i \end{cases}, \tag{6.9.12}$$

[8]Note that we have used $\hat{\mathbf{w}}_i = -K_{w,i}\mathbf{x}_i$ to rewrite the state equation as

$$\mathbf{x}_{i+1} = F_i\mathbf{x}_i + G_{1,i}\mathbf{w}_i + G_{2,1}\check{u}_i = (F_i - G_{1,i}K_{w,i})\mathbf{x}_i + G_{1,i}(\mathbf{w}_i - \hat{\mathbf{w}}_i) + G_{2,1}\check{u}_i.$$

with

$$\left\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{w}_i - \hat{\mathbf{w}}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{w}_j - \hat{\mathbf{w}}_j \\ \mathbf{v}_j \end{bmatrix} \right\rangle = \begin{bmatrix} (\Pi_0^{-1} - \gamma^{-2} P_0^c)^{-1} & 0 \\ 0 & \begin{bmatrix} Q_i & S_i \\ S_i^* & R_i \end{bmatrix} \delta_{ij} \end{bmatrix}. \tag{6.9.13}$$

We can now use the results of Theorem 2.7.4 and Lemma 2.7.3 to minimize $J_N^m$ over the variables, $\{x_0, w_0, \ldots w_N\}$ and to check the conditions for a minimum. Note that the condition for a minimum is that,

(i) $\Pi_0^{-1} - \gamma^{-2} P_0^c > 0$

(ii) $Q_i = -\gamma^2 \Delta_i^{-1} > 0$, for all $j = 0, \ldots, N$

(iii) $R_i - S^* Q_i^{-1} S$ and $R_{e,i}$ have the same inertia, for all $j = 0, \ldots, N$

where

$$R_{e,i} = R_i + \begin{bmatrix} -K_{u,i} \\ H_i \end{bmatrix} P_i \begin{bmatrix} -K_{u,i}^* & H_i^* \end{bmatrix}, \tag{6.9.14}$$

and $P_i$ satisfies the Riccati recursion,

$$P_{i+1} = (F_i - G_{1,i} K_{w,i}) P_i (F_i - G_{1,i} K_{w,i})^* + -\gamma^2 G_{1,i} \Delta_i^{-1} G_{1,i}^* - K_{p,i} R_{e,i} K_{p,i}^*, \tag{6.9.15}$$

initialized with $P_0 = (\Pi_0^{-1} - \gamma^{-2} P_0^c)^{-1}$, and with

$$K_{p,i} = \left\{ (F_i - G_{1,i} K_{w,i}) P_i \begin{bmatrix} -K_{u,i}^* & H_i^* \end{bmatrix} + G_{1,i} S_i \right\} R_{e,i}^{-1}. \tag{6.9.16}$$

Now some simple algebra shows that

$$R_i - S^* Q_i^{-1} S = \begin{bmatrix} -\gamma^2 \left[ Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} \right]^{-1} & 0 \\ 0 & R_i^v \end{bmatrix}, \tag{6.9.17}$$

so that using the facts that $R_i^v > 0$ and $Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} > 0$,[9] we readily see that condition (iii) can be replaced by the condition that

(iii) $(-I_{m_2}) \oplus I_p$ and $R_{e,i}$ have the same inertia, for all $j = 0, \ldots, N$.

[9]This follows from the full information control problem.

Now from Theorem 2.7.4 the value of $J_N^m$ at its minimum is given by

$$J_N^m(\min) = \sum_{i=0}^{N} \begin{bmatrix} \check{u}_i + K_{u,i}\hat{x}_i \\ y_i - H_i\hat{x}_i \end{bmatrix}^* R_{e,i}^{-1} \begin{bmatrix} \check{u}_i + K_{u,i}\hat{x}_i \\ y_i - H_i\hat{x}_i \end{bmatrix}, \tag{6.9.18}$$

where $\hat{x}_i$ obeys the Krein space Kalman filter recursions,

$$\hat{x}_{i+1} = (F_i - G_{1,i}K_{w,i})\hat{x}_i + K_{p,i} \begin{bmatrix} \check{u}_i + K_{u,i}\hat{x}_i \\ y_i - H_i\hat{x}_i \end{bmatrix} + G_{2,i}\check{u}_i, \quad \hat{x}_0 = 0. \tag{6.9.19}$$

Now any choice of control strategy, $\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_i)$ that renders $J(\min)$ positive is an acceptable control. To see that, under the aforementioned conditions, such a control strategy always exists since, via a UDL factorization of $R_{e,i}$ or a completion of squares argument, we can write $J(\min)$ as

$$J_N^m(\min) = \sum_{i=0}^{N}(\check{u}_i - \bar{u}_i)^*\Delta_{R,i}^{-1}(\check{u}_i - \bar{u}_i) + \sum_{i=0}^{N}(y_i - H_i\hat{x}_i)^*(R_i^v + H_iP_iH_i^*)^{-1}(y_i - H_i\hat{x}_i), \tag{6.9.20}$$

where

$$\bar{u}_i = -K_{u,i}\hat{x}_i - K_{u,i}P_iH_i^*(R_i^v + H_iP_iH_i^*)^{-1}(y_i - H_i\hat{x}_i), \tag{6.9.21}$$

and $\Delta_{R,i}$ is the Schur complement of the $(2,2)$ entry $R_i^v + H_iP_iH_i^*$ in $R_{e,i}$. Now in view of the inertia condition (iii) on the $R_{e,i}$ (and the positivity of $P_i$ and $R_i^v$) we have

$$R_i^v + H_iP_iH_i^* > 0 \quad \text{and} \quad \Delta_{R,i} < 0. \tag{6.9.22}$$

Thus the choice $\check{u}_i = \bar{u}_i$ always renders $J_N^m$ positive. This control strategy is referred to as the central controller. There are, of course, many other acceptable strategies.

We can now summarize the results obtained so far in the following theorem.

**Theorem 6.9.1 (Finite Horizon Measurement Feedback $H^\infty$ Control)** *Consider the state-space model (6.9.1) and define $\mathcal{T}(\mathcal{F})$ as the transfer operator that maps the unknown disturbances $\{\Pi_0^{-\frac{*}{2}}x_0, \{(Q_i^w)^{\frac{1}{2}}w_i, (R_i^v)^{-\frac{*}{2}}v_i\}_{i=0}^N\}$ to the variables*

$$\left\{(P_{N+1}^c)^{\frac{1}{2}}x_{N+1}, \{(R_i^c)^{\frac{1}{2}}s_i, (Q_i^c)^{\frac{1}{2}}z_i\}_{i=0}^N\right\},$$

*where* $\Pi_0$, $P^c_{N+1}$, $Q^w_i$, $Q^c_i$ $R^v_i$ *and* $R^c_i$ *are positive (semi)definite weighting matrices. Then for any given* $\gamma > 0$, *a measurement feedback* $H^\infty$ *control strategy* $\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_i)$ *that yields*

$$\| \mathcal{T}(\mathcal{F}) \|_\infty < \gamma,$$

*exists if, and only if,*

*(i)* $\Pi_0^{-1} - \gamma^{-2} P^c_0 > 0$

*(ii)* $\Delta_i = -\gamma^2 Q^w_i + G^*_{1,i} P^c_{i+1} G_{1,i} - G^*_{1,i} P^c_{i+1} G_{2,i} R^{-1}_{G^c,i} G^*_{2,i} P^c_{i+1} G_{1,i} < 0 \quad i = 0, \ldots, N$

*(iii) the matrices*

$$\begin{bmatrix} -I_{m_2} & 0 \\ 0 & I_p \end{bmatrix} \quad \text{and} \quad R_{e,i} = R_i + \begin{bmatrix} -K_{u,i} \\ H_i \end{bmatrix} P_i \begin{bmatrix} -K^*_{u,i} & H^*_i \end{bmatrix}$$

*have the same inertia, for all* $j = 0, \ldots, N$

*where* $P^c_{i+1}$ *and* $P_i$ *satisfy the Riccati recursions*

$$\begin{cases} P^c_i &= F^*_i P^c_{i+1} F_i + L^*_i R^c_i L_i - K^*_{c,i} R^c_{e,i} K_{c,i}, \\ P_{i+1} &= (F_i - G_{1,i} K_{w,i}) P_i (F_i - G_{1,i} K_{w,i})^* + -\gamma^2 G_{1,i} \Delta_i^{-1} G^*_{1,i} - K_{p,i} R_{e,i} K^*_{p,i}, \end{cases}$$

*initialized with* $P^c_{N+1}$ *and* $P_0 = (\Pi_0^{-1} - \gamma^{-2} P^c_0)^{-1}$, *and where*

$$\begin{cases} R_{G^c,i} &= Q^c_i + G^*_{2,i} P^c_{i+1} G_{2,i} \\ K_{c,i} &= (R^c_{e,i})^{-1} \begin{bmatrix} G^*_{1,i} \\ G^*_{2,i} \end{bmatrix} P^c_{i+1} F_i \\ R^c_{e,i} &= \begin{bmatrix} Q^c_i + G^*_{1,i} P^c_{i+1} G_{1,i} & G^*_{1,i} P^c_{i+1} G_{2,i} \\ G^*_{2,i} P^c_{i+1} G_{1,i} & -\gamma^2 Q^w_i + G^*_{2,i} P^c_{i+1} G_{2,i} \end{bmatrix} \\ \begin{bmatrix} K_{w,i} \\ K_{u,i} \end{bmatrix} &= K_{c,i} \\ K_{p,i} &= \left[ (F_i - G_{1,i} K_{w,i}) P_i \begin{bmatrix} -K^*_{u,i} & H^*_i \end{bmatrix} + G_{1,i} S_i \right] R^{-1}_{e,i} \\ R_i &= \begin{bmatrix} (\Delta'_i)^{-1} & 0 \\ 0 & R^v_i \end{bmatrix} \\ \Delta'_i &= Q^c_i + G^*_{2,i} P^c_{i+1} G_{2,i} - G^*_{2,i} P^c_{i+1} G_{1,i} (R'_{G^c,i})^{-1} G^*_{1,i} P^c_{i+1} G_{2,i} \\ R'_{G^c,i} &= -\gamma^2 Q^w_i + G^*_{1,i} P^c_{i+1} G_{1,i} \end{cases}$$

*If this is the case, then all possible measurement feedback $H^\infty$ control strategies, $\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_i)$, are given by those that satisfy*

$$\sum_{i=0}^{N}(\check{u}_i - \bar{u}_i)^* \Delta_{R,i}^{-1}(\check{u}_i - \bar{u}_i) + \sum_{i=0}^{N}(y_i - H_i\hat{x}_i)^*(R_i^v + H_iP_iH_i^*)^{-1}(y_i - H_i\hat{x}_i) > 0, \quad (6.9.23)$$

*where $\hat{x}_i$ satisfies the recursion,*

$$\hat{x}_{i+1} = (F_i - G_{1,i}K_{w,i})\hat{x}_i + K_{p,i}\begin{bmatrix} \check{u}_i + K_{u,i}\hat{x}_i \\ y_i - H_i\hat{x}_i \end{bmatrix} + G_{2,i}\check{u}_i, \quad \hat{x}_0 = 0$$

*where*

$$\Delta_{R,i} = -\gamma^2(\Delta_i')^{-1} + K_{u,i}P_iK_{u,i}^* - K_{u,i}P_iH_i^*(R_i^v + H_iP_iH_i^*)^{-1}H_iK_{u,i}^*,$$

*and where*

$$\bar{u}_i = -K_{u,i}\hat{x}_i - K_{u,i}P_iH_i^*(R_i^v + H_iP_iH_i^*)^{-1}(y_i - H_i\hat{x}_i).$$

*The so-called central controller is given by*

$$\check{u}_i = \bar{u}_i \triangleq -K_{u,i}\hat{x}_{i|i}, \quad (6.9.24)$$

*where $\hat{x}_{i|i}$ satisfies the recursion,*

$$\hat{x}_{i+1|i+1} = (F_i - G_{1,i}K_{w,i} - K_{s,i+1}H_{i+1}F_i)\hat{x}_{i|i} + K_{s,i+1}y_{i+1} + G_{2,i}\check{u}_i, \quad \hat{x}_{-1|-1} = 0,$$

*with $K_{s,i} = P_iH_i^*(R_i^v + H_iP_iH_i^*)^{-1}$.*

**Remarks:**

(i) Note that the above Theorem is simply the finite-horizon time-variant counterpart of Theorem 1.7.4. (See also the comments at the end of Sec. 1.7.3.)

(ii) There are three conditions for the existence of $H^\infty$ controllers: the first is a certain positivity check for the solution of the backwards Riccati recursion at each time instant, the second is a certain inertia check for the solution of the forwards Riccati recursion at each time instant, and the third is a coupling condition between the end-point solution of the backwards Riccati recursion and the weighting on the initial state $x_0$.

(iii) Note that, the Riccati recursions for $P_i^c$ and $P_i$ are coupled. Indeed, the recursion for $P_i$ depends on the solution of the recursion for $P_i^c$ (but not vice versa). We should mention that it is possible, through a suitable change of variables (essentially a bilinear transformation involving $P_i$ and $P_i^c$) to come up with an auxiliary variable, say $P_i^d$, that satisfies a recursion which is independent of $P_i^c$. However, the price we have to pay is a certain coupling condition on the spectral radius of $P_i^c P_i^d$ at each time instant. In fact, the first solutions to the measurement feedback $H^\infty$ and risk-sensitive control problems (see [DGKF89] and [Whi90]) presented such Riccati equations. However, we shall not gain much by presenting that algebra here.

(iv) The central full information controllers given by Eq. (6.9.24) have the same basic properties that the central filters of $H^\infty$ estimation and full information control had. They are thus risk-sensitive optimal with risk-sensitivity parameter, $\theta = -\gamma^{-2}$, they are the solution to certain quadratic dynamic games, and they are maximum entropy controllers.

We can also consider the measurement feedback problem with strictly causal controllers,

$$\check{u}_i = \mathcal{F}_i(y_0, \ldots y_{i-1}). \tag{6.9.25}$$

The development is exactly the same as that given prior to the statement of Theorem 6.9.1 with the exception that in the expression for $J_N^m(\min)$,

$$J_N^m(\min) = \sum_{i=0}^{N} \begin{bmatrix} \check{u}_i + K_{u,i}\hat{x}_i \\ y_i - H_i\hat{x}_i \end{bmatrix}^* R_{e,i}^{-1} \begin{bmatrix} \check{u}_i + K_{u,i}\hat{x}_i \\ y_i - H_i\hat{x}_i \end{bmatrix}, \tag{6.9.26}$$

instead of performing the UDL factorization of $R_{e,i}$, we need to perform its LDU factorization. This allows us to write,

$$J_N^m(\min) = \sum_{i=0}^{N}(\check{u}_i - \hat{u}_i)^* \left(R_{e,i}(1,1)\right)^{-1} (\check{u}_i - \hat{u}_i) + \sum_{i=0}^{N}(y_i - H_i\bar{x}_i)^*(\Delta_i^{'})^{-1}(y_i - H_i\bar{x}_i), \tag{6.9.27}$$

where

$$R_{e,i}(1,1) = -\gamma^2 \left[Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i}\right]^{-1} + K_{u,i} P_i K_{u,i}^*,$$

is the block $(1,1)$ entry of $R_{e,i}$,

$$\Delta_i^{'} = R_i^v + H_i P_i H_i^* - H_i P_i K_{u,i}^* \left(R_{e,i}(1,1)\right)^{-1} K_{u,i} P_i H_i^*, \tag{6.9.28}$$

is its Schur complement, and

$$\bar{x}_i = \hat{x}_i - P_i K_{u,i}^* \left(R_{e,i}(1,1)\right)^{-1} (\check{u}_i - \hat{u}_i). \tag{6.9.29}$$

We thus can show the following result.

**Theorem 6.9.2 (Strictly Causal Measurement Feedback $H^\infty$ Control)** *Consider the state-space model (6.9.1) and define $\mathcal{T}(\mathcal{F})$ as the transfer operator that maps the unknown disturbances* $\{\Pi_0^{-\frac{*}{2}} x_0, \{(Q_i^w)^{\frac{1}{2}} w_i, (R_i^v)^{-\frac{*}{2}} v_i\}_{i=0}^N\}$ *to the variables*

$$\left\{(P_{N+1}^c)^{\frac{1}{2}} x_{N+1}, \{(R_i^c)^{\frac{1}{2}} s_i, (Q_i^c)^{\frac{1}{2}} z_i\}_{i=0}^N\right\},$$

*where* $\Pi_0$, $P_{N+1}^c$, $Q_i^w$, $Q_i^c$ $R_i^v$ *and* $R_i^c$ *are positive (semi)definite weighting matrices. Then for any given* $\gamma > 0$, *a strictly causal measurement feedback* $H^\infty$ *control strategy* $\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_{i-1})$ *that yields*

$$\| \mathcal{T}(\mathcal{F}) \|_\infty < \gamma,$$

*exists if, and only if,*

*(i)* $\Pi_0^{-1} - \gamma^{-2} P_0^c > 0$

*(ii)* $\Delta_i = -\gamma^2 Q_i^w + G_{1,i}^* P_{i+1}^c G_{1,i} - G_{1,i}^* P_{i+1}^c G_{2,i} R_{G^c,i}^{-1} G_{2,i}^* P_{i+1}^c G_{1,i} < 0 \quad i = 0, \ldots, N$

*(iii) all leading submatrices of*

$$\begin{bmatrix} -I_{m_2} & 0 \\ 0 & I_p \end{bmatrix} \quad \textit{and} \quad R_{e,i} = R_i + \begin{bmatrix} -K_{u,i} \\ H_i \end{bmatrix} P_i \begin{bmatrix} -K_{u,i}^* & H_i^* \end{bmatrix}$$

*have the same inertia, for all* $j = 0, \ldots, N$

*where all the variables are as in Theorem 6.9.1. If this is the case, then all possible strictly causal measurement feedback* $H^\infty$ *control strategies,* $\check{u}_i = \mathcal{F}_i(y_0, \ldots, y_{i-1})$, *are given by those that satisfy*

$$\sum_{i=0}^N (\check{u}_i + K_{u,i}\hat{x}_i)^* \left(R_{e,i}(1,1)\right)^{-1} (\check{u}_i + K_{u,i}\hat{x}_i) + \sum_{i=0}^N (y_i - H_i\bar{x}_i)^* (\Delta_i^{'})^{-1} (y_i - H_i\bar{x}_i) > 0, \tag{6.9.30}$$

*where*

$$\begin{cases} R_{e,i}(1,1) &= -\gamma^2 \left[ Q_i^c + G_{2,i}^* P_{i+1}^c G_{2,i} \right]^{-1} + K_{u,i} P_i K_{u,i}^* \\ \Delta_i' &= R_i^v + H_i P_i H_i^* - H_i P_i K_{u,i}^* \left( R_{e,i}(1,1) \right)^{-1} K_{u,i} P_i H_i^* \end{cases}$$

*and*

$$\bar{x}_i = \hat{x}_i - P_i K_{u,i}^* \left( R_{e,i}(1,1) \right)^{-1} (\check{u}_i + K_{u,i} \hat{x}_i), \tag{6.9.31}$$

*with $\hat{x}_i$ satisfying the recursion,*

$$\hat{x}_{i+1} = (F_i - G_{1,i} K_{w,i}) \hat{x}_i + K_{p,i} \begin{bmatrix} \check{u}_i + K_{u,i} \hat{x}_i \\ y_i - H_i \hat{x}_i \end{bmatrix} + G_{2,i} \check{u}_i, \quad \hat{x}_0 = 0.$$

*The so-called central controller is given by*

$$\check{u}_i = -K_{u,i} \hat{x}_{i|i}. \tag{6.9.32}$$

## 6.10 Conclusion

In this chapter we studied duality in linear spaces through the notion of a dual basis. This study led to certain dual approaches for solving (definite and indefinite) quadratic problems with various applications. We, in particular, used duality to solve the LQR control problem with indefinite weighting matrices, and the closely-related problems of full information and measurement feedback $H^\infty$ control.

# Chapter 7

# Infinite Horizon Results

In this chapter we study the celebrated discrete-time algebraic Riccati equation (DARE) which arises in an impressive range of applications in systems and control theory. Although a great deal is known about the Riccati equation when the coefficient matrices are positive semi-definite, much less is known when these coefficients are indefinite matrices. Here we shall consider the DARE in the full generality of this, so-called, indefinite case and shall then particularize the results to some important special cases.

The approach taken in this chapter is given by the introduction of a certain (so-called) Popov function whose factorizations are intimately related to solutions of the DARE. The main result is that solutions to the DARE, or more more precisely a system of algebraic Riccati equations (SDARE), exists if, and only if, a certain proper factorization of the Popov function exists. Additional conditions are then given under which the solution to the DARE becomes stabilizing, Hermitian, positive semi-definite, etc. We also relate the solutions of the DARE to a so-called Hamiltonian matrix, and remark that the famous invariant subspace method can be used to compute solutions of the DARE in the indefinite case as well. Some examples are also included to illustrate the significance of the results.

## 7.1 Introduction

Undoubtedly one of the most important concepts in linear systems and control, both from a practical and a theoretical point of view, is the algebraic Riccati equation. Although the origin of the Riccati equation goes back to Count Riccati in 1724 [Ric24], it was first introduced into control theory by Kalman in 1960 [Kal60a]. Since then the algebraic Riccati equation has known an impressive range of applications such as linear quadratic (LQ) optimal control [AM71, KS72], stability theory [Kal63a, Yak62, Pop64, Wil76], stochastic filtering and stochastic control [Jaz70, AM79, Kai81, Ast70, BS78], stochastic realization theory [And69, Fau76, Pic76], synthesis of linear passive networks [KB61, AV73], dynamic games [Isa65, BO82] and, most recently, $H^\infty$ optimal control and filtering [GD88, DGKF89, KN91, GL95].

There currently exists a vast and growing literature on the algebraic Riccati equation. (The survey papers [SH83, Dor83, Sha83, And88, LR91, Kuc91, Lau91] and the monograph [LR95] provide a good perspective of the research activity in this field.) However, most of the results available concern the special case where the coefficient matrices of the Riccati equation are positive semi-definite. This is quite natural because the Riccati equations encountered in LQ optimal control and filtering (which is where these equations first appeared) are of this, so-called positive, type. However, the Riccati equations that arise in the more recent fields of $H^\infty$ and risk-sensitive (see *e.g.*, [Whi90]) optimal control and filtering are of the type that have (possibly) indefinite coefficient matrices. Therefore there is now growing interest in studying Riccati equations of this latter indefinite type.

In this chapter we shall be interested in the discrete-time algebraic Riccati equation in the general case where the coefficient matrices are possibly indefinite. Although we shall confine our attention here to the discrete-time case, one should be able to generalize the results to the continuous-time algebraic Riccati equation (CARE) and its associated Riccati differential equation.

Earlier approaches to studying the algebraic Riccati equation in the indefinite case essentially generalize the idea of invariant subspaces of the Hamiltonian matrix, first introduced by Potter [Pot66]. Here we shall take a different approach to studying the indefinite DARE that follows the line of the earlier chapters of this thesis on generalizing linear estimation theory to indefinite metric spaces (see Chapter 2). In order to do so, we shall introduce a so-called Popov function, that can be regarded as the generalization of the usual power spectral density function to indefinite metric spaces. In another sense, the Popov function can be regarded as a generalization of the concept of a "supply rate", introduced and used by Willems to study the positive DARE [Wil71b, Wil72], to indefinite metric spaces. In our framework, the Popov function is a natural object to study since its factorizations are intimately related to solutions of the DARE. Therefore, we can replace the problem of studying solutions to the DARE with the problem of finding factorizations of the Popov function. This latter problem can be studied by considering the Smith-Mcmillan canonical form of the Popov function, or by other means.

The major result of the chapter is that solutions to the DARE, or more more precisely a system of algebraic Riccati equations (SDARE), exists if, and only if, a certain proper factorization of the Popov function exists. Additional conditions are then given under which the solution to the DARE becomes stabilizing, Hermitian, positive semi-definite, etc.

The remainder of the chapter is organized as follows. In Sec. 7.2 we begin the study of the DARE by motivating the Popov function. We then proceed to establish some properties of the Popov function in Sec. 7.3. We first establish an equivalent class for the center matrices that yield the same Popov function and then study factorizations of the Popov function via the Smith-Mcmillan form. In particular, we define the notions of proper, proper canonical, and proper canonical Hermitian factorizations of the Popov function. These are then used to prove general existence results for the DARE in Sec. 7.4. These results essentially state that, solutions to the SDARE (with various properties) exist if, and only if, certain proper factorizations of the Popov function (with various properties) exist. Sec. 7.5 considers three special cases: the positive DARE, the case where the state matrix is invertible, and the case that frequently arises in $H^\infty$ problems. Sec. 7.6 then introduces the Hamiltonian matrix and shows that the wellknown method of invariant subspaces, for finding solutions to the DARE, can be generalized to the indefinite case. Some examples that illustrate the concepts presented are given in Sec. 7.7, and the chapter concludes with Sec. 7.8.

## 7.2 The Discrete-time Algebraic Riccati Equation

In this section we shall study the properties of the discrete-time algebraic Riccati equation (DARE),

$$P = FPF^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^* + GQG^*, \tag{7.2.1}$$

where $F \in \mathcal{C}^{n\times n}$, $G \in \mathcal{C}^{n\times m}$ and $H \in \mathcal{C}^{p\times n}$ are given, $Q = Q^* \in \mathcal{C}^{m\times m}$, $R = R^* \in \mathcal{C}^{p\times p}$ and $S \in \mathcal{C}^{m\times p}$ are known, and the unknown variable is the (possibly) Hermitian matrix, $P \in \mathcal{C}^{n\times n}$. Moreover, it is assumed that the Hermitian matrices

$$R \quad \text{and} \quad \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix},$$

are nonsingular.

We shall often find it convenient to introduce the notations

$$K_p = (FPH^* + GS)(R + HPH^*)^{-1} \quad \text{and} \quad R_e = R + HPH^*, \tag{7.2.2}$$

and, when $P$ is Hermitian, to rewrite the DARE (7.2.1) as

$$P = FPF^* - K_p R_e K_p^* + GQG^*. \tag{7.2.3}$$

When, $P$, the solution to the DARE is not Hermitian, we can define

$$K_q = (FP^*H^* + GS)(R + HP^*H^*)^{-1}, \tag{7.2.4}$$

and rewrite the DARE as

$$P = FPF^* - K_p R_e K_q^* + GQG^*. \tag{7.2.5}$$

Solutions to the DARE turn out to be intimately related to factoring the following so-called *Popov function*,

$$\Sigma(z) = \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{7.2.6}$$

The connection comes through the fact that for *any* $n \times n$ matrix, $Z$, we can write:

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* - Z + FZF^* & GS + FZH^* \\ S^*G^* + HZF^* & R + HZH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{7.2.7}$$

Now if we choose $Z = P$, a solution to the DARE, the center matrix in (7.2.7) drops rank, *i.e.*,

$$\begin{bmatrix} GQG^* - Z + FZF^* & GS + FZH^* \\ S^*G^* + HZF^* & R + HZH^* \end{bmatrix} = \begin{bmatrix} K_pR_eK_q^* & K_pR_e \\ R_eK_q^* & R_e \end{bmatrix} = \begin{bmatrix} K_p \\ I \end{bmatrix} R_e \begin{bmatrix} K_q^* & I \end{bmatrix},$$

where we have made use of (7.2.2), (7.2.4) and (7.2.5). Thus, it is straightforward to see that the Popov function (7.2.7) can be written as

$$\Sigma(z) = [H(zI-F)K_p + I]\, R_e \left[H(z^{-*}I - F)K_q + I\right]^*, \tag{7.2.8}$$

which is the factorization we claimed earlier.

**Remark:** If, for future reference, we define the center matrix in (7.2.7) as

$$N(Z) = \begin{bmatrix} GQG^* - Z + FZF^* & GS + FZH^* \\ S^*G^* + HZF^* & R + HZH^* \end{bmatrix}, \tag{7.2.9}$$

we may note that the DARE given by (7.2.1) is simply obtained by setting the *Schur complement* of the $(2,2)$ block entry in $N(Z)$ equal to zero. In this sense, we shall often refer to (7.2.1) as being the DARE corresponding to the Popov function $\Sigma(z)$, or similarly the DARE corresponding to the matrix $N(Z)$.

## 7.3 Properties of the Popov Function

With this motivation of the Popov function, we will now begin to establish some of its properties. These properties will be useful in our study of the DARE. But first, we remark that we shall henceforth implicitly assume that $F$ is stable. However, in all the results given below we have replaced the stability condition on $F$ with the weaker condition that $\{F, H\}$ is detectable[1]. The reason why this can be done is given below.

[1] There are various characterizations of detectability. The one that we shall use is that $\{F, H\}$ is detectable if, and only, if there exists a constant matrix $K$ such that $F - KH$ is stable.

**Lemma 7.3.1 (Detectable $\{F, H\}$)** *When $\{F, H\}$ is detectable, the Popov function $\Sigma(z)$ can be written as*

$$\Sigma(z) = \left(I + H(zI - F)^{-1}K\right) \Sigma_k(z) \left(I + K^*(z^{-1}I - F^*)^{-1}H^*\right), \tag{7.3.1}$$

*where*

$$\Sigma_k(z) = \begin{bmatrix} H(zI - F + KH)^{-1} & I \end{bmatrix} N_k(Z) \begin{bmatrix} (z^{-1}I - F^* + H^*K^*)^{-1} \\ I \end{bmatrix}, \tag{7.3.2}$$

*and where $K$ is a constant matrix such that $F - KH$ is stable. Moreover $N_k(Z)$ is given by*

$$N_k(Z) =$$

$$\begin{bmatrix} -Z + (F-KH)Z(F-KH)^* + \begin{bmatrix} G & -K \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K^* \end{bmatrix} & (F-KH)ZH^* + \begin{bmatrix} G & -K \end{bmatrix} \begin{bmatrix} S \\ R \end{bmatrix} \\ HZ(F-KH)^* + \begin{bmatrix} S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K^* \end{bmatrix} & R + HZH^* \end{bmatrix} \tag{7.3.3}$$

*where $Z$ is any $n \times n$ matrix. Finally, the discrete-time algebraic Riccati equation corresponding to $N_k(Z)$ is the same as the original DARE given by (7.2.1).*

**Remark:** The point of Lemma 7.3.1 is that when $\{F, H\}$ is detectable we can always work instead with the Popov function $\Sigma_k(z)$ whose system matrix, $F - KH$, is stable. Thus, there is no loss of generality in assuming that $F$ is stable, as long as we have the appropriate detectability condition.

**Proof of Lemma 7.3.1:** Since $\{F, H\}$ is detectable, we can choose a constant gain matrix, $K$, such that $F - KH$ is stable. Now we may write

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} N(Z) \begin{bmatrix} I & 0 \\ -K^* & I \end{bmatrix} \begin{bmatrix} I & 0 \\ K^* & I \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

for any $Z \in \mathcal{C}^{n \times n}$. But some algebra shows that

$$\begin{bmatrix} I & -K \\ 0 & I \end{bmatrix} N(Z) \begin{bmatrix} I & 0 \\ -K^* & I \end{bmatrix} = N_k(Z),$$

and likewise that

$$\begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} = \left(I + H(zI-F)^{-1}K\right) \begin{bmatrix} H(zI - F + KH)^{-1} & I \end{bmatrix},$$

which yield (7.3.1), (7.3.2) (7.3.3).

To show the last statement, using some algebra, it is straightforward to show that the original DARE given by (7.2.1) can be rewritten as

$$P=(F-KH)P(F-KH)^*+\begin{bmatrix} G & -K \end{bmatrix}\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}\begin{bmatrix} G^* \\ -K^* \end{bmatrix}-(K_p-K)R_e(K_q-K)^*.$$

But this is simply the DARE associated with $N_k(Z)$. (This can be seen by setting the Schur complement of the $(2,2)$ entry in $N_k(Z)$ equal to zero — see the remark preceding Sec. 7.3.)

■

The next result concerns the nonuniqueness of the center matrices in (7.2.6) that yield the same Popov function, $\Sigma(z)$.

**Lemma 7.3.2 (Equivalence Class for Center Matrices)** *(a) For any $n\times n$ matrix $Z$, the Popov function,*

$$\Sigma(z)=\begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix}\begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix}\begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix},$$

*is invariant under the transformation*

$$\begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix}\rightarrow\begin{bmatrix} GQG^*-Z+FZF^* & GS+FZH^* \\ S^*G^*+HZF^* & R+HZH^* \end{bmatrix}. \tag{7.3.4}$$

*(b) Assume that the pair $\{F,H\}$ is detectable. If there exist Hermitian center matrices*

$$\begin{bmatrix} G_1Q_1G_1^* & G_1S_1 \\ S_1^*G_1^* & R_1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} G_2Q_2G_2^* & G_2S_2 \\ S_2^*G_2^* & R_2 \end{bmatrix}$$

*that yield the same Popov function,* i.e.,

$$\begin{aligned}\Sigma(z) &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix}\begin{bmatrix} G_1Q_1G_1^* & G_1S_1 \\ S_1^*G_1^* & R_1 \end{bmatrix}\begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix} \\ &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix}\begin{bmatrix} G_2Q_2G_2^* & G_2S_2 \\ S_2^*G_2^* & R_2 \end{bmatrix}\begin{bmatrix} (z^{-1}I-F^*)^{-1}H^* \\ I \end{bmatrix}\end{aligned}$$

*then there exists a* <u>*unique*</u> $n \times n$ *Hermitian matrix,* $Z$*, such that*

$$\begin{bmatrix} G_1 Q_1 G_1^* & G_1 S_1 \\ S_1^* G_1^* & R_1 \end{bmatrix} = \begin{bmatrix} G_2 Q_2 G_2^* - Z + FZF^* & G_2 S_2 + FZH^* + V \\ S_2^* G_2^* + HZF^* + V^* & R_2 + HZH^* \end{bmatrix}, \tag{7.3.5}$$

*where* $V$ *lies in the null-space of the observability matrix of* $\{F, H\}$*. In particular, if* $\{F, H\}$ *is observable, then there exists a unique Hermitian* $Z$ *such that*

$$\begin{bmatrix} G_1 Q_1 G_1^* & G_1 S_1 \\ S_1^* G_1^* & R_1 \end{bmatrix} = \begin{bmatrix} G_2 Q_2 G_2^* - Z + FZF^* & G_2 S_2 + FZH^* \\ S_2^* G_2^* + HZF^* & R_2 + HZH^* \end{bmatrix}. \tag{7.3.6}$$

**Remarks:**

(i) Note that Lemma 7.3.2 parametrizes the nonuniqueness of the center matrices that yield the same Popov function, in terms of an arbitrary $Z \in \mathcal{C}^{n \times n}$. It is this freedom in choosing the parameter $Z$ that allows us to reduce the rank of the center matrix, $N(Z)$, and thereby obtain a factorization of $\Sigma(z)$, via solutions to the DARE.

(ii) The above Lemma can be proven and interpreted by introducing linear time-invariant systems driven by inputs that lie in an indefinite, or so-called *Krein*, space. [This was actually done in Sec. 2.2.2.] However, we shall not adopt this route here, and shall give a different proof to Lemma 7.3.2.

**Proof of Lemma 7.3.2:** The proof of part (a) is via a direct calculation — simply check that the identity

$$\begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} -Z + FZF^* & FZH^* \\ HZF^* & HZH^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1} H^* \\ I \end{bmatrix} = 0,$$

holds for any $Z \in \mathcal{C}^{n \times n}$.

To prove part (b), we proceed as follows. First, note that since we have assumed $\{F, H\}$ detectable, using Lemma 7.3.1, there is no loss of generality in assuming that

$F$ is stable. With $F$ stable let us chose $Z_1$ and $Z_2$ as the unique (Hermitian) solutions of the following Lyapunov equations[2]

$$G_1Q_1G_1^* - Z_1 + FZ_1F^* = 0 \quad \text{and} \quad G_2Q_2G_2^* - Z_2 + FZ_2F^* = 0. \tag{7.3.7}$$

With these choices of $Z_1$ and $Z_2$, and using the result of part (a), we may write

$$\begin{aligned}\Sigma(z) &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & FZ_1H^* + G_1S_1 \\ HZ_1F^* + S_1^*G_1^* & R_1 + HZ_1H^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \\ &= \sum_{j=1}^{\infty} HF^{j-1}(FZ_1H^* + G_1S_1)z^{-j} + (R_1 + HZ_1H^*) + \sum_{j=1}^{\infty}(FZ_1H^* + G_1S_1)^*F^{*(j-1)}H^*z^j,\end{aligned}$$

and

$$\begin{aligned}\Sigma(z) &= \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & FZ_2H^* + G_2S_2 \\ HZ_2F^* + S_2^*G_2^* & R_2 + HZ_2H^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \\ &= \sum_{j=1}^{\infty} HF^{j-1}(FZ_2H^* + G_2S_1)z^{-j} + (R_2 + HZ_2H^*) + \sum_{j=1}^{\infty}(FZ_2H^* + G_2S_2)^*F^{*(j-1)}H^*z^j,\end{aligned}$$

where we have also used the fact that $F$ is stable to perform the above expansions.

Equating the coefficients of $z^j$ in the above two expansions for $\Sigma(z)$, yields

$$R_1 + HZ_1H^* = R_2 + HZ_2H^*, \tag{7.3.8}$$

and

$$HF^{j-1}(FZ_1H^* + G_1S_1) = HF^{j-1}(FZ_2H^* + G_2S_1), \quad j \geq 1.$$

The last of the above two equalities implies that

$$\mathcal{O}(FZ_1H^* + G_1S_1) = \mathcal{O}(FZ_2H^* + G_2S_1),$$

where $\mathcal{O}$ is the observability matrix corresponding to the pair $\{F, H\}$. Thus, $FZ_1H^* + G_1S_1$ and $FZ_2H^* + G_2S_1$ differ by some matrix $V$ in the null-space of $\mathcal{O}$, and we may write

$$FZ_1H^* + G_1S_1 = FZ_2H^* + G_2S_1 + V. \tag{7.3.9}$$

[2]In fact, for the solution of the Lyapunov equation $GQG^* - Z + FZF^* = 0$ to be unique, we only require that $F$ have no two eigenvalues such that $\lambda_i = \lambda_j^{-*}$ — see *e.g.*, [Kai80].

Combining (7.3.7)-(7.3.9) yields

$$\begin{bmatrix} G_1Q_1G_1^* & G_1S_1 \\ S_1^*G_1^* & R_1 \end{bmatrix} =$$
$$\begin{bmatrix} G_2Q_2G_2^* - (Z_2 - Z_1) + F(Z_2 - Z_1)F^* & G_2S_2 + F(Z_2 - Z_1)H^* + V \\ S_2^*G_2^* + H(Z_2 - Z_1)F^* + V^* & R_2 + H(Z_2 - Z_1)H^* \end{bmatrix},$$

so that defining the unique matrix $Z = Z_2 - Z_1$ proves (7.3.5).

To prove (7.3.6) we note that when $\{F, H\}$ is observable, $\mathcal{O}$ has full rank, which means $V = 0$ and leads to the desired result.

■

A very similar result holds when the center matrix in the Popov function is not restricted to be Hermitian. Since we shall make use of this result, we state it here as a corollary. The proof is exactly the same as the one given above and is therefore not repeated.

**Corollary 7.3.1 (Non-Hermitian Center Matrices)** *Assume that the pair $\{F, H\}$ is detectable. If there exist (possibly non-Hermitian) center matrices*

$$\begin{bmatrix} Q_1 & S_{a,1} \\ S_{b,1}^* & R_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} Q_2 & S_{a,2} \\ S_{b,2}^* & R_2 \end{bmatrix}$$

*that yield the same Popov function,* i.e.,

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_1 & S_{a,1} \\ S_{b,1}^* & R_1 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \\ &= \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} Q_2 & S_{a,2} \\ S_{b,2}^* & R_2 \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix} \end{aligned}$$

*then there exists a unique $n \times n$ matrix, $Z$, such that*

$$\begin{bmatrix} Q_1 & S_{a,1} \\ S_{b,1}^* & R_1 \end{bmatrix} = \begin{bmatrix} Q_2 - Z + FZF^* & S_{a,2} + FZH^* + V_a \\ S_{b,2}^* + HZF^* + V_b^* & R_2 + HZH^* \end{bmatrix}, \tag{7.3.10}$$

*where $V_a$ and $V_b$ lie in the null-space of the observability matrix of $\{F, H\}$. In particular, if $\{F, H\}$ is observable, then there exists a unique $Z$ such that*

$$\begin{bmatrix} Q_1 & S_{a,1} \\ S_{b,1}^* & R_1 \end{bmatrix} = \begin{bmatrix} Q_2 - Z + FZF^* & S_{a,2} + FZH^* \\ S_{b,2}^* + HZF^* & R_2 + HZH^* \end{bmatrix}. \tag{7.3.11}$$

### 7.3.1 Factorization of the Popov Function

Our next result concerns factorizations of the Popov function, $\Sigma(z)$. We shall essentially use the Smith-Mcmillan form of $\Sigma(z)$ to study the existence of various forms of factorization for $\Sigma(z)$. When coupled with Lemma 7.3.2, this result will allow us to give conditions for the existence of solutions to the DARE, in terms of the existence of certain factorizations, and to establish their various properties.

**Lemma 7.3.3 (Factorization of the Popov Function)** *Consider the Popov function*

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

*where $\{F, H\}$ is detectable. Then $\Sigma(z)$ can always be factorized in the fashion*

$$\Sigma(z) = M(z)N^*(z^{-*}), \tag{7.3.12}$$

*where $M(z)$ and $N(z)$ are $p \times q$ $(p \leq q)$ rational transfer matrices that have poles of magnitude less than one and zeros of magnitude less than or equal to one.*

**Proof:** The proof basically relies on the Smith-McMillan form (see *e.g.*, [McM52, Kai80]) of the Popov function $\Sigma(z)$. Since the Popov function is para-Hermitian, *i.e.*, $\Sigma(z) = \Sigma^*(z^{-*})$, then using elementary row and column operations, the Smith form of $\Sigma(z)$ may be written as

$$\Sigma(z) = U(z)\Lambda(z)V^*(z^{-*}),$$

where $U(z)$ and $V(z)$ are unimodular (here this means that both $U(z)$ and $U^{-1}(z)$ — likewise, $V(z)$ and $V^{-1}(z)$ — are polynomial matrices in $z$) and where the diagonal matrix

$$\Lambda(z) = \begin{bmatrix} \frac{b_1(z)}{a_1(z)} & & & & & \\ & \ddots & & & & \\ & & \frac{b_q(z)}{a_q(z)} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix},$$

is such that

$$a_i(z)|a_{i+1}(z) \quad (i.e., \ a_i(z) \text{ divides } a_{i+1}(z)), \quad \text{and} \quad b_{i+1}(z)|b_i(z), \quad i = 1, \ldots, q-1.$$

The roots of the polynomials $a_i(z)$ and $b_i(z)$ are the poles and zeros of $\Sigma(z)$, respectively.

Now note that, since $\Sigma(z)$ is a para-Hermitian, for every pole (or zero) of $\Sigma(z)$ at $z = \alpha$ there exists a pole (correspondingly, zero) at $z = \alpha^{-*}$. Due to the observability condition, we may assume that $F$ is stable which means that none of the poles of $\Sigma(z)$ (*i.e.*, the eigenvalues of $F$ and $F^{-*}$) lie on the unit circle. However, zeros on the unit circle may exist. Moreover, such zeros need not appear in pairs, since

$$\alpha = e^{j\omega_0} = 1/e^{-j\omega_0} = \alpha^{-*}.$$

With these observations, we can factorize the diagonal entries of $\Lambda(z)$ as

$$a_i(z) = a_{m,i}(z)a^*_{m,i}(z^{-*}) \quad \text{and} \quad b_i(z) = b_{m,i}(z)b^*_{n,i}(z^{-*}), \quad i = 1, \ldots q,$$

where $a_{m,i}(z)$ is a polynomial in $z$ with roots of magnitude less than one, and $b_{m,i}(z)$ and $b_{n,i}(z)$ are polynomials in $z$ with roots of magnitude less than or equal to one.

Using the aforementioned Smith-McMillan form, along with the above factorization of $\Lambda(z)$, allows us to write $\Sigma(z)$ as

$$\Sigma(z) = U(z) \underbrace{\begin{bmatrix} \frac{b_{m,1}(z)}{a_{m,1}(z)} & & & \\ & \ddots & & \\ & & \frac{b_{m,q}(z)}{a_{m,q}(z)} & \\ & & & \bigcirc \end{bmatrix}}_{\begin{bmatrix} M(z) & 0 \end{bmatrix}} \underbrace{\begin{bmatrix} \frac{b^*_{n,1}(z^{-*})}{a^*_{m,1}(z^{-*})} & & & \\ & \ddots & & \\ & & \frac{b^*_{n,q}(z^{-*})}{a^*_{m,q}(z^{-*})} & \\ & & & \bigcirc \end{bmatrix}}_{\begin{bmatrix} N^*(z^{-*}) \\ 0 \end{bmatrix}} U^*(z^{-*}),$$

where we have indicated the rational transfer matrices $M(z)$ and $N(z)$. Note that both $M(z)$ and $N(z)$ are of dimension $p \times q$ and that they have poles of magnitude less than one and zeros of magnitude less than or equal to one (since $U(z)$ and $V(z)$ do not have any poles or zeros). This establishes the claim of the lemma.

∎

The above result shows that under a certain detectability assumption the Popov function always admits a certain (stable-antistable) factorization. Since at the beginning of Sec. 7.2.1 we noted the connection between factorizations of the Popov function and solutions to the DARE, we may now suspect Lemma 7.3.3 to have implications to the DARE. Unfortunately, the situation is not quite as simple, and, in fact, to relate to the DARE, what is needed is not any factorization of the Popov function, but certain (so-called) proper factorizations. These factorizations, along with several others, are defined below. It turns out that, contrary to the factorization of Lemma 7.3.3, these factorizations do not necessarily exist, and that their existence is tied to the existence of certain solutions to the DARE.

**Definition 7.3.1 (Factorizations of the Popov Function)** *Consider the Popov function*

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

*where* $\{F, H\}$ *is detectable.*

(i) *$\Sigma(z)$ is said to have a "proper" factorization if, we can write,*

$$\Sigma(z) = M(z)N^*(z^{-*}), \tag{7.3.13}$$

*where $M(z)$ and $N(z)$ are $p \times q$ ($p \leq q$) proper rational transfer matrices, with Mcmillan degree less than or equal to $n$, that have poles of magnitude less than one.*

  (a) *If, in addition, $M(z)$ and $N(z)$ have zeros of magnitude less than or equal to one, then (7.3.13) is said to be a proper "semi-canonical" factorization of $\Sigma(z)$.*

  (b) *If, in addition, $M(z)$ and $N(z)$ have zeros of magnitude strictly less one, then (7.3.13) is said to be a proper "canonical" factorization of $\Sigma(z)$.*

(ii) *$\Sigma(z)$ is said to have a proper "Hermitian" factorization if, we can write,*

$$\Sigma(z) = M(z)JM^*(z^{-*}), \tag{7.3.14}$$

*where the $p \times q$ ($p \leq q$) proper rational transfer matrix $M(z)$ has Mcmillan degree less than or equal to $n$, and poles of magnitude less than one, and where $J$ is a signature matrix (*i.e.*, a diagonal matrix with $\pm 1$ on the main diagonal) that represents the inertia of $\Sigma(z)$.*

  (a) *If, in addition, $M(z)$ has zeros of magnitude less than or equal to one, then (7.3.14) is said to be a proper "semi-canonical" Hermitian factorization of $\Sigma(z)$.*

  (b) *If, in addition, $M(z)$ has zeros of magnitude strictly less than one, then (7.3.14) is said to be a proper "canonical" Hermitian factorization of $\Sigma(z)$.*

The reason why we have introduced the above factorizations is that the factors given by the solutions to the DARE are proper (see Eq. (7.2.8)). On the other hand, the factors obtained from the Smith-McMillan form, *e.g.*,

$$\begin{bmatrix} M(z) & 0 \end{bmatrix} = U(z) \begin{bmatrix} \frac{b_{m,1}(z)}{a_{m,1}(z)} & & & \\ & \ddots & & \\ & & \frac{b_{m,q}(z)}{a_{m,q}(z)} & \\ & & & \bigcirc \end{bmatrix},$$

are not necessarily proper. [Note that, although the diagonal matrix on the RHS is proper, its product with the polynomial matrix, $U(z)$, need not be so.]

We should also mention that, for arbitrary para-Hermitian rational transfer matrices, $\Sigma(z)$, the conditions required for the existence of the above proper factorizations are not known.[3] Moreover, there are many other questions in the factorization of rational matrix functions, and matrix polynomials, that are open. For example, the question of whether a para-Hermitian matrix polynomial, $P(z)$, admits a Hermitian factorization,

$$P(z) = P^{1/2}(z) J P^{*/2}(z^{-*}), \tag{7.3.15}$$

with $P^{1/2}(z)$ polynomial[4] and $J$ a signature matrix, is also open. A sufficient condition is due to Yakubovich [Yak70a], and is given below.

**Lemma 7.3.4 (Hermitian Factorizations)** *Suppose the multiplicity of the unit circle zeros of each of the invariant polynomials*[5] *of the para-Hermitian polynomial matrix $P(z)$ is even. Then $P(z)$ has a Hermitian factorization (7.3.15).*

The above condition is not necessary, as exemplified by the following factorization,

$$P(z) = \begin{bmatrix} e^{-j\frac{\theta_1-\theta_2}{2}}(z-e^{j\theta_1})(z^{-1}-e^{-j\theta_2}) & 0 \\ 0 & -e^{-j\frac{\theta_1-\theta_2}{2}}(z-e^{j\theta_1})(z^{-1}-e^{-j\theta_2}) \end{bmatrix}, \tag{7.3.16}$$

[3]In fact, in the next section we shall give such conditions when $\Sigma(z)$ can be represented by a Popov function.

[4]If $P^{1/2}(z)$ is not restricted to be polynomial the factorization (7.3.15) can always be achieved, say, by Hermitian Gaussian elimination, provided $P(z)$ has constant inertia almost everywhere on the unit circle.

[5]For the definition of the invariant polynomials of a polynomial matrix see [Kai80].

with

$$P^{1/2}(z) = \begin{bmatrix} z - e^{j\frac{\theta_1+\theta_2}{2}} & -(2+\cos\frac{\theta_1-\theta_2}{2})^{-1/2} \\ -(2+\cos\frac{\theta_1-\theta_2}{2})^{-1/2} & z - e^{-j\frac{\theta_1+\theta_2}{2}} \end{bmatrix} \quad (7.3.17)$$

and $J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Note that the $(z - e^{j\theta_1})(z^{-1} - e^{-j\theta_2})$'s, *i.e.*, the invariant polynomials of $P(z)$, have simple (odd) unit circle zeros.

All the aforementioned difficulties are due to the fact that $\Sigma(z)$ has been taken to be an arbitrary para-Hermitian rational transfer matrix. When $\Sigma(z)$ is positive semidefinite on the unit circle, *i.e.*,

$$\Sigma(e^{j\omega}) \geq 0, \quad \omega \in [0, 2\pi] \quad (7.3.18)$$

then we have the following result that is attributed to Youla [You61].

**Theorem 7.3.1 (Factorization of Positive Rational Matrices)** *Consider the proper $p \times p$ rational para-Hermitian matrix, $\Sigma(z)$, that is nonnegative definite on the unit circle. Then we may write,*

$$\Sigma(z) = M(z)M^*(z^{-*}), \quad (7.3.19)$$

*where $M(z)$ is a $p \times q$ ($q \leq p$) proper rational matrix function with poles of magnitude less than one and zeros of magnitude less than or equal to one. If, in addition, $\Sigma(z)$ has constant rank (equal to $q$) everywhere on the unit circle, then all the zeros of $M(z)$ lie strictly inside the unit circle.*

**Proof:** The proof is lengthy and will not be given here. The reader is referred to [You61] for the details. We just mention, in passing, that the proof uses the Smith-McMillan form of $\Sigma(z)$ along with a certain construction, due to Oono and Yasuura [OY54], for the factorization of positive-definite unimodular para-Hermitian matrices.[6]

∎

[6] In fact, a simple generalization of this ingenious construction has been used in [Yak70a] to study the factorization of indefinite para-Hermitian matrices.

Before closing this section we note that the factorization of Theorem 7.3.1 is a special case of the Hermitian factorization (7.3.14) when $J = I$. Moreover, in this positive case, we cannot have nonproper factors, since if $M(z)$ were nonproper then so would be the product, $M(z)M^*(z^{-*})$. However, this is no longer true in the indefinite case — we can still have nonproper factors, since the signature matrix $J$ in $M(z)JM^*(z^{-*})$ could allow for cancellations that render the product proper.

## 7.4 A General Existence Result

We now present the main result of this chapter that essentially gives necessary and sufficient conditions for the existence of a solution to the DARE (7.2.1) in terms of the existence of certain proper factorizations of the associated Popov function.

**Theorem 7.4.1 (Existence of Solutions to the DARE)** *Consider the system of discrete-time algebraic Riccati equations (SDARE)*

$$\left\{\begin{array}{rcl} P & = & FPF^* + GQG^* - K_pR_eK_q^* \\ K_pR_e & = & FPH^* + GS \quad , \quad K_qR_e^* = FP^*H^* + GS \\ R_e & = & R + HPH^* \end{array}\right. \tag{7.4.1}$$

*where $\{F, H\}$ is detectable. Then we have the following results.*

*(a) The SDARE (7.4.1) has a solution, $P$, if, and only if, the Popov function*

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

*admits a proper factorization of the form*

$$\Sigma(z) = M(z)R_eN^*(z^{-*}),$$

*where $M(z)$ and $N(z)$ are $p\times p$ proper rational transfer matrices, with Mcmillan degree less than or equal to $n$, that have poles of magnitude less than one, and where $R_e$ is a constant matrix that yields the normalizations, $M(\infty) = N(\infty) = I_p$. Moreover, we have*

$$M(z) = H(zI-F)^{-1}K_p + I_p \quad \textit{and} \quad N(z) = H(zI-F)^{-1}K_q + I_p.$$

(b) *The SDARE (7.4.1) has a "semi-stabilizing" solution, $P$,* i.e., *one for which,*

$$F - K_p H \quad \text{and} \quad F - K_q H,$$

*are semi-stable,*[7] *if, and only if, $\Sigma(z)$ admits a proper semi-canonical factorization,* i.e., *one for which $M(z)$ and $N(z)$ have, in addition, zeros of magnitude less than or equal to one. Moreover, the solution is stabilizing,* i.e., *$F - K_p H$ and $F - K_q H$ are both stable, if, and only if, $\Sigma(z)$ admits a proper canonical factorization.*

(c) *The SDARE (7.4.1) has a Hermitian solution, $P$, if, and only if, the Popov function admits a proper Hermitian factorization,*

$$\Sigma(z) = M(z) R_e M^*(z^{-*}),$$

*where $M(z)$ is a $p \times p$ proper rational transfer matrix with Mcmillan degree less than or equal to $n$ and poles of magnitude less than one, and where $R_e$ is a constant Hermitian matrix that yields the normalization, $M(\infty) = I_p$. In this case, we have*

$$M(z) = H(zI - F)^{-1} K_p + I_p.$$

(d) *The SDARE (7.4.1) has a (semi-)stabilizing Hermitian solution, $P$, (*i.e., *$F - K_p H$ is (semi)stable), if, and only if, the Popov function admits a proper (semi-)canonical Hermitian factorization (*i.e., *$M(z)$ has, in addition, zeros of magnitude less than (or equal to) one).*

(e) *The SDARE has a solution, $P$, such that $R + HPH^*$ is nonsingular, if, and only if, the Popov function $\Sigma(z)$ is nonsingular a.e.*[8] *on the unit circle, $|z| = 1$, and admits a proper factorization. In this case, $P$ is a solution to the DARE*

$$P = FPF^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^* + GQG^*.$$

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[7] We shall call a matrix semistable if all its eigenvalues have magnitude less than or equal to one.
[8] Almost everywhere.

**Proof:** We begin with the proof of part (a). We have already seen that if a solution to the SDARE exists then $\Sigma(z)$ allows for a proper factorization (see the arguments leading to Eq. (7.2.8)). Therefore let us assume that a proper factorization (with the properties mentioned in the statement of the Theorem) exists. This means we may write

$$\Sigma(z) = M(z)N^*(z^{-*}) = A(z)DB^*(z^{-*}),$$

where $A(z)$ and $B(z)$ are $p \times p$ matrices with poles and zeros less than or equal to one, and $D$ is a (possibly singular) $p \times p$ matrix that is chosen such that $A(z)$ and $B(z)$ satisfy the normalizations,

$$A(\infty) = I \quad \text{and} \quad B(\infty) = I.$$

Let us now construct minimal realizations for $A(z)$ and $B(z)$ as follows

$$A(z) = H_a(zI - F_a)^{-1}K_a + I \quad \text{and} \quad B(z) = H_b(zI - F_b)^{-1}K_b + I.$$

At this point we should remark that the zeros of $A(z)$ and $B(z)$ are given by the eigenvalues of $F_a - K_aH_a$ and $F_b - K_bH_b$, respectively. Therefore, since $A(z)$ and $B(z)$ have zeros of magnitude less than or equal to one, the matrices $F_a - K_aH_a$ and $F_b - K_bH_b$ are marginally (or semi) stable.

Now note that we may write the Popov function as

$$\begin{aligned}
\Sigma(z) &= \left[H_a(zI - F_a)^{-1}K_a + I\right] D \left[H_b(z^{-*}I - F_b)^{-1}K_b + I\right]^* \\
&= \begin{bmatrix} H_a(zI - F_a)^{-1} & I \end{bmatrix} \begin{bmatrix} K_aDK_b^* & K_aD \\ DK_b^* & D \end{bmatrix} \begin{bmatrix} (z^{-1}I - F_b^*)^{-1}H_b^* \\ I \end{bmatrix}.
\end{aligned}$$

Defining $Z_{ab}$ as the unique solution to the Lyapunov equation,[9]

$$Z_{ab} = F_aZ_{ab}F_b^* + K_aDK_b^*, \tag{7.4.2}$$

and using the, readily verified, identity that

$$\begin{bmatrix} H_a(zI - F_a)^{-1} & I \end{bmatrix} \begin{bmatrix} -Z + F_aZF_b^* & F_aZH_b^* \\ H_aZF_b^* & H_aZH_b^* \end{bmatrix} \begin{bmatrix} (z^{-1}I - F_b^*)^{-1}H_b^* \\ I \end{bmatrix} = 0, \tag{7.4.3}$$

[9] Since both $A(z)$ and $B(z)$ are stable, the matrices $F_a$ and $F_b$ are stable as well. Therefore the Lyapunov equation will have a unique solution since $F_a$ and $F_b$ have no two eigenvalues, $\lambda_a$ and $\lambda_b$, such that $\lambda_a = \lambda_b^{-*}$.

for any matrix $Z$ of the appropriate dimensions, allows us to write the Popov function as,

$$\Sigma(z) = \begin{bmatrix} H_a(zI - F_a)^{-1} & I \end{bmatrix} \begin{bmatrix} 0 & F_a Z_{ab} H_b^* + K_a D \\ H_a Z_{ab} F_b^* + D K_b^* & D + H_a Z_{ab} H_b^* \end{bmatrix} \begin{bmatrix} (z^{-1} I - F_b^*)^{-1} H_b^* \\ I \end{bmatrix}, \tag{7.4.4}$$

Now we can expand $\Sigma(z)$ using (7.4.4) to obtain

$$\Sigma(z) = \sum_{j=1}^{\infty} H_a F_a^{j-1}(F_a Z_{ab} H_b^* + K_a D) z^{-j} + (D + H_a Z_{ab} H_b^*) + \sum_{j=1}^{\infty} \left[ H_b F_b^{j-1}(F_b Z_{ab}^* H_a^* + K_b D^*) \right]^* z^j .$$

A similar expansion for the Popov function in terms of its original representation (7.2.6) yields

$$\Sigma(z) = \sum_{j=1}^{\infty} H F^{j-1}(F Z H^* + G S) z^{-j} + (R + H Z H^*) + \sum_{j=1}^{\infty} \left[ H F^{j-1}(F Z^* H^* + G S) \right]^* z^j ,$$

where $Z$ is the unique solution to the Lyapunov equation $Z = FZF^* + GQG^*$.

Equating the coefficients of the powers of $z$ in these two expansions yields

$$H_a F_a^{j-1}(F_a Z_{ab} H_b^* + K_a D) = H F^{j-1}(F Z H^* + G S)$$

and

$$H_b F_b^{j-1}(F_b Z_{ab}^* H_a^* + K_b D^*) = H F^{j-1}(F Z H^* + G S),$$

for all $j \geq 1$. The above two relations mean that the system with state-space representation $\{F, FZH^* + GS, H\}$, has the same Markov parameters as the systems with state-space representation $\{F_a, F_a Z_{ab} H_b^* + K_a D, H_a\}$ and $\{F_b, F_b Z_{ab}^* H_a^* + K_b D^*, H_b\}$. Thus, there must exist a similarity transformation $T_a$ such that

$$\begin{bmatrix} H_a & 0 & H_{a,c} & 0 \end{bmatrix} = H T_a \quad , \quad \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} = T_a^{-1} F T_a, \tag{7.4.5}$$

and

$$\begin{bmatrix} F_a Z_{ab} H_b^* + K_a D \\ N_{a,o} \\ 0 \\ 0 \end{bmatrix} = T_a^{-1}(F Z H^* + G S), \tag{7.4.6}$$

where the subscripts '$c$', '$o$' and '$co$' represent the uncontrollable, the unobservable, and the both uncontrollable and unobservable, modes of the system $\{F, FZH^* + GS, H\}$ (see [Kai80], p. 133). Correspondingly, there exists a similarity transformation, $T_b$, relating $\{F, FZH^* + GS, H\}$ to $\{F_b, F_b Z_{ab}^* H_a^* + K_b D^*, H_b\}$.

On the other hand, $A(z)$ and $B(z)$ can also be written as

$$\begin{aligned} A(z) &= \begin{bmatrix} H_a & 0 & H_{a,c} & 0 \end{bmatrix} \left( zI - \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} \right)^{-1} \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} + I \\ &= H(zI - F)^{-1} T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} + I, \end{aligned}$$

and

$$\begin{aligned} B(z) &= \begin{bmatrix} H_b & 0 & H_{b,c} & 0 \end{bmatrix} \left( zI - \begin{bmatrix} F_b & 0 & F_{b,13} & 0 \\ F_{b,21} & F_{b,o} & F_{b,23} & F_{b,24} \\ 0 & 0 & F_{b,c} & 0 \\ 0 & 0 & F_{b,43} & F_{b,co} \end{bmatrix} \right)^{-1} \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix} + I \\ &= H(zI - F)^{-1} T_a \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix} + I, \end{aligned}$$

for *any* matrices $X_a$ and $X_b$ of the appropriate dimensions. Thus, we may write

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^* & T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \\ D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^* & D \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix}. \tag{7.4.7}$$

Now (7.4.7) and

$$\Sigma(z) = \begin{bmatrix} H(zI-F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1}H^* \\ I \end{bmatrix},$$

are two different representations of the Popov function. Therefore, using Corollary 7.3.1, there exists a unique $P$ and $V_a$, $V_b$ in the unobservable subspace of $\{F, H\}$ such that

$$\begin{bmatrix} -P + FPF^* + GQG^* & FPH^* + GS - V_a \\ HPF^* + S^*G^* - V_b^* & R + HPH^* \end{bmatrix} = \begin{bmatrix} T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^* & T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \\ D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^* & D \end{bmatrix}. \tag{7.4.8}$$

We shall now show that $X_a$ and $X_b$ can be chosen such that $V_a = 0$ and $V_b = 0$, respectively. To this end, first note that $P$ is given by $P = Z - Z_1$, where

$$Z = FZF^* + GQG^*,$$

and

$$Z_1 = FZ_1F^* + T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^*. \tag{7.4.9}$$

In Appendix 7.A, we show that $Z_1$ has the form:

$$Z_1 = T_a \begin{bmatrix} Z_{ab} & Z_{ab,o} & 0 & 0 \\ Z_{ba,o} & Z_{ab,oo} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} T_b^*, \tag{7.4.10}$$

where

$$\begin{bmatrix} Z_{ab} & Z_{ab,o} \\ Z_{ba,o} & Z_{ab,oo} \end{bmatrix} = \begin{bmatrix} F_a & 0 \\ F_{a,21} & F_{a,o} \end{bmatrix} \begin{bmatrix} Z_{ab} & Z_{ab,o} \\ Z_{ba,o} & Z_{ab,oo} \end{bmatrix} \begin{bmatrix} F_b^* & F_{b,21}^* \\ 0 & F_{b,o}^* \end{bmatrix} + \begin{bmatrix} K_a \\ X_a \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \end{bmatrix}^*. \tag{7.4.11}$$

Equating the $(1,2)$ block entries in (7.4.8), we have

$$\begin{aligned} V_a &= FPH^* + GS - T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D = FZH^* + GS - FZ_1H^* - T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \\ &= T_a \begin{bmatrix} F_aZ_{ab}H_b^* + K_aD \\ N_{a,o} \\ 0 \\ 0 \end{bmatrix} - \\ &\quad T_a \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} \begin{bmatrix} Z_{ab} & Z_{ab,o} & 0 & 0 \\ Z_{ba,o} & Z_{ab,oo} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} H_b^* \\ 0 \\ H_{b,c}^* \\ 0 \end{bmatrix} - T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \end{aligned}$$

where in the second step we used the similarity transformation (7.4.6) to evaluate $FZH^* + GS$. Simplifying the above expression yields

$$V_a = T_a \begin{bmatrix} 0 \\ N_a - F_{a,21}Z_{ab}H_b^* - F_{a,o}Z_{ba,o}H_b^* - X_aD \\ 0 \\ 0 \end{bmatrix}.$$

Note that $V_a$ belongs to the unobservable space of $\{F, H\}$. Using (7.4.11), $Z_{ba,o}$ satisfies the Lyapunov equation $Z_{ba,o} = F_{a,o}Z_{ba,o}F_b^* + F_{a,21}Z_{ab}F_b^* + X_aDK_b^*$. Moreover, since $Z_{ab}$ satisfies (7.4.2) it does not depend on the free parameter $X_a$. Therefore, to demonstrate that $X_a$ can be chosen such that $V_a = 0$, the following system of linear equations must be solvable in $Z_{ba,o}$ and $X_aD$

$$\begin{cases} N_a &= F_{a,o}Z_{ba,o}H_b^* + X_aD + F_{a,12}Z_{ab}H_b^* \\ Z_{ba,o} &= F_{a,o}Z_{ba,o}F_b^* + X_aDK_b^* + F_{a,12}Z_{ab}F_b^* \end{cases}.$$

But post-multiplying the first of the above equations by $K_b^*$ and subtracting yields

$$Z_{ba,o} = F_{a,o}Z_{ba,o}(F_b - K_bH_b)^* + F_{a,12}Z_{ab}(F_b - K_bH_b)^* + N_aK_b^*.$$

This equation will have a unique solution in $Z_{ba,o}$, if $F_{a,o}$ and $F_b - K_bH_b$ have no two eigenvalues, $\lambda_{a,o}$ and $\lambda_b$, such that $\lambda_{a,o} = \lambda_b^{-*}$. But since $\lambda_{a,o}$ is an unobservable (stable) eigenvalue of $F$ and $\lambda_b$ is a zero of $B(z)$ we can assume this.[10] Thus, the unique solution for $X_aD$ is $X_aD = N_a - F_{a,o}Z_{ba,o}H_b^* - F_{a,12}Z_{ab}H_b^*$. With this choice of $X_a$ (and a similar choice for $X_b$), we have $V_a = 0$ (and similarly $V_b = 0$), and if we define

$$K_p = T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix}, \quad K_q = T_b \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad R_e = D,$$

Eq. (7.4.8) becomes

$$\begin{bmatrix} -P + FPF^* + GQG^* & FPH^* + GS \\ HPF^* + S^*G^* & R + HPH^* \end{bmatrix} = \begin{bmatrix} K_pR_eK_q^* & K_pR_e \\ R_eK_q^* & R_e \end{bmatrix},$$

[10] As mentioned earlier, since we have assumed the detectability of $\{F, H\}$, we may take $F$ as stable, in which case due to the similarity transformation (7.4.5) $F_{a,o}$ must be stable as well.

which means the SDARE (7.4.1) has a solution. This concludes the proof of part (a).

The proofs of the remaining parts of the Theorem are much shorter. To prove part (b) if we assume that $P$ is semistabilizing then, since the zeros of $M(z)$ and $N(z)$ are the eigenvalues of $F - K_pH$ and $F - K_qH$, it is straightforward to see that $M(z)$ and $N(z)$ are semistable. Therefore the main effort is to show the other direction. Thus, let us assume that a proper semi-canonical (semistabilizing) factorization of $\Sigma(z)$ exists. Now note that

$$T_a^{-1}(F - K_pH)T_a = \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} - \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} H_a & 0 & H_{a,c} & 0 \end{bmatrix}$$
$$\begin{bmatrix} F_a - K_aH_a & 0 & F_{a,13} - K_aH_{a,c} & 0 \\ F_{a,21} - X_aH_a & F_{a,o} & F_{a,23} - X_aH_{a,c} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix}$$

Thus the eigenvalues of $F - K_pH$ are the union of the eigenvalues of the matrices

$$\begin{bmatrix} F_a - K_aH_a & 0 \\ F_{a,21} - X_aH_a & F_{a,o} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} F_{a,c} & 0 \\ 0 & F_{a,co} \end{bmatrix},$$

or, similarly, the matrices

$$F_a - K_aH_a \quad , \quad F_{a,o} \quad , \quad F_{a,c} \quad \text{and} \quad F_{a,co}.$$

Since the first of the above matrices is semi-stable (by the assumption that a semi-stable factorization exists) and the remaining three are stable, we conclude that $F - K_pH$ is semi-stable. A similar reasoning shows that $F - K_qH$ is semi-stable, thus ending our proof of part (b).

We now continue with the proof of part (c). First, suppose that the SDARE has a Hermitian solution, $P = P^*$. Then we can write the Popov function as

$$\Sigma(z) = \left[H(zI - F)^{-1}K_p + I\right] R_e \left[H(z^{-*}I - F)^{-1}K_p + I\right]^*, \tag{7.4.12}$$

where $R_e = R + HPH^*$ is Hermitian. This readily shows the desired proper Hermitian factorization. [Note, moreover, that for all $|z| = 1$ that are neither poles nor zeros of $H(zI - F)^{-1}K_p + I$, the above relation shows that $\Sigma(z)$ is congruent to $R_e$. Therefore $\Sigma(z)$ will have constant inertia (equal to the inertia of $R_e$) a.e. on the unit circle, $|z| = 1$.]

In the other direction, suppose that we can write,

$$\Sigma(z) = M(z)JM^*(z^{-*}), \tag{7.4.13}$$

where $M(z)$ is a $p \times q$ proper rational transfer matrix with poles of magnitude less than one and where $J$ is a $q \times q$ signature matrix. Repeating the arguments of the proof of part (a) to the above factorization shows that the SDARE will have a Hermitian solution. This ends the proof of part (c).

The proof of part (d) readily follows from parts (b) and (c).

Finally, to prove part (d), we note that (7.4.12) (in fact, its non-Hermitian counterpart) shows that $\Sigma(z)$ is nonsingular a.e. on the unit circle, $|z| = 1$, if, and only if, the matrix $R_e$ is nonsingular. If this is the case, then $R_e$ can be inverted in the SDARE, and we can rewrite the SDARE as the DARE

$$P = FPF^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^* + GQG^*.$$

∎

We have thus established a result on the existence, and properties of, solutions to the discrete-time algebraic Riccati equation (DARE) and the closely associated system of discrete-time algebraic Riccati equations (SDARE). The existence of these solutions, and their various properties, turns out to depend on the existence of certain proper factorizations, and their various properties, of the Popov function from which these equations are derived. Since we normally are interested in solving the DARE, or SDARE, in order to check whether the Popov function $\Sigma(z)$ has certain properties and admits certain factorizations (such as being positive, or having the canonical factorizations required of $H^\infty$ estimation and control), it will be useful to have a separate method for solving these equations. One such method (based on the so-called Hamiltonian matrix) will be briefly studied in Sec. 7.6. However, before doing so, it will be useful to study the consequence of our results for some special cases of interest.

## 7.5 Special Cases

In this section we shall briefly review some special cases of the general DARE and SDARE that have been studied so far. These special cases are motivated by applications in linear quadratic control, linear least-squares (Wiener) filtering, and $H^\infty$ control and filtering (see Secs. 1.3 and 1.6 and Secs. 1.4 and 1.7).

### 7.5.1 The Case of $R > 0$ and $Q - SR^{-1}S^* > 0$

We first remark that the case we are considering corresponds to the case where

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0.$$

We should also note that there is no loss of generality in considering $Q - SR^{-1}S^* > 0$ instead of $Q - SR^{-1}S^* \geq 0$, since we can always go from the latter condition to the former one by a simple redefinition of the matrix $G$.

The so-called "positive" case for the DARE, that we are considering here, arises in linear quadratic control and linear least-squares filtering (see *e.g.*, Theorems 1.3.30 and 1.6.3) and is very well studied. Therefore, the results given below are well known (see *e.g.*, [AM79, Kuc74] and the references therein). However, for completeness, we will provide brief proofs using the approach presented so far.

**Theorem 7.5.1 (The Positive DARE)** *Consider the DARE*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*, \qquad (7.5.1)$$

*where* $\{F, H\}$ *is detectable and* $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0$. *Then we have the following results.*

*(a) The DARE always has a Hermitian semi-stabilizing solution such that the matrix* $F - K_pH$, *with* $K_p = (FPH^* + GS)(R + HPH^*)^{-1}$ *is semi-stable,* i.e., *has all its eigenvalues inside the closed unit disk.*

(b) *If, in addition,* $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ *is stabilizable, then the DARE always has a unique Hermitian and positive semi-definite stabilizing solution such that* $F - K_pH$ *is stable,* i.e., *has all its eigenvalues inside the open unit disk.*

(c) *If, in addition to the assumptions of part (b),* $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ *is controllable, then the unique stabilizing solution to the DARE is positive definite.*

**Proof:** To prove part (a), let us note that using the factorization

$$\begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} = \begin{bmatrix} I & GSR^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} GQG^* - GSR^{-1}S^*G^* & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I & 0 \\ R^{-1}S^*G^* & I \end{bmatrix},$$

we may write the Popov function, $\Sigma(z)$, as

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} H(zI-F)^{-1}G & H(zI-F)^{-1}GSR^{-1} + I \end{bmatrix} \begin{bmatrix} Q - SR^{-1}S^* & 0 \\ 0 & R \end{bmatrix} \\ &\qquad \begin{bmatrix} G^*(z^{-1}I - F^*)^{-1}H^* \\ R^{-1}S^*G^*(z^{-1}I - F^*)^{-1}H^* + I \end{bmatrix} \\ &= H(zI-F)^{-1}G\left[Q - SR^{-1}S^*\right]G^*(z^{-1}I - F^*)^{-1}H^* + \\ &\quad \left[H(zI-F)^{-1}GSR^{-1} + I\right]R\left[R^{-1}S^*G^*(z^{-1}I - F^*)^{-1}H^* + I\right]. \end{aligned}$$

The first term in the RHS of the above equation is a.e. positive semi-definite on the unit circle (since $Q - SR^{-1}S^* > 0$), whereas the second term is a.e. positive definite on the unit circle (since $R > 0$, and $H(zI-F)^{-1}GSR^{-1} + I$ is nonsingular for all $|z| = 1$ except, possibly, for its unit magnitude zeros). Therefore, $\Sigma(z)$ is positive definite a.e. on the unit circle, $|z| = 1$, so that, using Theorem 7.3.1, it admits a proper semi-canonical Hermitian factorization. Thus, from Theorem 7.2.1 part (d), the SDARE has a Hermitian semi-stabilizing solution, $P$. Moreover, using part (d) of the same Theorem, since $\Sigma(z)$ is nonsingular a.e. on the unit circle, then $R_e = R + HPH^*$ is nonsingular, and $P$ is also a solution to the DARE (7.5.1). This concludes our proof of part (a).

The proofs for parts (b) and (c) are standard. We present them below.

For part (b) suppose that $P$ is the Hermitian semi-stabilizing solution to the DARE, and that $x$ is any left eigenvector of $F - K_pH$ with eigenvalue $|\lambda| = 1$, *i.e.*,

$$x(F - K_pH) = \lambda x.$$

Moreover, note that the DARE can be rewritten as

$$P = (F - K_pH)P(F - K_pH)^* + \begin{bmatrix} G & -K_p \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K_p^* \end{bmatrix}.$$

Pre and post-multiplying the above expression by $x$ and $x^*$, respectively, we obtain

$$xPx^* = |\lambda|^2 xPx^* + x \begin{bmatrix} G & -K_p \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K_p^* \end{bmatrix} x^*,$$

from which we infer

$$x \begin{bmatrix} G & -K_p \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K_p^* \end{bmatrix} x^* = 0.$$

But this implies

$$x[G(Q - SR^{-1}S^*)G^* + (GSR^{-1} - K_p)R(GSR^{-1} - K_p)^*]x^* = 0,$$

from which we infer

$$xG(Q - SR^{-1}S^*)G^*x^* = 0 \quad \text{and} \quad xK_p = xGSR^{-1}. \tag{7.5.2}$$

We now see that

$$\lambda x = x(F - K_pH) = xF - xK_pH = xF - xGSR^{-1}H = x(F - GSR^{-1}H),$$

*i.e.* , $x$ is also a left eigenvector of $F - GSR^{-1}H$ with eigenvalue $\lambda$. The first equality in (7.5.2) along with the above result can be rewritten as

$$xG(Q - SR^{-1}S^*) = 0 \quad \text{and} \quad x(F - GSR^{-1}H) = \lambda x.$$

This of course contradicts the stabilizability condition. We thus have

$$|\lambda| < 1,$$

meaning that $F - K_pH$ is stable. To show the uniqueness of the stabilizing solution, suppose that we have two solutions $P_1$ and $P_2$, with corresponding gain vectors $K_{p,1}$ and $K_{p,2}$. Now some algebra shows that

$$P_2 - P_1 = (F - K_{p,1}H)(P_2 - P_1)(F - K_{p,2}H)^*.$$

Applying the above equality $i$ times, we have

$$P_2 - P_1 = (F - K_{p,1}H)^i(P_2 - P_1)(F - K_{p,2}H)^{i*}.$$

Now since the matrices $F - K_{p,1}H$ and $F - K_{p,2}H$ are both stable, as $i \to \infty$, the above equation becomes

$$P_2 - P_1 = 0$$

showing that the stabilizing solution is unique.

To prove part (c) we begin by writing the DARE as

$$P = (F-K_pH)P(F-K_pH)^*+G(Q-SR^{-1}S^*)G^*+(-K_p+GSR^{-1})R(-K_p+GSR^{-1})^*.$$

Since $F - K_pH$ is stable, using Lyapunov theory, $P$ will be positive definite if the pair

$$\left\{F - K_pH, \begin{bmatrix} GQ - GSR^{-1}S^* & -K_p + GSR^{-1} \end{bmatrix}\right\}$$

is controllable. But since

$$\begin{bmatrix} F - GSR^{-1}H & GQ - GSR^{-1}S^* & -K_p + GSR^{-1} \end{bmatrix} = \begin{bmatrix} F - K_pH & GQ - GSR^{-1}S^* - K_p + GSR^{-1} \end{bmatrix} X$$

where

$$X = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -H & 0 & I \end{bmatrix}$$

is a nonsingular matrix, the controllability of $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ will also yield a positive definite $P$. This establishes part (c).

■

### 7.5.2 The Case of Nonsingular $F - GSR^{-1}H$

In the previous section we saw that when $R > 0$ and $Q - SR^{-1}S^* > 0$ then each solution to the SDARE was a solution of the DARE as well. Finding conditions under which $R_e$ is nonsingular, so that solutions of the SDARE are solutions of the DARE as well, is, in the general case, quite difficult (see Ex. 4). However, it is possible to give a simple sufficient condition for this to be so.

**Theorem 7.5.2 (Nonsingular $F - GSR^{-1}H$)** *Suppose that $\{F, H\}$ is detectable and that $R$ and $F - GSR^{-1}H$ are nonsingular. Then any solution to the SDARE (7.4.1) is also a solution to the DARE*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

**Proof:** Let $P$ be the solution to the SDARE

$$\left\{ \begin{array}{rcl} P & = & FPF^* + GQG^* - K_pR_eK_q^* \\ K_pR_e & = & FPH^* + GS \quad , \quad K_qR_e^* = FP^*H^* + GS \\ R_e & = & R + HPH^* \end{array} \right. .$$

Suppose now that $R_e$ is singular. Then there exists an $x$ such that

$$R_ex = 0.$$

Therefore if we post-multiply the second set of equations in the SDARE by $x$ we obtain

$$(FPH^* + GS)x = 0.$$

But

$$FPH^*+GS = (F-GSR^{-1}H)PH^*+GSR^{-1}HPH^*+GS = (F-GSR^{-1}H)PH^*+GSR^{-1}R_e,$$

so that

$$(F - GSR^{-1}H)PH^*x = 0.$$

But since $F - GSR^{-1}H$ is nonsingular, this implies $PH^*x$. We can also conclude that $HPH^*x = 0$, so that

$$HPH^*x = (R_e - R)x = -Rx = 0,$$

which is a contradiction since $R$ is assumed invertible. Therefore $R_e$ must be nonsingular, and $P$ is a solution to the DARE.

■

### 7.5.3 The Case of Positive $Q - SR^{-1}S^*$

Here we assume that $S = 0$, so that the DARE becomes

$$P = FPF^* + GQG^* - FPH^*(R + HPH^*)^{-1}HPF^*. \tag{7.5.3}$$

There is, of course, no loss of generality in making this assumption since the general DARE

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS),$$

can always be transformed to one with $S = 0$, by rewriting it as

$$\begin{aligned} P &= (F - GSR^{-1}H)P(F - GSR^{-1}H)^* + G(Q - SR^{-1}S^*)G^* - \\ &\quad (F - GSR^{-1}H)PH^*(R + HPH^*)^{-1}HP(F - GSR^{-1}H)^*, \end{aligned}$$

and redefining the matrices as

$$F \to F - GSR^{-1}H \quad \text{and} \quad Q \to Q - SR^{-1}S^*.$$

We therefore will be studying (7.5.3) in the case where $Q > 0$. Moreover, we can always assume that the (possibly) indefinite matrix $R$ has the form

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix}. \tag{7.5.4}$$

The reason being that the DARE can be rewritten as

$$P = F(I + PH^*R^{-1}H)^{-1}PF^* + GQG^*,$$

so that what really enters the DARE is the matrix $H^*R^{-1}H$ and a simple redefinition of $H$ will allow $R$ to have the form described in (7.5.4). We shall also partition $H$ according to the partitioning of $R$ and write

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

We should also remark that the DARE with $Q > 0$ and $R$ indefinite arises in $H^\infty$ filtering and control where it plays a prominent role [GD88, DGKF89, KN91, GL95]. [See also Theorems 1.4.2 and 1.7.3.] The following result on the DARE is known in the $H^\infty$ literature, but is rarely stated in the form given below.

**Theorem 7.5.3 (Positive $Q$)** *Consider the DARE*

$$P = FPF^* + GQG^* - FPH^*(R + HPH^*)^{-1}HPF^*,$$

*where*

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix} \quad , \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

$\{F, H_1\}$ *is detectable,* $\{F, GQ^{1/2}\}$ *is stabilizable and* $Q > 0$. *Suppose, moreover, that the DARE has a Hermitian stabilizing solution,* $P$. *Then we have the following results.*

*(a)* $P \geq 0$.

*(b) If, in addition,* $R$ *and* $R_e = R + HPH^*$ *have the same inertia, then*

*(i)* $F - K_1 H$ *is stable, where*

$$K_1 = FPH_1^*(I_{p_1} + H_1PH_1^*)^{-1}.$$

*(ii) The transfer matrix*

$$M(z) = \begin{bmatrix} M_1(z) & M_2(z) \end{bmatrix},$$

*where*

$$M_1(z) = H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}GQ^{1/2}$$

*and*

$$M_2(z) = H_2PH_1^*(I_{p_1}+H_1PH_1^*)^{-1}+H_2(I+PH_1^*H_1)^{-1}(zI-F+K_1H_1)^{-1}K_1$$

*is strictly contractive on the unit circle,* i.e.,

$$M(z)M^*(z^{-*}) < I_{p_2}, \qquad \forall |z| = 1.$$

**Remarks:**

(a) Theorem 7.5.3 can be used to check whether the Popov function

$$\Sigma(z) = \left[ \begin{array}{cc} H(zI-F)^{-1} & I \end{array} \right] \left[ \begin{array}{cc} GQG^* & GS \\ S^*G^* & R \end{array} \right] \left[ \begin{array}{c} (z^{-1}I-F^*)^{-1}H^* \\ I \end{array} \right],$$

admits the factorization required for the existence of suboptimal $H^\infty$ filters. From Theorem 1.4.2, the condition for the existence of an $H^\infty$ filter of level $\gamma = 1$ is that the Popov function admit the following canonical factorization,

$$\Sigma(z) = \left[ \begin{array}{cc} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{array} \right] \left[ \begin{array}{cc} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{array} \right] \left[ \begin{array}{cc} L_{11}^*(z^{-*}) & L_{12}^*(z^{-*}) \\ L_{21}^*(z^{-*}) & L_{22}^*(z^{-*}) \end{array} \right], \qquad (7.5.5)$$

with $\left[ \begin{array}{cc} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{array} \right]$ and $L_{11}(z)$ minimum phase and proper, and $L_{12}(z)$ stable and strictly proper. Now the existence of a canonical Hermitian factorization is equivalent to the existence of a Hermitian stabilizing solution to the DARE. That the factorization has the appropriate inertia is equivalent to $R$ and $R_e$ having the same inertia, and the strictly proper property of $L_{12}(z)$ and inverse stability of $L_{11}(z)$ can be checked by performing the block LDU decomposition of $R_e$ and checking whether $F - K_1H$ is stable, where $K_1 = FPH_1^*(I_{p_1}+H_1PH_1^*)^{-1}$. Thus we have the following result.

"An $H^\infty$ filter of level $\gamma = 1$ exists if, and only if, there exists a Hermitian stabilizing solution to the DARE such that the matrices $R = I_{p_1} \oplus (-I_{p_2})$ and $R_e = R + HPH^*$ have the same inertia." But this is precisely the what is stated in the second part of Theorem 1.4.2.

(b) The transfer matrix $M(z)$ of Theorem 7.5.3 can be seen to be equal to $T_{K_{cen}}(z)$, the transfer operator that maps the disturbances to estimation errors in the $H^\infty$ filtering problem of level $\gamma = 1$, when the central filter is used. Statement (b), part (ii), of Theorem 7.5.3 states the (obvious) fact that when a solution to the $H^\infty$ filtering problem exists $T_{K_{cen}}(z)$ is strictly contractive.

**Proof of Theorem 7.5.3:** Let us first prove part (a) by establishing $P \geq 0$. To this end, consider, $P_\gamma$, the stabilizing solution to the DARE[11]

$$P_\gamma = FP_\gamma F^* + GQG^* - FP_\gamma H^*(R_\gamma + HP_\gamma H^*)^{-1}HP_\gamma F^*, \tag{7.5.6}$$

where

$$R_\gamma = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -\gamma I_{p_2} \end{bmatrix}, \quad \gamma \geq 1.$$

This DARE may be written as

$$P_\gamma = F(I + P_\gamma H_1^* H_1 - \gamma^{-1} P_\gamma H_2^* H_2)^{-1} P_\gamma F^* + GQG^*. \tag{7.5.7}$$

We can use the above expression to differentiate $P_\gamma$ with respect to $\gamma$ and obtain $\dot{P}_\gamma = dP_\gamma / d\gamma$. Using the identity,

$$\frac{dA^{-1}}{d\gamma} = -A^{-1}\frac{dA}{d\gamma}A^{-1},$$

and after some algebra, we may write

$$\dot{P}_\gamma = F(I+P_\gamma H_1^* H_1 - \gamma^{-1} P_\gamma H_2^* H_2)^{-1}(P_\gamma - \gamma^{-2} P_\gamma H_2^* H_2 P_\gamma)(I+P_\gamma H_1^* H_1 - \gamma^{-1} P_\gamma H_2^* H_2)^{-*}F^*. \tag{7.5.8}$$

But note that since $P_\gamma$ is stabilizing, the matrix

$$F(I + P_\gamma H_1^* H_1 - \gamma^{-1} P_\gamma H_2^* H_2)^{-1} = F(I + P_\gamma H^* R_\gamma^{-1} H^*)^{-1} = F - K_{p,\gamma} H,$$

is stable. Thus (7.5.8) becomes

$$\dot{P}_\gamma = (F - K_{p,\gamma} H) P_\gamma (F - K_{p,\gamma} H)^* - \gamma^{-2}(F - K_{p,\gamma} H) P_\gamma H_2^* H_2 P_\gamma (F - K_{p,\gamma} H)^*.$$

[11] The given DARE is simply the DARE corresponding to the $H^\infty$ filtering problem with level $\gamma \geq 1$. Since we have assumed that the level $\gamma = 1$ $H^\infty$ filtering problem has a solution, the same will be true for all $\gamma \geq 1$. Therefore, for $\gamma \geq 1$ the given DARE will have a Hermitian stabilizing solution, as stated.

Since the second term in the above equation is negative semi-definite and since $F - K_p H$ is stable, using Lyapunov theory, we have

$$\dot{P}_\gamma \leq 0.$$

Therefore, for all $\gamma$ such that $P_\gamma$ is stabilizing, the $P_\gamma$ form a nonincreasing function of Hermitian matrices. In other words, if $P_{\gamma_1}$ and $P_{\gamma_2}$ are two stabilizing solutions to the DARE (7.5.6), then

$$\gamma_1 \leq \gamma_2 \quad \text{implies} \quad P_{\gamma_1} \geq P_{\gamma_2}.$$

Now if we let $\gamma \to \infty$, then (7.5.7) becomes

$$P_\infty = F(I+P_\infty H_1^* H_1)^{-1} P_\infty F^* + GQG^* = FP_\infty F^* + GQG^* - FP_\infty H_1^*(I_{p_1} + H_1 P_\infty H_1^*)^{-1} H_1 P_\infty F^*.$$

But since $Q > 0$, this is an example of the positive DARE studied in Sec. 7.5.1. Therefore, since $\{F, H_1\}$ is detectable and $\{F, GQ^{1/2}\}$ is stabilizable, using Theorem 7.5.1 part (b), a stabilizing solution to this DARE exists and is such that $P_\infty \geq 0$. Now from the hypothesis of the Theorem we know that the DARE (7.5.6) has a stabilizing solution for $\gamma = 1$. As $\gamma = 1 < \infty$, using the (above established) monotonicity property of the stabilizing solutions to (7.5.6), we have

$$P = P_1 \geq P_\infty \geq 0,$$

which proves part (a).

To proceed with the proof of part (b), we now assume that the matrices

$$R_e = R + HPH^* = \begin{bmatrix} I_{p_1} + H_1 P H_1^* & H_1 P H_2^* \\ H_2 P H_1^* & -I_{p_2} + H_2 P H_2^* \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix},$$

have the same inertia. Since $P \geq 0$, we have $I_{p_1} + H_1 P H_1^* > 0$, and therefore the above inertia condition implies that the Schur complement of the $(1,1)$ block entry of $R_e$ must be negative definite, *i.e.*,

$$-I_{p_2} + H_2 P H_2^* - H_2 P H_1^*(I_{p_1} + H_1 P H_1^*)^{-1} H_1 P H_2^* = -I_{p_2} + H_2(I + PH_1^* H_1)^{-1} P H_2^* < 0.$$

Now we can write the DARE as

$$\begin{aligned}
P &= F(I+PH_1^*H_1-PH_2^*H_2)^{-1}PF^*+GQG^* \\
&= F(I+PH_1^*H_1)^{-1}PF^*+GQG^* \\
&\quad -F(I+PH_1^*H_1)^{-1}PH_2^*\left[H_2(I+PH_1^*H_1)^{-1}PH_2^*-I_{p_2}\right]^{-1}H_2(I+PH_1^*H_1)^{-1}PF^* \\
&= F(I+PH_1^*H_1)^{-1}PF^*+GQG^* \\
&\quad +F(I+PH_1^*H_1)^{-1}PH_2^*\underbrace{\left[I_{p_2}-H_2(I+PH_1^*H_1)^{-1}PH_2^*\right]^{-1}}_{=M>0}H_2P(I+H_1^*H_1P)^{-1}F^* \\
&= (F-K_1H)P(F-K_1H)^*+GQG^*+K_1K_1^*+\underbrace{F(I+PH_1^*H_1)^{-1}PH_2^*MH_2P(I+H_1^*H_1P)^{-1}F^*}_{=\bar{Q}\geq 0},
\end{aligned}$$

where in the second step we have expanded $(I+PH_1^*H_1-PH_2^*H_2)^{-1}$ using the matrix inversion lemma, and where in the fourth step we have used the expression $K_1 = FPH_1^*(I_{p_1}+H_1PH_1^*)^{-1}$. Now since $P\geq 0$, $\{F, G^{1/2}\}$ is stabilizable, and $GQG^*+K_1K_1^*+\bar{Q}\geq$, we can use an argument similar to the argument of Theorem 7.5.1 part (b) to show that $F-K_1H$ is stable. This proves part (b-i) of this Theorem.

To prove part (b-ii) we proceed as follows. First, note that using the stabilizing solution to the DARE we can factorize the Popov function as

$$\Sigma(z) = \begin{bmatrix} \mathcal{L}_{11}(z) & \mathcal{L}_{12}(z) \\ \mathcal{L}_{21}(z) & \mathcal{L}_{22}(z) \end{bmatrix} \begin{bmatrix} R_{e,1}I_{p_1} & 0 \\ 0 & -R_{e,2}I_{p_2} \end{bmatrix} \begin{bmatrix} \mathcal{L}_{11}(z^{-*}) & \mathcal{L}_{21}(z^{-*}) \\ \mathcal{L}_{21}(z^{-*}) & \mathcal{L}_{22}(z^{-*}) \end{bmatrix}^*, \quad (7.5.9)$$

where both $R_{e,1}$ and $R_{e,2}$ are positive definite. (Such a factorization is called a $J$-spectral factorization.) Indeed, this factorization is achieved via

$$\Sigma(z) = \left\{ \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} (zI-F)^{-1}K_p + \begin{bmatrix} I_{p_1} & 0 \\ 0 & I_{p_2} \end{bmatrix} \right\} \begin{bmatrix} I_{p_1}+H_1PH_1^* & H_1PH_2^* \\ H_2PH_1^* & -I_{p_2}+H_2PH_2^* \end{bmatrix} \{\}^*,$$

and the block triangular factorization of the central matrix

$$\begin{bmatrix} I_{p_1}+H_1PH_1^* & H_1PH_2^* \\ H_2PH_1^* & -I_{p_2}+H_2PH_2^* \end{bmatrix} =$$
$$\begin{bmatrix} I_{p_1} & 0 \\ H_2PH_1^*(I_{p_1}+H_1PH_1^*)^{-1} & I_{p_2} \end{bmatrix} \begin{bmatrix} I_{p_1}+H_1PH_1^* & 0 \\ 0 & -R_{e,2} \end{bmatrix} \begin{bmatrix} I_{p_1} & (I_{p_1}+H_1PH_1^*)^{-1}H_1PH_2^* \\ 0 & I_{p_2} \end{bmatrix},$$

where $R_{e,2} = I_{p_2}-H_2(I+PH_1^*H_1)^{-1}PH_2^* > 0$. Thus it is readily seen that

$$\begin{bmatrix} \mathcal{L}_{11}(z) & \mathcal{L}_{12}(z) \\ \mathcal{L}_{21}(z) & \mathcal{L}_{22}(z) \end{bmatrix} = \left\{ \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} (zI-F)^{-1}K_p + \begin{bmatrix} I_{p_1} & 0 \\ 0 & I_{p_2} \end{bmatrix} \right\} \begin{bmatrix} I_{p_1} & 0 \\ H_2PH_1^*(I_{p_1}+H_1PH_1^*)^{-1} & I_{p_2} \end{bmatrix}.$$

Some algebra now shows that

$$\mathcal{L}_{11}(z) = H_1(zI - F)^{-1}K_1 + I_{p_1}, \tag{7.5.10}$$

and

$$\mathcal{L}_{21}(z) = H_2(zI - F)^{-1}K_1 + H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1}. \tag{7.5.11}$$

We may also remark that since $F - K_1H$ is stable, the transfer matrix $\mathcal{L}_{11}(z)$ has a stable inverse. Now using the factorization (7.5.9) it is straightforward to see

$$\begin{gathered}\begin{bmatrix} -\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z) & I_{p_2} \end{bmatrix} \Sigma(z) \begin{bmatrix} \mathcal{L}_{11}^{-*}(z^{-*})\mathcal{L}_{21}^*(z^{-*}) \\ I_{p_2} \end{bmatrix} = \\ -\left(\mathcal{L}_{22}(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{L}_{12}(z)\right) R_{e,2} \left(\mathcal{L}_{22}(z^{-*}) - \mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\mathcal{L}_{12}(z^{-*})\right)^* < 0,\end{gathered} \tag{7.5.12}$$

for all $|z| = 1$. Note that the inequality is strict since $\mathcal{L}_{22} - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{L}_{12}(z)$ has no unit circles zeros (in turn because $\begin{bmatrix} \mathcal{L}_{11}(z) & \mathcal{L}_{12}(z) \\ \mathcal{L}_{21}(z) & \mathcal{L}_{22}(z) \end{bmatrix}$ has no unit circle zeros – since $P$ is stabilizing – and $\mathcal{L}_{11}(z)$ has a stable inverse). On the other hand, if we define

$$\mathcal{H}_1(z) = H_1(zI - F)^{-1}GQ^{1/2} \quad \text{and} \quad \mathcal{H}_2(z) = H_2(zI - F)^{-1}GQ^{1/2}, \tag{7.5.13}$$

we may write

$$\Sigma(z) = \begin{bmatrix} I_{p_1} + \mathcal{H}_1(z)\mathcal{H}_1^*(z^{-*}) & \mathcal{H}_1(z)\mathcal{H}_2^*(z^{-*}) \\ \mathcal{H}_2(z)\mathcal{H}_1^*(z^{-*}) & -I_{p_2} + \mathcal{H}_2(z)\mathcal{H}_2^*(z^{-*}) \end{bmatrix}.$$

With the above expression for the Popov function we readily see that

$$\begin{gathered}\begin{bmatrix} -\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z) & I_{p_2} \end{bmatrix} \Sigma(z) \begin{bmatrix} \mathcal{L}_{11}^{-*}(z^{-*})\mathcal{L}_{21}^*(z^{-*}) \\ I_{p_2} \end{bmatrix} = \\ \left(\mathcal{H}_2(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{H}_1(z)\right) \left(\mathcal{H}_2(z^{-*}) - \mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\mathcal{H}_1(z^{-*})\right)^* \\ + \left(\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\right) \left(\mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\right)^* - I_{p_1}.\end{gathered}$$

Coupling the above equation with the strict inequality of (7.5.12) we obtain

$$\begin{gathered}\left(\mathcal{H}_2(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{H}_1(z)\right) \left(\mathcal{H}_2(z^{-*}) - \mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\mathcal{H}_1(z^{-*})\right)^* \\ + \left(\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\right) \left(\mathcal{L}_{21}(z^{-*})\mathcal{L}_{11}^{-1}(z^{-*})\right)^* < I_{p_1},\end{gathered}$$

for all $|z| = 1$. Therefore the transfer matrix

$$\left[ \begin{array}{cc} \mathcal{H}_2(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{H}_1(z) & \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z) \end{array} \right],$$

is strictly contractive on the unit circle. Using the expressions for $\mathcal{L}_{11}(z)$ and $\mathcal{L}_{21}(z)$ (see (7.5.10-7.5.11)) and those for $\mathcal{H}_1(z)$ and $\mathcal{H}_2(z)$ (see (7.5.13)), it is not too difficult to show that

$$\mathcal{H}_2(z) - \mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z)\mathcal{H}_1(z) = H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}GQ^{1/2},$$

and

$$\mathcal{L}_{21}(z)\mathcal{L}_{11}^{-1}(z) = H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1} + H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}K_1,$$

thus proving part (b-ii).

∎

## 7.6 The Hamiltonian Matrix

In the previous sections we developed general conditions for the existence of solutions to the DARE, and the closely related SDARE, in terms of the existence of certain proper factorizations of the associated Popov function. We then characterized the properties of these solutions in terms of the properties of the aforementioned factorizations. However, in most applications, such as in $H^\infty$ problems, one is interested in establishing the properties of the Popov function itself, *e.g.*, whether a canonical factorization of a certain type exists or not. Therefore it would be very useful (and indeed essential) to have an independent means of checking whether a solution to the DARE exists, and if so, of actually computing this solution. If this is possible, then one can then infer the properties of the Popov function from the properties of the (computed) Riccati solution.

There now exists a vast (and growing) literature on how to obtain solutions to algebraic Riccati equations, albeit mostly in the positive case (see the survey paper [Lau91] and the references therein). Although it is beyond the scope of this chapter to study (or even mention) all of the various methods for computing solutions to the DARE, in this section we shall investigate one possible method for computing the semi-stabilizing solution to the DARE. We shall essentially show that the wellknown method of invariant subspaces of the Hamiltonian matrix [Pot66, Mac63, Lau79] extends to the case of DARE's with indefinite $Q$ and $R$.

We also remark that in what follows, without loss of generality, we shall assume that $S = 0$.

### 7.6.1 The Case of Nonsingular $F$

We begin by assuming that $F$ is nonsingular. This simplifies some of the arguments. More importantly, from Theorem 7.5.2, it implies that, whenever a solution to the SDARE exists, a solution to the DARE also exists. Moreover, it also implies that the matrix

$$F - K_p H = F - FPH^*(R + HPH^*)^{-1} = F(I + PH^*R^{-1}H)^{-1},$$

is always invertible. Studying the general case of a (possibly) singular $F$ is more complicated. We mention what may amount to a promising route for this problem in the next section.

We continue by attempting to write the DARE as a quadratic form in $P$. To this end, we first note that the DARE may be rewritten as

$$-P + F(I + PH^*R^{-1}H)^{-1}PF^* + GQG^* = 0,$$

or, upon premultiplying by $(I + PH^*R^{-1}H)F^{-1}$, as

$$(I + PH^*R^{-1}H)F^{-1}(-P + GQG^*) + PF^* = 0.$$

Gathering terms yields

$$\begin{bmatrix} I & -P \end{bmatrix} \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} P \\ I \end{bmatrix} = 0, \qquad (7.6.1)$$

which is the desired quadratic form. The center matrix appearing in the above expression is referred to as the *Hamiltonian* matrix and is denoted by

$$M = \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix}. \tag{7.6.2}$$

It is convenient to expand the block row and column vectors appearing in (7.6.1) into upper triangular matrices, to obtain

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}.$$

Due to (7.6.1), the $(1,2)$ block entry in the above resulting product is zero. Some simple algebra shows that the remaining block entries are

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P \\ 0 & I \end{bmatrix} = \begin{bmatrix} F_p^{-1} & 0 \\ -H^*R^{-1}HF^{-1} & F_q^* \end{bmatrix} \tag{7.6.3}$$

where we have defined

$$F_p = F - K_pH \quad \text{and} \quad F_q = F - K_qH. \tag{7.6.4}$$

The identity (7.6.3) has special significance since it leads to the following result on the eigenvalue structure of the Hamiltonian matrix $M$.

**Lemma 7.6.1 (Eigenvalues of the Hamiltonian Matrix)** *Consider the Hamiltonian matrix*

$$M = \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix}.$$

*Then $\lambda$ is an eigenvalue of $M$ if, and only if, $\lambda^{-*}$ is an eigenvalue of $M$. Moreover, $M$ can be decomposed as*

$$T^{-1}MT = \begin{bmatrix} \Lambda_1 & \times \\ 0 & \Lambda_2^{-*} \end{bmatrix}, \tag{7.6.5}$$

*where $\Lambda_1$ and $\Lambda_2$ have eigenvalues of magnitude less than or equal to one, and '$\times$' denotes irrelevant entries. Finally, if $\lambda$ is an eigenvalue of $\Lambda_1$, such that $|\lambda| < 1$, then $\lambda$ is an eigenvalue of $\Lambda_2$ as well.*

**Proof:** The results are a direct consequence of (7.6.3). Note that since

$$\begin{bmatrix} I & -P \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}^{-1},$$

the Hamiltonian $M$ is similar to the block diagonal matrix

$$\begin{bmatrix} F_p^{-1} & 0 \\ -H^*R^{-1}HF^{-1} & F_q^* \end{bmatrix}.$$

Therefore the set of eigenvalues of $M$ is the union of the set of eigenvalues of $F_p^{-1}$ and the set of eigenvalues of $F_q^*$. Now from the factorization of the Popov function,

$$\Sigma(z) = \left[H(zI-F)^{-1}K_p + I\right] R_e \left[H(z^{-*}I-F)^{-1}K_q + I\right]^*,$$

we see that the zeros of $M(z)$ are the eigenvalues of $F_p$ and $F_q^{-*}$. Therefore, we conclude that the eigenvalues of $M$ are simply the *inverse* of the zeros of the Popov function, $\Sigma(z)$. Since $\Sigma(z)$ is para-Hermitian, for every zero at $\sigma$ there exists a zero at $\sigma^{-*}$. The same is therefore true of the eigenvalues of $M$. This establishes the first claim of the Lemma.

The decomposition (7.6.5) follows immediately from (7.6.3). We can then readily identify the $\Lambda_1$ with the eigenvalues of $F_q^*$ and $\Lambda_2$ with the eigenvalues of $F_p^*$, which means that they both have eigenvalues of magnitude less than or equal to one (since $F_p$ and $F_q$ are semi-stable). Due to the above factorization of the para-Hermitian Popov function, $\Sigma(z)$, we see that $F_p$ and $F_q$ have the same eigenvalues of magnitude strictly less than one. The same is therefore true of $\Lambda_1$ and $\Lambda_2$, thus finishing the proof of this Lemma.

■

The above Lemma furnishes a method for computing solutions to the DARE (when they exist). To see why, let us suppose that $P$ is a solution of the DARE and write (7.6.3) as

$$M\begin{bmatrix} I & P \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & P \\ 0 & I \end{bmatrix}\begin{bmatrix} F_p^{-1} & 0 \\ -H^*R^{-1}HF^{-1} & F_q^* \end{bmatrix},$$

from which we may conclude

$$M \begin{bmatrix} P \\ I \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix} F_q^*. \tag{7.6.6}$$

Now from the proof of Lemma 7.6.1, we have identified the eigenvalue structure of $F_q^*$ with $\Lambda_1$. Therefore there must exist a similarity transformation relating these two matrices, *i.e.*,

$$F_q^* = W\Lambda_1 W^{-1},$$

so that we may write (7.6.6) as

$$M \begin{bmatrix} PW \\ W \end{bmatrix} = \begin{bmatrix} PW \\ W \end{bmatrix} \Lambda_1.$$

If we now define $T_{21} = W$ and $T_{11} = PW$, then we may write

$$M \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} \Lambda_1. \tag{7.6.7}$$

Note that in the above expression the matrix $T_{21}$ is invertible. This means that $P$, the solution to the DARE, may be found via

$$P = T_{11}T_{21}^{-1}. \tag{7.6.8}$$

This is an important result since it has allowed us to write the stabilizing solution of the DARE in terms of elements of the (generalized) eigenvalue decomposition of $M$. We formalize this result in the following Theorem.

**Theorem 7.6.1 (The Invariant Subspace Method)** *Consider the DARE*

$$P = FPF^* + GQG^* = FPH^*(R + HPH^*)^{-1}HPF^*,$$

*and suppose that $F$ is invertible and $\{F, H\}$ is detectable. Then we have the following results.*

(a) *The DARE has a solution if, and only if, there exists a basis,* $\begin{bmatrix} T_{11}^* & T_{21}^* \end{bmatrix}^*$, *for some $n$-dimensional invariant subspace of the Hamiltonian matrix, $M$, such that $T_{21}$ is invertible. In other words, if, and only if, there exist $T_{11}$ and $T_{21}$ given by*

$$\begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} = \begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} \Lambda_1, \quad (7.6.9)$$

*where $\Lambda_1$ is an $n \times n$ matrix that represents $n$ (of the $2n$) eigenvalues of $M$, and $T_{21}$ is invertible. In this case, the solution of the DARE is given by*

$$P = T_{11}T_{21}^{-1}. \qquad (7.6.10)$$

(b) *The DARE has a semistabilizing solution if, and only if, there exists a basis,* $\begin{bmatrix} T_{11}^* & T_{21}^* \end{bmatrix}^*$, *for some $n$-dimensional semistable invariant subspace of the Hamiltonian matrix, $M$, such that $T_{21}$ is invertible. In other words, if, and only if, there exist $T_{11}$ and $T_{21}$ given by (7.6.9), where now $\Lambda_1$ is an $n \times n$ matrix that has all its eigenvalues inside the closed unit disk, and $T_{21}$ is invertible. In this case, $P$ is given by (7.6.10).*

(c) *The DARE has a stabilizing solution if, and only if, the Hamiltonian matrix, $M$, has no unit circle eigenvalues, and,* $\begin{bmatrix} T_{11}^* & T_{21}^* \end{bmatrix}^*$, *the $n$-dimensional stable invariant subspace of $M$ is such that $T_{21}$ is invertible. In this case, $P$ is given by (7.6.10).*

**Proof:** The proof is quite straightforward. We have already shown that if a solution to the DARE exists then a basis for some invariant subspace of $M$ with $T_{21}$ invertible exists. On the other hand, it is easy to show that if a basis, with the aforementioned properties, for some invariant subspace of $M$ exists, then $T_{11}T_{21}^{-1}$ is a solution to the DARE.

Finally, the claims of parts (b) and (c) follow from the fact that the eigenvalues of $\Lambda_1$ (associated with the eigenvalues of the invariant subspace under consideration) coincide with the eigenvalues of $F_q^*$. Moreover, $M$ can only have a stable invariant subspace if it has no unit circle eigenvalues. (Otherwise we only have semistable invariant subspaces.)

■

The connection between an apparently nonlinear matrix Riccati equation of order $n$ and a linear eigenvalue problem of order $2n$ is classical and dates back at least to Von Escherich in 1898 [Esc98]. The above "eigenvector" solution was popularized in the control literature by MacFarlane in 1963 [Mac63] and Potter in 1966 [Pot66]. Schur methods for the solution of this eigenvalue problem have been proposed by Laub in 1978 [Lau79], and have resulted in reliable and efficient numerical solutions to algebraic Riccati equations in the positive case.

The above result essentially claims that solutions to the DARE can also be found using the invariant subspace method even in the indefinite case. However, the study of reliable numerical methods for obtaining this solution is beyond the scope of this chapter and is worthy of further scrutiny.

### 7.6.2 The Case of Singular $F$

We note that when $F$ is invertible, the Hamiltonian matrix $M$ may be written as

$$M = \begin{bmatrix} I & 0 \\ -H^*R^{-1}H & F^* \end{bmatrix} \begin{bmatrix} F & GQG^* \\ 0 & I \end{bmatrix}^{-1}.$$

This therefore suggests that in the case of singular $F$, instead of computing the eigenvalues and eigenvectors of $M$, we should compute the generalized eigenvalues and eigenvectors of the pair

$$\left\{ \begin{bmatrix} I & 0 \\ -H^*R^{-1}H & F^* \end{bmatrix}, \begin{bmatrix} F & GQG^* \\ 0 & I \end{bmatrix} \right\}.$$

In the positive case, this approach has been used for computing solutions to the DARE when $F$ is singular [TPS80]. We believe such an approach should extend to the indefinite case although we shall not pursue it here.

## 7.7 Some Examples

In this section we shall consider some simple examples that will demonstrate the results obtained so far. Our first example concerns how to use solutions of the DARE to compute bounds for the $H^\infty$ norm of a certain transfer function.

**Example 1 (Computation of the $H^\infty$ Norm)** Consider the transfer function

$$A(z) = \frac{1}{z - 1/2}, \tag{7.7.1}$$

and suppose we would like to find constants $\gamma_u$ and $\gamma_l$ such that

$$\gamma_l^2 \leq |A(e^{j\omega})|^2 \leq \gamma_u^2, \qquad \forall \omega \in [0, 2\pi]. \tag{7.7.2}$$

Note that the smallest possible $\gamma_u$ (*i.e.*, one that makes the above inequality tight) is the $H^\infty$ norm of $A(z)$. The expression (7.7.2) suggests that we introduce the Popov function

$$\Sigma(z) = \gamma^2 - A(z)A^*(z^{-*}). \tag{7.7.3}$$

The inertia of $\Sigma(z)$ on the unit circle, for different values of $\gamma$, will allow us to find $\gamma_u$ and $\gamma_l$ in (7.7.2). But let us first write $\Sigma(z)$ in the standard form

$$\Sigma(z) = \begin{bmatrix} (z - 1/2)^{-1} & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & \gamma^2 \end{bmatrix} \begin{bmatrix} (z^{-1} - 1/2)^{-1} \\ 1 \end{bmatrix}. \tag{7.7.4}$$

Comparing with the general Popov function (7.2.6), we see that we can identify

$$F = 1/2 \ , \quad G = 1 \ , \quad H = 1 \ , \quad Q = -1 \ , \quad S = 0 \ , \quad R = \gamma^2.$$

Thus the DARE associated with $\Sigma(z)$ is given by

$$p = p/4 - 1 - \frac{p^2/4}{\gamma^2 + p},$$

or, after some rearrangement of terms,

$$4p^2 + (3\gamma^2 + 4)p + 4\gamma^2 = 0. \tag{7.7.5}$$

Note that, being a quadratic equation in $p$, the above DARE always has a solution (see Theorem 7.4.1). Now solutions to the DARE are given by

$$p_{1,2} = \frac{1}{8} \cdot \left( -3\gamma^2 - 4 \pm \sqrt{9\gamma^4 - 40\gamma^2 + 16} \right).$$

The DARE will therefore have real (corresponding to Hermitian) solutions if, and only if,

$$9\gamma^4 - 40\gamma^2 + 16 \geq 0,$$

or equivalently, if, and only if,

$$\gamma^2 \geq 4 \quad \text{or} \quad \gamma^2 \leq \frac{4}{9}.$$

Thus for the above values of $\gamma$, the Popov function, $\Sigma(z)$, admits Hermitian factorizations, and will have constant inertia on the unit circle. By Theorem 7.4.1, this constant inertia is given by the inertia of $R_e$. But

$$R_{e_{1,2}} = \gamma^2 + p_{1,2} = \frac{1}{8} \cdot \left[ 5\gamma^2 - 4 \pm \sqrt{(5\gamma^2 - 4)^2 - 16\gamma^4} \right].$$

Therefore for $\gamma^2 \geq 4$, we have $R_e > 0$, and for $\gamma^2 \leq \frac{4}{9}$, we have $R_e < 0$. Hence $\Sigma(z)$ is non-negative on the unit circle for all $\gamma^2 \geq 4$, and is non-positive on the unit circle for all $\gamma^2 \leq \frac{4}{9}$.[12] Thus, we conclude that

$$\gamma_u^2 = \sup_{\omega \in [0, 2\pi]} |A(e^{j\omega})|^2 = 4 \quad \text{and} \quad \gamma_l^2 = \inf_{\omega \in [0, 2\pi]} |A(e^{j\omega})|^2 = \frac{4}{9},$$

which is what we had set out to find.

We should also mention that for intermediate values of $\gamma$, *i.e.*, $\frac{4}{9} < \gamma^2 < 4$, the DARE does not have a real (Hermitian) solution, and that therefore the Popov function $\Sigma(z)$ does not have constant inertia on the unit circle, $|z| = 1$.

■

**Example 2 (Properties of the DARE)** We now continue with the DARE of Example 1 and establish some of its properties. First, we verify that it always has a semi-stabilizing solution. To this end, note that we may write

$$F_p = F - K_p H = F(I + PH^* R^{-1} H)^{-1} = \frac{FR}{R + HPH^*} = \frac{FR}{R_e}.$$

[12] Correspondingly, $\Sigma(z)$ is positive and negative on the unit circle for all $\gamma^2 > 4$ and $\gamma^2 < \frac{4}{9}$, respectively.

Therefore

$$F_{p_{1,2}} = \frac{\gamma^2/2}{\frac{1}{8}\cdot\left[5\gamma^2 - 4 \pm \sqrt{(5\gamma^2-4)^2 - 16\gamma^4}\right]} = \frac{4\gamma^2}{5\gamma^2 - 4 \pm \sqrt{(5\gamma^2-4)^2 - 16\gamma^4}}.$$

Now it is straightforward to see that

$$F_{p_1} F_{p_2} = 1.$$

Therefore at least one of $F_{p_1}$ or $F_{p_2}$ is semi-stable, meaning that at least one of $p_1$ or $p_2$ is semi-stabilizing. To be more specific, when $\gamma^2 > 4$, we have

$$F_{p_1} = \frac{4\gamma^2}{4\gamma^2 + (\gamma^2 - 4) + \sqrt{(5\gamma^2-4)^2 - 16\gamma^4}} < 1,$$

so that $p_1$ is the stabilizing solution. Likewise, when $\gamma^2 < \frac{4}{9}$, we have

$$F_{p_2} = \frac{4\gamma^2}{-4\gamma^2 - (4 - 9\gamma^2) - \sqrt{(5\gamma^2-4)^2 - 16\gamma^4}} > -1,$$

so that $-1 < F_{p_2} < 0$, and $p_2$ is the stabilizing solution. In either case, we have $K_p = K_q$ (where $K_p$ corresponds to the stabilizing solution), so that the spectral factorization of $\Sigma(z)$ is given by

$$\Sigma(z) = \left[\frac{K_p}{z - 1/2} + 1\right] R_e \left[\frac{K_p}{z^{-1} - 1/2} + 1\right] = \frac{z - 1/2 + K_p}{z - 1/2} R_e \frac{z^{-1} - 1/2 + K_p}{z^{-1} - 1/2}.$$

In the case where $\frac{4}{9} < \gamma^2 < 4$, we have

$$F_{p_{1,2}} = \frac{4\gamma^2}{5\gamma^2 - 4 \pm j\sqrt{16\gamma^4 - (5\gamma^2-4)^2}}.$$

Therefore, it is easy to see that

$$|F_{p_{1,2}}| = 1,$$

so that *both* $p_1$ and $p_2$ are semi-stabilizing solutions. In this case, we have $F_{p_1} = F_{p_2}^*$, so that the spectral factorization of $\Sigma(z)$ is given by

$$\Sigma(z) = \frac{z - F_p}{z - 1/2} R_e \frac{z^{-1} - F_p}{z^{-1} - 1/2}, \qquad |F_p| = 1.$$

If we compare this factorization with the factorization $\Sigma(z) = M(z)N^*(z^{-*})$, given in Lemma 7.3.3, we can identify

$$M(z) = \frac{z - F_p}{z - 1/2} \quad \text{and} \quad N(z) = R_e^* \frac{z - F_p^*}{z - 1/2}.$$

Therefore the zero of $N(z)$ is the complex conjugate of the zero of $M(z)$.

■

**Example 3 (The Hamiltonian Matrix)** We once more return to the DARE of Example 1, and now study its associated Hamiltonian matrix

$$M = \begin{bmatrix} 2 & 2 \\ -2\gamma^{-2} & 1/2 - 2\gamma^{-2} \end{bmatrix}. \tag{7.7.6}$$

It is easy to see that the characteristic equation corresponding to $M$ is

$$\lambda^2 - (5/2 - 2\gamma^{-2})\lambda + 1 = 0,$$

whose solutions are

$$\lambda_{1,2} = \frac{5 - 4\gamma^{-2} \pm \sqrt{(5 - 4\gamma^{-2})^2 - 16}}{4}.$$

Suppose, for example, that $\gamma^2 < \frac{4}{9}$. In this case the stable eigenvalue of $M$ is given by

$$-1 < \lambda_1 = \frac{5 - 4\gamma^{-2} + \sqrt{(5 - 4\gamma^{-2})^2 - 16}}{4} < 0.$$

A basis for the stable invariant subspace of $M$ can be found by solving the equations

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 \begin{bmatrix} x \\ y \end{bmatrix},$$

which yields

$$x = 1 \quad \text{and} \quad y = \frac{-3 - 4\gamma^{-2} + \sqrt{(5 - 4\gamma^{-2})^2 - 16}}{8}.$$

Thus, the stabilizing solution to the DARE is given by

$$p = xy^{-1} = \frac{-3\gamma^2 - 4 - \sqrt{(5\gamma^2 - 4)^2 - 16\gamma^2}}{8},$$

which corresponds to the solution obtained in Example 1.

As an example of the case where $\frac{4}{9} \leq \gamma^2 \leq 4$, let us consider $\gamma = 1$. In this case we have,

$$\lambda_{1,2} = \frac{1 \pm j\sqrt{15}}{4},$$

and therefore a basis for a semi-stable invariant subspace of $M$ can be found by solving the equations

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1 + j\sqrt{15}}{4} \begin{bmatrix} x \\ y \end{bmatrix},$$

which yields

$$x = 1 \quad \text{and} \quad y = \frac{-7 + j\sqrt{15}}{8},$$

and from which we infer that a semi-stabilizing solution to the DARE is given by

$$p = xy^{-1} = \frac{-7 - j\sqrt{15}}{8}.$$

Finally, another case of interest is when $\gamma^2 = 4$, since in this case $M$ has repeated eigenvalues and may not be diagonalizable. With this choice of $\gamma$, we have

$$M = \begin{bmatrix} 2 & 2 \\ -\frac{1}{2} & 0 \end{bmatrix} \quad \text{and} \quad \lambda_{1,2} = 1,$$

so that $M$ is obviously non-diagonalizable. Nonetheless, we can still find a basis for the semi-stable invariant subspace of $M$ by solving the equations

$$M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix},$$

which yields,

$$x = 1 \quad \text{and} \quad y = -\frac{1}{2}.$$

The semi-stabilizing solution to the DARE is therefore given by

$$p = xy^{-1} = -2,$$

which corresponds to the solution that can be obtained from solving (7.7.5) when $\gamma^2 = 4$. The spectral factorization of $\Sigma(z)$ now becomes

$$\Sigma(z) = \frac{z - 1}{z - 1/2} \cdot 2 \cdot \frac{z^{-1} - 1}{z^{-1} - 1/2}.$$

$\blacksquare$

Our next example concerns the case where $\Sigma(z)$, and hence $R_e$, is singular. This example shows that $R_e$ may be singular even when $\{F, G, H\}$ is minimal, and hence serves to demonstrate that, unlike the positive case, it is very difficult to give necessary and sufficient conditions for the invertibility of $R_e$ in the indefinite case.

**Example 4 (Singular $\Sigma(z)$)** Consider the SDARE

$$\left\{\begin{array}{rcl} P & = & FPF^* + GQG^* - K_p R_e K_p^* \\ FPH^* & = & K_p R_e \\ R_e & = & R + HPH^* \end{array}\right. \tag{7.7.7}$$

where

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad S = 0 \quad , \quad R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

It is easy to check that one solution to the above SDARE is given by

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad R_e = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad K_p = \begin{bmatrix} 1/2 & 0 \\ 0 & \cos\theta \end{bmatrix}$$

where $\theta$ is arbitrary. Note that this solution is stabilizing since

$$F - K_p H = \begin{bmatrix} 1/2 & 0 \\ 0 & -\cos\theta \end{bmatrix},$$

is stable for all $\theta$.

This example is of interest since it shows that in the indefinite case, even when $\{F, G, H\}$ is minimal, the solution to the DARE may result in a singular $R_e$. The reason is that the Popov function associated with this SDARE is singular. Indeed,

$$\Sigma(z) \quad = \quad \begin{bmatrix} H(zI - F)^{-1} & I \end{bmatrix} \begin{bmatrix} GQG^* & GS \\ S^*G^* & R \end{bmatrix} \begin{bmatrix} (z^{-1}I - F^*)^{-1} \\ I \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0 & z \end{bmatrix}^{-1} & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} z^{-1}-1 & 0 \\ 0 & z^{-1} \end{bmatrix}^{-1} \\ I \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \cdot \frac{1}{(z-1)(z^{-1}-1)} + 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

■

**Example 5 (DARE Without Semi-Stabilizing Solution)** Consider the DARE with $F = 1$, $G = 1$, $H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Q = 3$, $R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Using the fact that the general DARE (with $S = 0$) can be written as

$$P = F(I + PH^*R^{-1}H)^{-1}PF^* + GQG^*,$$

allows us to write the DARE, in this special case, as

$$P = 4P + 3,$$

from which we infer $P = -1$. Thus the DARE has a unique (Hermitian solution). It is now straightforward to see that,

$$R_e = \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \quad , \quad R_e^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix} \quad , \quad K_p = \begin{bmatrix} 2 & -2 \end{bmatrix},$$

from which we can infer,

$$F_p = F - K_pH = F - 0 = 2.$$

Therefore the solution to the DARE is not (semi)stabilizing. Indeed, although $\{F, H\}$ is detectable, and although the DARE has a Hermitian solution, this solution is not stable. This, of course, means that the Popov function,

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{3}{(z-2)(z^{-1}-2)} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{3}{(z-2)(z^{-1}-2)} & \frac{3}{(z-2)(z^{-1}-2)} \\ \frac{3}{(z-2)(z^{-1}-2)} & -1 + \frac{3}{(z-2)(z^{-1}-2)} \end{bmatrix} \end{aligned}$$

does not admit a proper canonical Hermitian factorization. Using the above solution to the DARE, the proper factorization that it does admit is given by

$$\Sigma(z) = \begin{bmatrix} \frac{z-4}{(z-2)(z^{-1}-2)} & \frac{2}{(z-2)(z^{-1}-2)} \\ \frac{-2}{(z-2)(z^{-1}-2)} & \frac{z}{(z-2)(z^{-1}-2)} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} \frac{z^{-1}-4}{(z-2)(z^{-1}-2)} & \frac{-2}{(z-2)(z^{-1}-2)} \\ \frac{2}{(z-2)(z^{-1}-2)} & \frac{z^{-1}}{(z-2)(z^{-1}-2)} \end{bmatrix},$$

which is a "dual" canonical factorization. [Note also that, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, the stable invariant subspace of

$$M = \begin{bmatrix} \frac{1}{2} & -\frac{3}{2} \\ 0 & 2 \end{bmatrix},$$

does not have an invertible $\mathcal{T}_{21}$, whereas the antistable invariant subspace, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ does.]

∎

**Example 6 (DARE Without Solution)** Consider the DARE with $F = 1$, $G = 1$, $H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Q = 1$, $R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Using the fact that the general DARE (with $S = 0$) can be written as

$$P = F(I + PH^*R^{-1}H)^{-1}PF^* + GQG^*,$$

allows us to write the DARE, in this special case, as

$$P = P + 1,$$

from which we infer that the DARE has no solution! This implies that, despite the fact that $\{F, H\}$ is detectable, the Popov function,

$$\begin{aligned} \Sigma(z) &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{(z-1)(z^{-1}-1)} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{1}{(z-1)(z^{-1}-1)} & \frac{1}{(z-1)(z^{-1}-1)} \\ \frac{1}{(z-1)(z^{-1}-1)} & -1 + \frac{1}{(z-1)(z^{-1}-1)} \end{bmatrix} \end{aligned}$$

does not admit any proper factorization. [Note that in this case, the only invariant subspace of the Hamiltonian matrix,

$$M = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ which does not have an invertible $\mathcal{T}_{21}$.]

∎

## 7.8 Conclusion

In this chapter we studied the discrete-time algebraic Riccati equation (DARE), and showed that the existence of solutions to the DARE, or more precisely the system of discrete-time algebraic Riccati equations (SDARE), is equivalent to the existence of a proper factorization of the associated Popov function. Additional properties of the DARE solution, such as being Hermitian or stabilizing, were then related to additional properties of the factorization of the Popov function. A method for computing the solution to the DARE, and, in fact, for checking its existence was also given in terms of invariant subspaces of the so-called Hamiltonian matrix. We also particularized these results to some important special cases that are encountered in LQR and $H^\infty$ estimation and control. Finally, we should mention that the results presented here are extensions of some wellknown results on discrete-time algebraic Riccati equations with positive (semi)definite coefficient matrices to the indefinite (or Krein space) setting.

## 7.A Structure of $Z_1$ in Theorem 7.4.1

In this Appendix we shall show that if $Z_1$ satisfies the Lyapunov equation

$$Z_1 = FZ_1F^* + T_a \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* T_b^*, \tag{7.A.1}$$

where

$$F = T_a \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} T_a^{-1},$$

and

$$F = T_b \begin{bmatrix} F_b & 0 & F_{b,13} & 0 \\ F_{b,21} & F_{b,o} & F_{b,23} & F_{b,24} \\ 0 & 0 & F_{b,c} & 0 \\ 0 & 0 & F_{b,43} & F_{b,co} \end{bmatrix} T_b^{-1},$$

then $Z_1$ has the form

$$Z_1 = T_a \begin{bmatrix} Z_{ab} & Z_{ab,o} & 0 & 0 \\ Z_{ba,o} & Z_{ab,oo} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} T_b^*, \tag{7.A.2}$$

where

$$\begin{bmatrix} Z_{ab} & Z_{ab,o} \\ Z_{ba,o} & Z_{ab,oo} \end{bmatrix} = \begin{bmatrix} F_a & 0 \\ F_{a,21} & F_{a,o} \end{bmatrix} \begin{bmatrix} Z_{ab} & Z_{ab,o} \\ Z_{ba,o} & Z_{ab,oo} \end{bmatrix} \begin{bmatrix} F_b^* & F_{b,21}^* \\ 0 & F_{b,o}^* \end{bmatrix} + \begin{bmatrix} K_a \\ X_a \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \end{bmatrix}^*. \tag{7.A.3}$$

To this end, let us pre- and post-multiply (7.A.1) by $T_a^{-1}$ and $T_b^{-*}$, respectively, to obtain

$$T_a^{-1}Z_1T_b^{-*} = \begin{bmatrix} F_a & 0 & F_{a,13} & 0 \\ F_{a,21} & F_{a,o} & F_{a,23} & F_{a,24} \\ 0 & 0 & F_{a,c} & 0 \\ 0 & 0 & F_{a,43} & F_{a,co} \end{bmatrix} Z_1 \begin{bmatrix} F_b & 0 & F_{b,13} & 0 \\ F_{b,21} & F_{b,o} & F_{b,23} & F_{b,24} \\ 0 & 0 & F_{b,c} & 0 \\ 0 & 0 & F_{b,43} & F_{b,co} \end{bmatrix} + \begin{bmatrix} K_a \\ X_a \\ 0 \\ 0 \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \\ 0 \\ 0 \end{bmatrix}^* . \tag{7.A.4}$$

If we now partition $T_a^{-1}Z_1T_b^{-*}$ as

$$T_a^{-1}Z_1T_b^{-*} = \begin{bmatrix} Z_1^{11} & Z_1^{12} \\ Z_1^{21} & Z_1^{22} \end{bmatrix},$$

and define

$$F_a^{11} = \begin{bmatrix} F_a & 0 \\ F_{a,21} & F_{a,o} \end{bmatrix}, \quad F_a^{12} = \begin{bmatrix} F_{a,13} & 0 \\ F_{a,23} & F_{a,24} \end{bmatrix}, \quad F_a^{22} = \begin{bmatrix} F_{a,c} & 0 \\ F_{a,43} & F_{a,co} \end{bmatrix}$$

then we may write (7.A.4) as

$$\begin{bmatrix} Z_1^{11} & Z_1^{12} \\ Z_1^{21} & Z_1^{22} \end{bmatrix} = \begin{bmatrix} F_a^{11} & F_a^{12} \\ 0 & F_a^{22} \end{bmatrix} \begin{bmatrix} Z_1^{11} & Z_1^{12} \\ Z_1^{21} & Z_1^{22} \end{bmatrix} \begin{bmatrix} F_b^{11} & F_b^{12} \\ 0 & F_b^{22} \end{bmatrix}^* + \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}, \tag{7.A.5}$$

where we have made similar definitions for $F_b^{11}$, $F_b^{12}$ and $F_b^{22}$, and where $U$ is given by

$$U = \begin{bmatrix} K_a \\ X_a \end{bmatrix} D \begin{bmatrix} K_b \\ X_b \end{bmatrix}^* .$$

Now (7.A.5) may be rewritten as

$$\begin{bmatrix} Z_1^{11} & Z_1^{12} \\ Z_1^{21} & Z_1^{22} \end{bmatrix} = \begin{bmatrix} F_a^{11}Z_1^{11}F_b^{11*} + F_a^{12}Z_1^{21}F_b^{11*} + F_a^{11}Z_1^{11}F_b^{12*} + F_a^{12}Z_1^{22}F_b^{12*} + U & \times \\ F_a^{22}Z_1^{21}F_b^{11*} + F_a^{22}Z_1^{22}F_b^{12*} & F_a^{22}Z_1^{22}F_b^{22*} \end{bmatrix}. \tag{7.A.6}$$

Equating the $(2,2)$ block entries in the above equality yields

$$Z_1^{22} = F_a^{22}Z_1^{22}F_b^{22*},$$

from which we conclude (since both $F_a^{22}$ and $F_b^{22}$ are stable) that

$$Z_1^{22} = 0. \tag{7.A.7}$$

Using the above result and equating the $(2,1)$ block entries yields

$$Z_1^{21} = F_a^{22} Z_1^{21} F_b^{11*},$$

from which we conclude (since $F_a^{22}$ is stable and $F_b^{11}$ is marginally stable) that

$$Z_1^{21} = 0. \tag{7.A.8}$$

A similar argument shows that

$$Z_1^{12} = 0. \tag{7.A.9}$$

Eqs. (7.A.7-7.A.9) show that $Z_1$ has the desired structure given in (7.A.2). Finally, (7.A.3) follows from equating the $(1,1)$ block entries in (7.A.6).

■

# Chapter 8

# Asymptotic Behaviour

In this chapter we shall focus on the behaviour of the Riccati recursion as time progresses to infinity. In order to do so, we shall confine our attention to time-invariant state-space models. Our main interest is to find conditions under which, for a given initial condition, the solution to the Riccati recursion converges to a solution of the DARE. The main result uses an identity that relates the solution of the Riccati recursion for one initial condition to the solution for another initial condition, and states that if a certain inertia condition is met, then the Riccati recursion converges to the unique stabilizing solution (assuming such a solution exists) of the DARE. In the general case, the aforementioned inertia conditions need to be recursively checked, however, in some special cases they may be reduced to more simple and more explicit requirements on the initial condition. In particular, when the coefficient matrices of the Riccati recursion are positive semi-definite, we can guarantee the convergence of the Riccati recursion for some indefinite, and even negative semi-definite, initial conditions (provided they are bounded below by a certain negative semi-definite matrix). Moreover, in the case frequently encountered in $H^\infty$ filtering and control, we can guarantee convergence for all positive semi-definite initial conditions that are less than or equal to the unique positive semi-definite solution of a related Lyapunov equation. Finally, we mention some implications of the results obtained here to different problems in which these Riccati equations occur.

## 8.1 The Riccati Recursion

In this section we shall consider the discrete-time algebraic Riccati recursion (DARR)

$$P_{i+1} = FP_iF^* + GQG^* - (FP_iH^* + GS)(R + HP_iH^*)^{-1}(FP_iH^* + GS), \quad P_0 = \Pi_0 \tag{8.1.1}$$

where $F \in \mathcal{C}^{n\times n}$, $G \in \mathcal{C}^{n\times m}$ and $H \in \mathcal{C}^{p\times n}$ are given, $Q = Q^* \in \mathcal{C}^{m\times m}$, $R = R^* \in \mathcal{C}^{p\times p}$ and $S \in \mathcal{C}^{m\times p}$ are known, and it is assumed that the Hermitian matrices

$$R \quad \text{and} \quad \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix},$$

are nonsingular. The (possibly Hermitian) sequence of $n \times n$ matrices $\{P_i\}$ that satisfies (8.1.1) is referred to as the solution to the DARR with given initial condition, $P_0 = \Pi_0$.

As before, we shall often find it convenient to introduce the notations

$$K_{p,i} = (FP_iH^* + GS)(R + HP_iH^*)^{-1} \quad \text{and} \quad R_{e,i} = R + HP_iH^*, \tag{8.1.2}$$

and, when $P_i$ is Hermitian, to rewrite the DARR (8.1.1) as

$$P_{i+1} = FP_iF^* - K_{p,i}R_{e,i}K_{p,i}^* + GQG^*. \tag{8.1.3}$$

When, $P_i$, the solution to the DARR is not Hermitian, we can define

$$K_{q,i} = (FP_i^*H^* + GS)(R + HP_i^*H^*)^{-1}, \tag{8.1.4}$$

and rewrite the DARR as

$$P_{i+1} = FP_iF^* - K_{p,i}R_{e,i}K_{q,i}^* + GQG^*. \tag{8.1.5}$$

The main question of interest in this section is to clarify conditions under which, for a given initial condition $P_0$, the Riccati variable $P_i$ converges to $P$, some solution of the DARE (7.2.1). This question, of the asymptotic behaviour of the Riccati recursion, is important since it gives conditions under which time-varying least-squares (or $H^\infty$) controllers and filters converge to time-invariant ones.

In this section we shall give what we believe is a very direct approach to establishing the convergence of the Riccati recursion. Moreover, to the best of our knowledge, the results obtained here are more general than those to have appeared in the literature and subsume as special cases all of the earlier given results [Wil71b, LK76, Kuc72, Won68a, BRC72, CW81]. Since the section has quite a few results, it will be useful to begin with a brief overview of what shall be presented.

## 8.2 Overview of Results

We begin by proving two important identities relating the solution of the Riccati recursion for one initial condition to the solution of the Riccati recursion for another initial condition. The first of these is a local identity (see Lemma 8.3.1) while the second is a global identity (see Lemma 8.3.2). The result of Lemma 8.3.2 is then used in Sec. 8.4 to compare the solution of the Riccati recursion for an arbitrary initial condition, $P_0$, with any solution of the corresponding DARE. The ensuing result is given in Theorem 8.4.1 where it is shown that if a certain sequence of matrices (that depend on $P_0$) are uniformly bounded for all $i \geq 0$, then the solution to the Riccati recursion will *exponentially* converge to the unique stabilizing solution of the DARE (assuming such a solution exists). A sufficient condition for convergence turns out to be the condition that the matrices,

$$R_{e,i} = R + HP_iH^*,$$

have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$, where $P_i$ is the solution of the Riccati recursion with initial condition $P_0$.

Theorems 8.5.1 and 8.5.2 prove the convergence of the zero-initial-condition Riccati recursion (corresponding to $P_0 = 0$) for the positive case and the case where $Q - SR^{-1}S^* > 0$, respectively. The proof in both cases follows from showing that the sequence of matrices,

$$R_{e,i}^0 = R + HP_i^0H^*,$$

where $P_i^0$ is the zero-initial-condition Riccati solution, has constant inertia for all $i \geq 0$.

The fact that we can establish the convergence of $\{P_i^0\}$ to $P$, the unique stabilizing solution of the DARE, allows us to obtain more simple and more explicit conditions for convergence than the aforementioned inertia condition on the $\{R_{e,i}\}$. Indeed we can use Lemma 8.3.2 to compare $\{P_i\}$, the solution of the Riccati recursion for an arbitrary initial condition, to $\{P_i^0\}$, the zero initial condition solution, for which we have already established convergence. The relationship between $\{P_i\}$ and $\{P_i^0\}$ can be expressed more concisely in terms of quantities defined via the *dual* Riccati recursion

$$P_{i+1}^a = F^* P_i^a F + H^* R^{-1} H - F^* P_i^a G (Q^{-1} + G^* P_i^a G)^{-1} G^* P_i^a F, \qquad (8.2.1)$$

defined in Sec. 8.6.1. The exact relationship is given by

$$P_{i+1} - P_{i+1}^0 = \Phi_p^0(i+1, 0) \left[ I + P_0 P_{i+1}^{a,0} \right]^{-1} \Pi_0 \Phi_p^0(i+1, 0)^*, \qquad (8.2.2)$$

where $P_i^{a,0}$ is the zero initial condition solution to the dual Riccati recursion (8.2.1) and $\Phi_p^0(i+1,0)$ is a certain state transition matrix. Since it can be shown that, under certain stabilizability conditions, the state transition matrix $\Phi_p^0(i+1,0)$ tends to zero as $i \to \infty$, to show the convergence of $P_i$ to $P_i^0$ (and hence to $P$) we need only show that the matrix $\left[ I + P_0 P_{i+1}^{a,0} \right]^{-1}$ is uniformly bounded for all $i$. In the positive case (where $R > 0$ and $Q - SR^{-1}S^* > 0$) we establish that a sufficient condition for this is that the initial condition $P_0$ be such that

$$I + (P^a)^{*/2} P_0 (P^a)^{1/2} > 0, \qquad (8.2.3)$$

where $(P^a)^{1/2}(P^a)^{*/2} = P^a$, is the unique positive semi-definite solution to the dual DARE. The above result, whose precise statement is given in Theorem 8.7.1, gives, in the positive case, a very general description of the basin of attraction of the stabilizing solution $P$ to the Riccati recursion. Note that convergence for $P_0 \geq 0$ follows immediately from (8.2.3). However, the condition (8.2.3) also allows for some indefinite (and even negative semi-definite) initial conditions $P_0$. Indeed, under a certain controllability assumption, it can be shown that $P^a$ is invertible so that (8.2.3) can be replaced by the more revealing condition

$$P_0 > -(P^a)^{-1}. \qquad (8.2.4)$$

Finally, in the case where $Q - SR^{-1}S^* > 0$ it does not seem to be possible to replace the condition that $R_{e,i}$ should have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$, with a single condition such as (8.2.3). [Nonetheless, the inertia condition on $R_{e,i}$ is quite intriguing since it is equivalent to the condition required for the existence of $H^\infty$ filters and full-information controllers.] However, it is still possible to give a simple characterization of the basin of attraction of the stabilizing solution, $P$, of the DARE in this case as well. Indeed, Theorem 8.7.2 shows that if the initial condition is chosen such that

$$0 \leq P_0 \leq \Pi,$$

where $\Pi$ is the unique positive semidefinite solution to the Lyapunov equation, $\Pi = F\Pi F^* + GQG^*$, then $\{P_i\}$ converges to $P$.

## 8.3 Solutions to the Riccati Recursion for Different Initial Conditions

We first give a certain algebraic identity concerning the Riccati recursion (8.1.1). These identities are wellknown [Nis67, MPG88, dS89], and are very useful for obtaining comparison results for solutions to the Riccati recursion.

**Lemma 8.3.1 (Riccati Solutions for Different Initial Conditions)** *Suppose $P_i^{(1)}$ and $P_i^{(2)}$ are two solutions to the discrete-time Riccati recursion with the same $\{F, G, H\}$ and $\{Q, R, S\}$ matrices, but with different initial conditions $\Pi_0^{(1)}$ and $\Pi_0^{(2)}$, respectively. Then we have the following identities*

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})F_{p,i}^{(2)*} \tag{8.3.1}$$

*and*

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = F_{p,i}^{(1)}\left[(P_i^{(2)} - P_i^{(1)}) - (P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1}H(P_i^{(2)} - P_i^{(1)})\right]F_{p,i}^{(1)*}, \tag{8.3.2}$$

*where we have defined*

$$F_{p,i}^{(m)} = F - K_{p,i}^{(m)}H, \quad K_{p,i}^{(m)} = (FP_i^{(m)}H^* + GS)(R_{e,i}^{(m)})^{-1} \quad \text{and} \quad R_{e,i}^{(m)} = R + HP_i^{(m)}H^*, \ \ m = 1,2.$$

**Remark:** Note that (8.3.1) gives a simpler expression for $P_{i+1}^{(2)} - P_{i+1}^{(1)}$ than (8.3.2). However, (8.3.2) is often more useful since it gives a *symmetric* representation for $P_{i+1}^{(2)} - P_{i+1}^{(1)}$. In the following sections we shall make use of both identities.

**Proof of Lemma 8.3.1:** The proof involves only algebraic manipulations. We shall outline the steps. We first mention that in what follows we shall, without loss of generality, assume that $S = 0$. It is now straightforward to verify the identities

$$R_{e,i}^{(2)} - R_{e,i}^{(1)} = H(P_i^{(2)} - P_i^{(1)})H^*, \tag{8.3.3}$$

and

$$K_{p,i}^{(2)} - K_{p,i}^{(1)} = F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1}. \tag{8.3.4}$$

Now subtracting the two Riccati recursions for $P_i^{(1)}$ and $P_i^{(2)}$ yields

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = F(P_i^{(2)} - P_i^{(1)})F^* - K_{p,i}^{(2)}R_{e,i}^{(2)}K_{p,i}^{(2)*} + K_{p,i}^{(1)}R_{e,i}^{(1)}K_{p,i}^{(1)*}. \tag{8.3.5}$$

Now applying the identities (8.3.3)-(8.3.4), along with a rearrangement of the formula $F_{p,i} = F - K_{p,i}H$, to the above expression, we may write

$$\begin{aligned} P_{i+1}^{(2)} - P_{i+1}^{(1)} &= (F_{p,i}^{(1)} + K_{p,i}^{(1)}H)(P_i^{(2)} - P_i^{(1)})(F_{p,i}^{(2)} + K_{p,i}^{(2)}H)^* \\ &\quad -(F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1})R_{e,i}^{(2)}K_{p,i}^{(2)*} + K_{p,i}^{(1)}R_{e,i}^{(1)}(F_{p,i}^{(2)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(1)})^{-1})R_{e,i}^{(1)}, \end{aligned}$$

which, thanks to nice cancellations, yields (8.3.1).

To prove (8.3.1), we rewrite (8.3.5) as

$$\begin{aligned} P_{i+1}^{(2)} - P_{i+1}^{(1)} &= (F_{p,i}^{(1)} + K_{p,i}^{(1)}H)(P_i^{(2)} - P_i^{(1)})(F_{p,i}^{(1)} + K_{p,i}^{(1)}H)^* \\ &\quad -(F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1})(R_{e,i}^{(2)}(F_{p,i}^{(1)}(P_i^{(2)} - P_i^{(1)})H^*(R_{e,i}^{(2)})^{-1})^* \\ &\quad +K_{p,i}^{(1)}(R_{e,i}^{(2)} - H(P_i^{(2)} - P_i^{(1)})H^*)K_{p,i}^{(1)*}, \end{aligned}$$

and simplify.

∎

The next identity that we shall establish is just the statement of the fact that if we know a solution to the Riccati recursion for one initial condition, then under certain conditions, solutions for other initial conditions can be expressed in terms of the first solution. Although formulas of this type exist in continuous-time (see [San59, Rei72]), we have not encountered their discrete-time analogs in the literature. We have already provided a *local* identity relating solutions of the Riccati recursion for different initial conditions in Lemma 8.3.1. What we shall now present is a *global* identity. The precise statement follows.

**Lemma 8.3.2 (Riccati Solutions for Different Initial Conditions)** *Suppose $P_i^{(1)}$ and $P_i^{(2)}$ are two solutions to the discrete-time Riccati recursion (8.1.1) with the same $\{F, G, H\}$ and $\{Q, R, S\}$ matrices, but with different initial conditions $\Pi_0^{(1)}$ and $\Pi_0^{(2)}$, respectively. We then have the following identity:*

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \Phi_p^{(1)}(i+1,0)\left[I + (\Pi_0^{(2)} - \Pi_0^{(1)})\mathcal{O}_i^{(1)}\right]^{-1}(\Pi_0^{(2)} - \Pi_0^{(1)})\Phi_p^{(1)}(i+1,0)^*, \quad (8.3.6)$$

*where*

$$\Phi_p^{(1)}(i,0) = \begin{cases} \prod_{j=0}^{i-1} F_{p,j}^{(1)} & i > 0 \\ I & i = 0 \end{cases} \quad (8.3.7)$$

*is the state transition matrix of $F_{p,j}^{(1)} = F - K_{p,j}^{(1)} H$, and*

$$\mathcal{O}_i^{(1)} = \sum_{j=0}^{i} \Phi_p^{(1)}(j,0)^* H^* (R_{e,j}^{(1)})^{-1} H \Phi_p^{(1)}(j,0), \quad (8.3.8)$$

*is the observability Gramian of $\{F_{p,j}^{(1)}, (R_{e,j}^{(1)})^{-1/2} H\}$.*

**Proof:** We shall prove (8.3.6) by induction. Define $\delta P_i \triangleq P_i^{(2)} - P_i^{(1)}$. Then for $i = 0$, using (8.3.2), we have

$$\begin{aligned}
P_1^{(2)} - P_1^{(1)} &= F_{p,0}^{(1)}\left[\delta P_0 - \delta P_0 H^* (R_{e,0}^{(2)})^{-1} H \delta P_0\right] F_{p,0}^{(1)*} \\
&= \Phi_p^{(1)}(1,0)\left[I - \delta P_0 H^* (R_{e,0}^{(2)})^{-1} H\right] \delta P_0 \Phi_p^{(1)}(1,0)^* \\
&= \Phi_p^{(1)}(1,0)\left[I - \delta P_0 H^* (R_{e,0}^{(1)} + H \delta P_0 H^*)^{-1} H\right] \delta P_0 \Phi_p^{(1)}(1,0)^* \\
&= \Phi_p^{(1)}(1,0)\left[I + \delta P_0 H^* (R_{e,0}^{(1)})^{-1} H\right]^{-1} \delta P_0 \Phi_p^{(1)}(1,0)^* \\
&= \Phi_p^{(1)}(1,0)\left[I + \delta P_0 \mathcal{O}_0^{(1)}\right]^{-1} \delta P_0 \Phi_p^{(1)}(1,0)^*
\end{aligned}$$

as desired.

Now suppose that (8.3.6) is true for $i$. We shall show that the identity is true for $i+1$ as well. Indeed, using (8.3.2) and the above arguments we have

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = F_{p,i}^{(1)} \left[\delta P_i - \delta P_i H^*(R_{e,i}^{(2)})^{-1} H \delta P_i\right] F_{p,i}^{(1)*} = F_{p,i}^{(1)} \left[I + \delta P_i H^*(R_{e,i}^{(1)})^{-1} H\right]^{-1} \delta P_i F_{p,i}^{(1)*}. \tag{8.3.9}$$

On the other hand,

$$\begin{aligned}
I + \delta P_i H^*(R_{e,i}^{(1)})^{-1} H &= I + \Phi_p^{(1)}(i,0)\left[I + \delta P_0 \mathcal{O}_{i-1}^{(1)}\right]^{-1} \delta P_0 \Phi_p^{(1)}(i,0)^* H^* (R_{e,i}^{(1)})^{-1} H \\
&= I + \Phi_p^{(1)}(i,0)\left[I + \delta P_0 \mathcal{O}_{i-1}^{(1)}\right]^{-1} \delta P_0 \underbrace{\Phi_p^{(1)}(i,0)^* H^* (R_{e,i}^{(1)})^{-1} H \Phi_p^{(1)}(i,0)}_{\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)}} \Phi_p^{(1)}(i,0)^{-1} \\
&= I + \Phi_p^{(1)}(i,0)\left[I + \delta P_0 \mathcal{O}_{i-1}^{(1)}\right]^{-1} \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)}) \Phi_p^{(1)}(i,0)^{-1} \\
&= \Phi_p^{(1)}(i,0)\left[I + (I + \delta P_0 \mathcal{O}_{i-1}^{(1)})^{-1} \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)})\right] \Phi_p^{(1)}(i,0)^{-1}.
\end{aligned}$$

Using the last of the above expressions in (8.3.9) yields

$$\begin{aligned}
P_{i+1}^{(2)} - P_{i+1}^{(1)} &= \Phi_p^{(1)}(i+1,0)\left[I + (I + \delta P_0 \mathcal{O}_{i-1}^{(1)})^{-1} \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)})\right]^{-1} \Phi_p^{(1)}(i,0)^{-1} \delta P_i F_{p,i}^{(1)*} \\
&= \Phi_p^{(1)}(i+1,0)\left[I + (I + \delta P_0 \mathcal{O}_{i-1}^{(1)})^{-1} \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)})\right]^{-1} (I + \delta P_0 \mathcal{O}_{i-1}^{(1)})^{-1} \delta P_0 \Phi_p^{(1)}(i+1,0)^* \\
&= \Phi_p^{(1)}(i+1,0)\left[I + \delta P_0 \mathcal{O}_{i-1}^{(1)} + \delta P_0 (\mathcal{O}_i^{(1)} - \mathcal{O}_{i-1}^{(1)})\right]^{-1} \delta P_0 \Phi_p^{(1)}(i+1,0)^* \\
&= \Phi_p^{(1)}(i+1,0)\left[I + \delta P_0 \mathcal{O}_i^{(1)}\right]^{-1} \delta P_0 \Phi_p^{(1)}(i+1,0)^*,
\end{aligned}$$

which is the desired result.

■

The result of Lemma 8.3.2 will prove to be very useful in establishing convergence results for the Riccati recursion (8.1.1). To see why, suppose is $P$ is some solution to the DARE (7.2.1). Then using (8.3.6), we may write

$$P_{i+1} - P = F_p^{i+1}\left[I + (P_0 - P)\mathcal{O}_i^p\right]^{-1} (P_0 - P) F_p^{(i+1)*}, \tag{8.3.10}$$

where $P_i$ is the solution to the Riccati recursion with initial condition $P_0$, and $\mathcal{O}_i^p$ is the observability Gramian of the pair $\{F_p, R_e^{-1/2} H\}$, and satisfies the recursion

$$\mathcal{O}_{i+1}^p = F_p^* \mathcal{O}_i^p F_p + H^* R_e H, \quad \mathcal{O}_{-1}^p = 0.$$

Now if $P$ is chosen as the stabilizing solution to the DARE (assuming such a solution exists), then the matrix $F_p = F - K_p H$ is stable and we have $\lim_{i\to\infty} F_p^i = 0$.

Then using (8.3.10), we see that if the initial condition $P_0$ is such that the matrix $[I + (P_0 - P)\mathcal{O}_i^p]^{-1}(P_0 - P)$ remains uniformly bounded for all $i$, then $P_i$ will converge to $P$, *i.e.*,

$$\lim_{i\to\infty} P_{i+1} - P = 0.$$

The above line of reasoning will be used in the next section to establish general convergence results for the Riccati recursion.

## 8.4 Some General Convergence Results

In this section we shall obtain some general results concerning the convergence of the Riccati recursion (8.1.1). We shall essentially give requirements on the initial condition, $P_0$, such that the solution of the Riccati recursion converges to the unique stabilizing solution (when a stabilizing solution exists) of the discrete-time algebraic Riccati equation. Although the requirements on $P_0$ may be difficult to verify in the general case, in later sections we shall see that, when specialized to some very important cases of interest in least-squares and $H^\infty$ control and estimation, they result in easy to verify requirements and yield basins of attraction for the initial condition, $P_0$, that are more general than those currently available in the literature.

The results presented in this section all use the identity (8.3.10). However, it will be useful to introduce a more symmetric representation of (8.3.10). To this end, let us introduce the generalized square-root factorization of the (possibly indefinite) Hermitian matrix $P_0 - P$, as follows

$$P_0 - P = L_0 J L_0^*, \tag{8.4.1}$$

where $J$ is a signature matrix of appropriate dimensions that represents the inertia of $P_0 - P$. Then using the identity

$$[I + L_0 J L_0^* \mathcal{O}_i^p]^{-1} L_0 J L_0^* = L_0 [J + L_0^* \mathcal{O}_i^p L_0]^{-1} L_0^*,$$

we may write (8.3.10) as

$$P_{i+1} - P = F_p^{i+1} L_0 [J + L_0^* \mathcal{O}_i^p L_0]^{-1} L_0^* F_p^{(i+1)*}, \tag{8.4.2}$$

which is the form that we shall more often use.

**Theorem 8.4.1 (A General Convergence Result)** *Consider the discrete-time Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - (FP_iH^* + GS)(R + HP_iH^*)^{-1}(FP_iH^* + GS), \quad (8.4.3)$$

*with initial condition, $P_0$, and suppose that the associated DARE (7.2.1) has a stabilizing solution, $P$. Then if the initial condition $P_0$ is chosen such that the matrices*

$$[J + L_0^*\mathcal{O}_i^p L_0]^{-1},$$

*are uniformly bounded for all $i \geq 0$, where*

$$P_0 - P = L_0 J L_0^*,$$

*and $\mathcal{O}_i^p$ satisfies the recursion*

$$\mathcal{O}_{i+1}^p = F_p^*\mathcal{O}_i^p F_p + H^* R_e^{-1} H, \qquad \mathcal{O}_{-1}^p = 0,$$

*then $P_i$ converges to, $P$, the unique stabilizing solution of the DARE. Moreover, we have*

$$\|P_i - P\| \leq \lambda^{2i} m, \quad (8.4.4)$$

*where $\|\cdot\|$ denotes the spectral radius,*

$$\lambda = \|F - K_pH\| \leq 1, \quad (8.4.5)$$

*and*

$$m = \sup_{i \geq 0} \left\| L_0 \left[J + L_0^*\mathcal{O}_i^p L_0\right]^{-1} L_0^* \right\|, \quad (8.4.6)$$

i.e., *the convergence of $P_i$ to $P$ is exponential.*

*Two equivalent conditions that ensure the boundedness of $[J + L_0^*\mathcal{O}_i^p L_0]^{-1}$ are the following:*

*(a) The matrices $J + L_0^*\mathcal{O}_i^p L_0$ have constant inertia (equal to the inertia of $J$) for all $i \geq 0$.*

*(b) The matrices $R_{e,i} = R + HP_iH^*$ have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$.*

**Remark:** The condition given in the above Theorem for convergence to the stabilizing solution of the DARE is essentially that the sequence of matrices $T_i^{-1} = [J + L_0^* \mathcal{O}_i^p L_0]^{-1}$ be uniformly bounded for all $i \geq 0$. This condition is interesting since it only depends on the initial condition through the matrices $L_0$ and $J$ (the $\mathcal{O}_i^p$ do not depend on $P_0$). However, this condition may be quite difficult to verify in the general case. The sufficient conditions (a) and (b) for boundedness seem to be much more restrictive, since instead of requiring that the $T_i$ be uniformly nonsingular for all $i$, they require that the $T_i$ have constant inertia for all $i$. Despite this fact, as we shall see in subsequent sections, the conditions (a) and (b) are much easier to verify, and are still quite general since they yield conditions for convergence that are quite more general than those that have appeared in the literature so far.

**Proof of Theorem 8.4.1:** To prove the first part we need only consider (8.4.2). Since $P$ is stabilizing, so that $F_p = F - K_p H$ is stable and $\lim_{i \to \infty} F_p^i = 0$, and since $[J + L_0^* \mathcal{O}_i^p L_0]^{-1}$ is uniformly bounded for all $i \geq 0$, we immediately deduce that

$$\lim_{i \to \infty} P_i - P = 0,$$

as desired.

To show the exponential convergence of $P_i$, we can use (8.4.2) to compute $\|P_i - P\|$. Thus, using the submultiplicative property of the spectral radius ($\|AB\| \leq \|A\| \cdot \|B\|$), we have

$$\|P_i - P\| = \|F - K_p H\|^i \cdot \left\| L_0 \left[J + L_0^* \mathcal{O}_i^p L_0\right]^{-1} L_0^* \right\| \cdot \|F - K_p H\|^i.$$

The desired result (8.4.4) now follows from (8.4.5) and (8.4.6).

Now the fact that condition (a) leads to uniformly bounded $[J + L_0^* \mathcal{O}_i^p L_0]^{-1}$ for all $i$ is obvious, since if the $J + L_0^* \mathcal{O}_i^p L_0$ have constant inertia (equal to the inertia of $J$) for all $i$, then they are uniformly nonsingular as well. Therefore, what remains to be shown is that the conditions (a) and (b) are equivalent. This requires somewhat more effort.

To this end, let us first write

$$P_i - P = L_i D_i L_i^*,$$

where $D_i$ is a nonsingular Hermitian matrix of the appropriate dimensions. Now using (8.3.2) and the above equation, we may write

$$P_{i+1} - P = F_p L_i \left[ D_i - D_i L_i^* H R_{e,i}^{-1} H^* L_i^* D_i \right]^{-1} L_i^* F_p^*, \tag{8.4.7}$$

from which we may infer

$$L_{i+1} = F_p L_i = F_p^{i+1} L_0,$$

and

$$\begin{aligned} D_{i+1} &= D_i - D_i L_i^* H R_{e,i}^{-1} H^* L_i^* D_i \\ &= D_i - D_i L_i^* H (R_{e,i} + H L_i D_i L_i^* H^*)^{-1} H^* L_i^* D_i \\ &= \left[ D_i^{-1} + H^* R_{e,i}^{-1} H \right]^{-1}. \end{aligned}$$

Thus, (8.4.7) may be written as

$$P_{i+1} - P = F_p^{i+1} L_0 D_{i+1} L_0^* F_p^{(i+1)*}.$$

Comparing the above expression with (8.4.2), we note that the matrices $[J + L_0^* \mathcal{O}_i^p L_0]^{-1}$ will have constant inertia for all $i$, if, and only if, the matrices $D_i$ have constant inertia for all $i$.

Now consider the matrix

$$T = \begin{bmatrix} D_i & D_i L_i^* H^* \\ H L_i D_i & R_{e,i} \end{bmatrix}.$$

Two different (block lower-upper and block upper-lower) triangular factorizations of $T$ show that the matrices

$$\begin{bmatrix} D_i & 0 \\ 0 & \underbrace{R_{e,i} - H L_i D_i L_i^* H^*}_{R_{e,i}} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \underbrace{D_i - D_i L_i^* H R_{e,i}^{-1} H^* L_i^* D_i}_{D_{i+1}} & 0 \\ 0 & R_{e,i} \end{bmatrix},$$

are congruent and have the same inertia. Thus $D_{i+1}$ will have the same inertia as $D_i$ if, and only if, $R_{e,i}$ has the same inertia as $R_{e,i}$. Therefore we conclude that the $D_i$ have constant inertia for all $i \geq 0$ if, and only if, the $R_{e,i}$ have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$. This establishes the equivalence of (a) and (b).

■

## 8.5 Convergence of the Riccati Recursion with Zero Initial Condition

From the discussions of the prior section we conclude that to verify the convergence of the Riccati recursion for some given initial condition, $P_0$, we need to check the conditions of Theorem 8.4.1. In this section we shall do this for the special case $P_0 = 0$. The motivation for doing so is not so much to illustrate an example of the application of Theorem 8.4.1, but rather because understanding the convergence of the Riccati recursion for $P_0 = 0$ will greatly assist us in obtaining more explicit convergence results for arbitrary initial conditions, $P_0$. This matter will be taken up in Sec. 8.6.

### 8.5.1 The Case of $R > 0$ and $Q - SR^{-1}S^* > 0$

Recall from the discussion at the beginning of Sec. 7.5.1 that there is no loss of generality in assuming $Q - SR^{-1}S^* > 0$ as opposed to $Q - SR^{-1}S^* \geq 0$.

Now the result given below on the convergence of $P_i^0$, the zero-initial-condition solution to the Riccati recursion, is wellknown (see *e.g.*, [Wil71b, Kuc72, Won68a, BRC72]). The standard proof of convergence involves showing that the sequence $\{P_i^0\}$ is nondecreasing and, given the detectability condition, bounded from above. Here, however, we shall give a different proof that uses Theorem 8.4.1.

**Theorem 8.5.1 (Convergence of Riccati Recursion with $P_0 = 0$)** *Consider the Riccati recursion with zero initial condition*

$$P_{i+1}^0 = FP_i^0F^* + GQG^* - (FP_i^0H^* + GS)(R + HP_i^0H^*)^{-1}(FP_i^0H^* + GS)^*, \quad P_0^0 = 0, \tag{8.5.1}$$

*and suppose that $\{F, H\}$ is detectable, $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable, and*

$$\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0.$$

*Then $P_i^0$ converges to $P$, the unique stabilizing and positive semi-definite solution of the discrete-time algebraic Riccati equation*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

**Proof:** We first remark that, from Theorem 7.5.1, since $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable, the DARE has a unique stabilizing and positive semi-definite solution, $P$. Now to prove the Theorem we shall show that $P_i^0 \geq 0$, for all $i \geq 0$. Since $R > 0$, this will imply that

$$R_{e,i}^0 = R + HP_i^0H^* > 0, \qquad \forall i,$$

*i.e.*, that $R_{e,i}$ has constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$. Therefore using Theorem 8.4.1, the sequence $\{P_i^0\}$ must converge to $P$.

To prove that $P_i^0 \geq 0$, for all $i \geq 0$, we proceed by induction. The case $i = 0$ is obvious since $P_0^0 = 0 \geq 0$. So let us assume that $P_i^0 \geq 0$, for some given $i$. We then may write the Riccati recursion as

$$P_{i+1}^0 = F_{p,i}^0 P_i F_{p,i}^{0*} + \begin{bmatrix} G & -K_{p,i}^0 \end{bmatrix} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} G^* \\ -K_{p,i}^{0*} \end{bmatrix}.$$

Now since $P_i \geq 0$ and $\begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} > 0$, the above expression readily shows that $P_{i+1}^0 \geq 0$. This finishes our induction and thus proves the Theorem.

■

### 8.5.2 The Case of Positive $Q - SR^{-1}S^*$

As mentioned at the beginning of Sec. 7.5.3, there is no loss of generality in assuming $S = 0$ and that the (possibly) indefinite matrix $R$ has the form

$$R = \begin{bmatrix} I_{p1} & 0 \\ 0 & -I_{p2} \end{bmatrix}.$$

Therefore, in this case our assumption will reduce to $Q > 0$. Moreover, we shall partition the matrix $H$ according to the partitioning of $R$ and write

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

In this case, as in the positive case studied in the previous section, our approach will be to find conditions for which the matrices

$$R^0_{e,i} = R + HP^0_i H^*,$$

have constant inertia for all $i \geq 0$, where $P^0_i$ is the zero-initial-condition solution to the Riccati recursion (8.1.1).

Now since $R^0_{e,0} = R$ and $R^0_{e,\infty} = R_e$ (assuming we have convergence), then a necessary condition for $R^0_{e,i}$ to have constant inertia for all $i \geq 0$, is that $R$ and $R_e$ have the same inertia. The surprising result, given below, is that this is sufficient as well. More precisely, if a stabilizing solution to the DARE with $Q > 0$ exists, then if $R$ and $R_e$ have the same inertia, the $R^0_{e,i}$ will have constant inertia for all $i$.

**Theorem 8.5.2 (Zero-Initial-Condition Convergence for $Q > 0$)** *Consider the Riccati recursion with zero initial condition*

$$P^0_{i+1} = FP^0_i F^* + GQG^* - FP^0_i H^*(R + HP^0_i H^*)^{-1} HFP^0_i F^*, \quad P^0_0 = 0, \tag{8.5.2}$$

*where*

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix} \quad , \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

*$\{F, H_1\}$ is detectable, $\{F, GQ^{1/2}\}$ is stabilizable and $Q > 0$. Suppose, moreover, that the DARE*

$$P = FPF^* + GQG^* - FPH^*(R + HPH^*)^{-1} HPF^*, \tag{8.5.3}$$

*has a stabilizing solution, $P$. Then if the matrices*

$$R \quad \text{and} \quad R_e = R + HPH^*$$

*have the same inertia, the sequence of matrices $\{P^0_i\}$ will converge to $P$.*

**Proof:** The main effort of this proof is to show, under the hypotheses of the Theorem, that the matrices $R_{e,i}^0$ have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$. Once this fact is established, the convergence of $P_i^0$ to $P$ follows immediately from Theorem 8.4.1.

To show that the $R_{e,i}^0$ have constant inertia we shall begin with what may appear as a digression. To this end, consider the time-invariant state-space model

$$\begin{cases} x_{i+1} &=& Fx_i + GQ^{1/2}u_i \\ y_i &=& H_1x_i + v_i \end{cases} \tag{8.5.4}$$

and suppose we would like to estimate the linear combination of the states,

$$s_i = H_2x_i, \tag{8.5.5}$$

using the observations $\{y_j\}_{j=-\infty}^{i}$. For this purpose, let us construct the estimator

$$\begin{cases} \hat{x}_{i+1} &=& F\hat{x}_i + K_1(y_i - H_1\hat{x}_i) \\ \hat{s}_i &=& H_2\hat{x}_i + H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1}(y_i - H_1\hat{x}_i) \end{cases} \tag{8.5.6}$$

where $P$ is the stabilizing solution to the DARE (8.5.3) and $K_1 = FPH_1^*(I_{p_1} + H_1PH_1^*)^{-1}$. It is now straightforward to write down a state-space model for the estimation errors $\tilde{s}_i = s_i - \hat{s}_i$ using (8.5.4) and (8.5.6), as follows

$$\begin{cases} \tilde{x}_{i+1} &=& (F - K_1H)\tilde{x}_i + GQ^{1/2}u_i - K_1v_i \\ \tilde{s}_i &=& H_2P(I + PH_1^*H_1)^{-1}(F - K_1H)\tilde{x}_i - H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1}v_i \end{cases} \tag{8.5.7}$$

where we have defined $\tilde{x}_i = x_i - \hat{x}_i$. Taking $z$-transforms, we may write the above as

$$\tilde{s}(z) = \begin{bmatrix} M_1(z) & M_2(z) \end{bmatrix} \begin{bmatrix} u(z) \\ v(z) \end{bmatrix}, \tag{8.5.8}$$

where

$$M_1(z) = H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}GQ^{1/2}$$

and

$$M_2(z) = -H_2PH_1^*(I_{p_1} + H_1PH_1^*)^{-1} - H_2(I + PH_1^*H_1)^{-1}(zI - F + K_1H_1)^{-1}K_1.$$

But from Theorem 7.5.3, since the stabilizing solution to the DARE is such that $R$ and $R_e$ have the same inertia, the transfer matrix $M(z)$ is strictly contractive on the unit circle, $|z|=1$, *i.e.*,

$$M(z)M^*(z^{-*}) < 1, \qquad \forall |z| = 1.$$

This implies that the mapping from the inputs $\{u_j, v_j\}_{j=-\infty}^{\infty}$ to the estimation error $\{\tilde{z}_j\}_{j=-\infty}^{\infty}$ is strictly contractive. To be more precise, the time-domain representation of this fact is that

$$\sup_{u,v \in l^2} \frac{\|\tilde{s}\|_2^2}{\|u\|_2^2 + \|v\|_2^2} < 1, \tag{8.5.9}$$

where the notation $\|\cdot\|_2$ denotes the 2-norm of a sequence

$$\|a\|_2 = \sum_{j=-\infty}^{\infty} |a_i|_2^2,$$

and $l^2$ is the space of all square-summable sequences.

Let us now fix a constant $i > 0$, and consider the space of those inputs $\{u_j\}$ and $\{v_j\}$ that are zero for all $j > i$. For this class of inputs we may write (8.5.9) as

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{\sum_{j=-\infty}^{i} |u_j|^2 + \sum_{j=-\infty}^{i} |v_j|^2} < 1,$$

or

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{\sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.5.10}$$

We now fix a value for $x_0$, the state vector at time zero, and choose the negative part of the inputs, *i.e.*, $\{u_j^m, v_j^m\}_{j=-\infty}^{-1}$, such that

$$\{u_j^m, v_j^m\}_{j=-\infty}^{-1} = \operatorname{argmin} \left[ \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 \right],$$

where the $\{u_j, v_j\}$ satisfy the state-space constraints

$$x_{i+1} = Fx_i + GQ^{1/2}u_i, \qquad -\infty < i < 0$$

and generate the given state vector $x_0$. The solution to the above minimization problem is wellknown in LQR control (see *e.g.*, [AM79, KS72]). For our purposes, it suffices to know that the value of the LQ (linear quadratic) cost function at its minimum is given by

$$\sum_{j=-\infty}^{-1} |u_j^m|^2 + \sum_{j=-\infty}^{-1} |v_j^m|^2 = \min_{\{u_j,v_j\}} \left[ \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 \right] = x_0^* \Pi^{-1} x_0,$$

where $\Pi$ is the unique positive semidefinite solution to the Lyapunov equation[1]

$$\Pi = F \Pi F^* + GQG^*. \tag{8.5.11}$$

With this particular choice of inputs, we may write (8.5.10) as

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1,$$

or, after discarding the portions of the numerator that involve $j < 0$ and $j > i$,

$$\frac{\sum_{j=0}^{i} |\tilde{s}_j|^2}{x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.5.12}$$

Note that in the above inequality the $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ are arbitrary. Moreover, it also readily follows that in (8.5.12) we may replace $\Pi$ with any matrix $P_0^{-1}$, such that

$$P_0^{-1} > \Pi \geq 0, \tag{8.5.13}$$

and write

$$\frac{\sum_{j=0}^{i} |\tilde{s}_j|^2}{x_0^* P_0^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.5.14}$$

Now it is wellknown in $H^\infty$ theory that if the above inequality holds for all $\{x_0, \{u_j, v_j\}_{j=0}^i\}$, then the matrices

$$R_{e,j} = R + HP_jH^*,$$

will have constant inertia for all $0 \leq j \leq i$, where $P_j$ satisfies the Riccati recursion

$$P_{j+1} = FP_jF^* + GQG^* - FP_jH^*(R + HP_jH^*)^{-1}HP_jF^*, \quad P_0.$$

(See Chapter 3. For completeness, a proof of this fact is also given in the Appendix.) Now since our choice of the integer, $i > 0$, was arbitrary we can conclude that the

[1] We have, for simplicity, assumed that $\Pi$ is invertible. The end result does not need this assumption.

$R_{e,j}$ must have constant inertia for *all* $j \geq 0$. Finally, it is straightforward to see that one choice of $P_0$ that satisfies (8.5.13) is $P_0 = 0$. With this choice of initial condition, we readily see that the matrices $R^0_{e,j}$ must have constant inertia for all $j \geq 0$, thus finishing the proof of the Theorem.

■

## 8.6 Convergence of the Riccati Recursion with Arbitrary $P_0$

We shall now study the problem of convergence of the Riccati recursion for the case of an arbitrary initial covariance $P_0$. To do so, instead of comparing, $P_i$, the solution to the Riccati recursion with arbitrary given initial condition, to $P$, the stabilizing solution of the DARE, which resulted in Theorem 8.4.1, we shall compare $P_i$ with the (converging) zero-initial-condition solution, $P_i^0$. The reason for this, as we shall presently see, is that it allows one to obtain more explicit, and easier to verify, conditions on $P_0$ for convergence than those that were obtained in Theorem 8.4.1.

Since we have already proven the convergence of $P_i^0$, the solution to the Riccati recursion (8.1.1) with zero initial condition, we can use the result of Lemma 8.3.2 to compare $P_i^0$ to $P_i$, the solution of the Riccati recursion to an arbitrary initial condition $P_0 = \Pi_0$. In this case we may write

$$P_{i+1} - P^0_{i+1} = \Phi^0_p(i+1,0)\left[I + \Pi_0 \mathcal{O}^0_i\right]^{-1} \Pi_0 \Phi^0_p(i+1,0)^*, \tag{8.6.1}$$

where $\Phi^0_p(i+1,0)$ is the state transition matrix for $F^0_{p,i} = F - K^0_{p,i}H$, and $\mathcal{O}^0_i$ is the observability Gramian for $\{F^0_{p,j}, (R^0_{e,j})^{-1/2}H\}$. Eq. (8.6.1) suggests that we need to study the structure of $\Phi^0_p(i+1,0)$ and $\mathcal{O}^0_i$ in more detail.

In order to do so, we shall prove a series of identities concerning the Riccati recursion and its so-called dual Riccati recursion. Although these identities will be useful for convergence studies, they are also of independent interest since they shed further light on the properties of the Riccati recursion itself.

### 8.6.1 The Dual Riccati Recursion

In this section, for simplicity, we shall assume that $S = 0$. In this case, the Riccati recursion (8.1.1) can be simply written as

$$P_{i+1} = FP_iF^* + GQG^* - FP_iH^*(R + HP_iH^*)^{-1}HP_iF^*. \tag{8.6.2}$$

If $S \neq 0$, we can always write the Riccati recursion as

$$\begin{aligned} P_{i+1} &= (F - GSR^{-1}H)P_i(F - GSR^{-1}H)^* + G(Q - SR^{-1}S^*)G^* - \\ &\quad (F - GSR^{-1}H)P_iH^*(R + HP_iH^*)^{-1}HP(F - GSR^{-1}H)^*, \end{aligned}$$

The above expression implies that by making the transformations

$$F \;\to\; F - GSR^{-1}H \quad , \quad Q \;\to\; Q - SR^{-1}S^* \quad \text{and} \quad R \;\to\; R$$

we can always reduce the Riccati recursion to one with $S = 0$. Thus, there is no loss of generality in making this assumption.

To present some identities concerning $\Phi_p^0(i{+}1, 0)$ and $\mathcal{O}_i^0$, we now define $P_i^{a,0}$ as the solution to the so-called *adjoint* or *dual* Riccati recursion with zero initial condition, *i.e.*,

$$P_{i+1}^{a,0} = F^*P_i^{a,0}F + H^*R^{-1}H - F^*P_i^{a,0}G(Q^{-1} + G^*P_i^{a,0}G)^{-1}G^*P_i^{a,0}F, \quad P0^{a,0} = 0. \tag{8.6.3}$$

Note that the dual Riccati is obtained by applying the transformations

$$F \to F^* \quad , \quad G \to H^* \quad , \quad H \to G^* \quad , \quad Q \to R^{-1} \quad \text{and} \quad R \to Q^{-1},$$

to the original Riccati (8.6.2).

We can now give our first identity. This identity is known, and can be motivated, in scattering theory [FKL76].

**Lemma 8.6.1 (State Transition Matrix for Dual Riccati)** *Denote by*

$$\Phi_p^{a,0}(i,0) = \begin{cases} \prod_{j=0}^{i-1}(F^* - K_{p,i}^{a,0}G^*) & i > 0 \\ I & i = 0 \end{cases} , \tag{8.6.4}$$

*the state transition matrix of* $F^* - K_{p,i}^{a,0} G^*$. *Then we have*

$$\Phi_p^0(i,0)^* = \Phi_p^{a,0}(i,0). \tag{8.6.5}$$

We shall defer the proof of Lemma 8.6.1 to the Appendix. For the time being, let us note an important consequence of Lemma 8.6.1 that relates the state transition matrices $\Phi_p^0(i,0)$ (corresponding to the zero-initial-condition solution of the Riccati recursion) and $\Phi_p(i,0) = (F - K_p H)^{i-1}$ (corresponding to any solution of the DARE).

**Lemma 8.6.2 (Relation Between $\Phi_p^0(i,0)$ and $\Phi_p(i,0) = (F - K_p H)^{i-1}$)** *We have the following identity*

$$\Phi_p^0(i,0) = (F - K_p H)^i \left[ I + P P_i^{a,0} \right], \tag{8.6.6}$$

*where*

$$\Phi_p^0(i,0) = (F - K_{p,i-1}^0 H) \ldots (F - K_{p,1}^0 H) F$$

*is the state transition matrix of* $F - K_{p,j}^0 H$ *corresponding to* $P_j^0$, *the zero initial condition solution to the Riccati recursion (8.1.1),* $P$ *is any solution of the DARE (7.2.1), and* $P_i^{a,0}$ *is the zero initial condition solution to the dual Riccati recursion*

$$P_{i+1}^{a,0} = F^* P_i^{a,0} F + H^* R^{-1} H - F^* P_i^{a,0} G (Q^{-1} + G^* P_i^{a,0} G)^{-1} G^* P_i^{a,0} F, \quad P_0^{a,0} = 0.$$

**Proof:** In this proof we assume that $S = 0$. As mentioned earlier, the results readily extend to the case $S \neq 0$. Let us first note the identity

$$F - K_p H = F - FPH^*(R + HPH^*)^{-1} = F(I + PH^* R^{-1} H)^{-1}. \tag{8.6.7}$$

We now proceed to prove (8.6.6) by induction. For $i = 1$ we have

$$\Phi_p^0(1,0) = F - K_{p,0}^0 H = F = \underbrace{F(I + PH^* R^{-1} H)^{-1}}_{F - K_p H} (I + P \underbrace{H^* R^{-1} H}_{P_1^{a,0}}) = (F - K_p H) \left[ I + P P_1^{a,0} \right],$$

as desired.

Let us now suppose that (8.6.6) is true for $i$. We then have

$$\begin{aligned}\Phi_p^0(i+1,0) = \Phi_p^{a,0}(i+1,0)^* &= \Phi_p^{a,0}(i,0)^* F_{p,i}^{a,0*} \\ &= \Phi_p^0(i,0) F_{p,i}^{a,0*} \\ &= (F-K_pH)^i (I+PP_i^{a,0}) F_{p,i}^{a,0*}, \qquad (8.6.8)\end{aligned}$$

where we have twice made use of (8.6.5). On the other hand, using the dual Riccati recursion and an argument similar to (8.6.7) we have

$$F_{p,i}^{a,0} = F^*(I + P_i^{a,0} GQG^*)^{-1}.$$

Using the above expression and proceeding with care yields

$$\begin{aligned}(I+PP_i^{a,0})F_{p,i}^{a,0*} &= (I+PP_i^{a,0})(I+GQG^*P_i^{a,0})^{-1}F \\ &= (I+PP_i^{a,0})\left[I + \underbrace{(P - F(I+PH^*R^{-1}H)^{-1}PF^*)}_{GQG^*} P_i^{a,0}\right]^{-1} F \\ &= (I+PP_i^{a,0})(I+PP_i^{a,0} - F(I+PH^*R^{-1}H)^{-1}PF^*P_i^{a,0})^{-1}F \\ &= F + \underbrace{F(I+PH^*R^{-1}H)^{-1}}_{F-K_pH} PF^*P_i^{a,0}(\underbrace{I+PP_i^{a,0} - F(I+PH^*R^{-1}H)^{-1}PF^*P_i^{a,0}}_{I+GQG^*P_i^{a,0}})^{-1}F \\ &= F + (F-K_pH)P\underbrace{F^*P_i^{a,0}(I+GQG^*P_i^{a,0})^{-1}F}_{P_{i+1}^{a,0}-H^*R^{-1}H} \\ &= F + (F-K_pH)P(P_{i+1}^{a,0} - H^*R^{-1}H) \\ &= F + (F-K_pH)PP_{i+1}^{a,0} - \underbrace{F(I+PH^*R^{-1}H)^{-1}PH^*R^{-1}H}_{F-(F-K_pH)} \\ &= (F-K_pH)PP_{i+1}^{a,0} + (F-K_pH) \\ &= (F-K_pH)(I+PP_{i+1}^{a,0}).\end{aligned}$$

If we now use the last of the above expressions in (8.6.8) we obtain

$$\Phi_p^0(i+1,0) = (F-K_pH)^{i+1}(I+PP_{i+1}^{a,0}),$$

which is the desired result.

∎

Our next identity concerns $\mathcal{O}_i^0$.

**Lemma 8.6.3 (Identity for $\mathcal{O}_i^0$)** *The quantity $\mathcal{O}_i^0$, defined in Lemma 8.3.2, can be expressed as*

$$\mathcal{O}_i^0 = P_{i+1}^{a,0}. \tag{8.6.9}$$

**Proof:** We begin by noting the identity

$$\begin{aligned} H^*(R_{e,j}^0)^{-1}H &= H^*(R + HP_j^0H^*)^{-1}H \\ &= H^*\left[R^{-1} - R^{-1}H(I + P_j^0H^*R^{-1}H)^{-1}P_j^0H^*R^{-1}\right]H \\ &= \left[I - H^*R^{-1}H(I + P_j^0H^*R^{-1}H)^{-1}P_j^0\right]H^*R^{-1}H \\ &= \left[I - H^*(R + HP_j^0H^*)^{-1}HP_j^0\right]H^*R^{-1}H \\ &= \left[F^* - H^*(R + HP_j^0H^*)^{-1}HP_j^0F^*\right]F^{-*}H^*R^{-1}H \\ &= F_{p,i}^{0*}F^{-*}H^*R^{-1}H \end{aligned}$$

Now plugging the last of the above equalities into (8.3.8), the expression for $\mathcal{O}_i^0$, yields

$$\begin{aligned} \mathcal{O}_i^0 &= \sum_{j=0}^{i} \Phi_p^0(j,0)^*F_{p,i}^{0*}F^{-*}H^*R^{-1}H\Phi_p^0(j,0) \\ &= \sum_{j=0}^{i} \Phi_p^0(j+1,0)^*F^{-*}H^*R^{-1}H\Phi_p^0(j,0) \\ &= \sum_{j=0}^{i} \Phi_p^{a,0}(j+1,0)F^{-*}H^*R^{-1}H\Phi_p^{a,0}(j,0)^* \\ &= \sum_{j=0}^{i} \Phi_p^{a,0}(j+1,1)H^*R^{-1}H\Phi_p^{a,0}(j,0)^*. \end{aligned} \tag{8.6.10}$$

On the other hand, using the dual Riccati recursion (8.6.3), we may write

$$\begin{aligned} P_{i+1}^{a,0} - P_i^{a,0} &= F_{p,i}^{a,0}(P_i^{a,0} - P_{i-1}^{a,0})F_{p,i-1}^{a,0*} \\ &= \Phi_p^{a,0}(i+1,1)\underbrace{(P_1^{a,0} - P_0^{a,0})}_{H^*R^{-1}H}\Phi_p^{a,0}(i,0)^*, \end{aligned}$$

which implies that

$$P_{i+1}^{a,0} = \sum_{j=0}^{i} \Phi_p^{a,0}(j+1,1)H^*R^{-1}H\Phi_p^{a,0}(j,0)^*.$$

Comparing this last expression with (8.6.10) yields the desired result.

∎

## 8.7 Conditions on $P_0$ for Convergence

We now have most of the ingredients necessary to study the convergence of the solution of the Riccati recursion for an arbitrary initial condition $P_0 = \Pi_0$, by comparing it with the zero-initial condition solution, $P_i^0$. However, in order to obtain explicit conditions on $P_0$ for convergence, as opposed to the generic conditions of Theorem 8.4.1, we need to pay the price of specializing to the positive case, or the case where $Q - SR^{-1}S^* > 0$. Although these may seem as restrictions, we should remark that they are very important special cases and include the classes of Riccati recursions encountered in both least-squares and $H^\infty$ estimation and control.

### 8.7.1 The Case Positive Case: $R > 0$ and $Q - SR^{-1}S^* > 0$

As frequently mentioned earlier, we may take $S = 0$, so that our assumption becomes $R > 0$ and $Q > 0$. In this case, we can make further assertions regarding $P_i^{a,0}$ the zero-initial-condition solution to the dual Riccati recursion.

**Lemma 8.7.1 (Monotonicity and Boundedness of $P_i^{a,0}$)** *Consider the zero-initial-condition dual Riccati recursion*

$$P_{i+1}^{a,0} = F^*P_i^{a,0}F + H^*R^{-1}H - F^*P_i^{a,0}G(Q^{-1} + G^*P_i^{a,0}G)^{-1}G^*P_i^{a,0}F, \quad P_0^{a,0} = 0$$

*and suppose that $Q > 0$, $R > 0$ and $\{F, GQ^{1/2}\}$ is stabilizable. Then we have the following results:*

(a) *The $\{P_i^{a,0}\}$ form a nondecreasing sequence of Hermitian matrices.*

(b) *The $\{P_i^{a,0}\}$ are bounded from above,* i.e.*, there exists a Hermitian matrix $\Pi$ such that*

$$P_i^{a,0} \leq \Pi,$$

*for all $i \geq 0$.*

**Proof:** To prove the monotonicity we shall proceed by induction. For $i = 0$, we have

$$P_1^{a,0} = H^*R^{-1}H \geq 0 = P_0^{a,0}.$$

Now suppose $P_i^{a,0} \geq P_{i-1}^{a,0}$. We shall show $P_{i+1}^{a,0} \geq P_i^{a,0}$. To this end, let us apply (8.3.2) to the dual Riccati recursion and write

$$P_i^{a,0} - P_{i+1}^{a,0} = F_{p,i+1}^{a,0}\left[(P_{i-1}^{a,0} - P_i^{a,0}) - (P_{i-1}^{a,0} - P_i^{a,0})(R_{e,i}^{a,0})^{-1}(P_{i-1}^{a,0} - P_i^{a,0})\right]F_{p,i+1}^{a,0*}.$$

Now since $P_i^{a,0} \geq P_0^{a,0} = 0$, we have $R_{e,i}^{a,0} = R + HP_i^{a,0}H^* > 0$. Therefore both terms on the RHS of the above expression are negative semi-definite. Thus, we can conclude

$$P_{i+1}^{a,0} \geq P_i^{a,0},$$

which shows that the $\{P_i^{a,0}\}$ form a nondecreasing sequence of Hermitian matrices.

To prove boundedness, we shall use the stabilizability of $\{F, GQ^{1/2}\}$. Since $\{F, GQ^{1/2}\}$ is stabilizable, $\{F^*, Q^{*/2}G^*\}$ is detectable, meaning that we can choose a gain matrix $K$ such that $F^* - KQ^{*/2}G^*$ is stable. With this choice of $K$, we can write the dual Riccati recursion as

$$P_{i+1}^{a,0} = (F^*-KQ^{*/2}G^*)P_i^{a,0}(F^*-KQ^{*/2}G^*)^*+H^*R^{-1}H+K^*K-(K_{p,i}^{a,0}-KQ^{*/2})R_{e,i}^{a,0}(K_{p,i}^{a,0}-KQ^{*/2})^*.$$

Since $F^*-KQ^{*/2}G^*$ is stable, and $H^*R^{-1}H+K^*K \geq 0$, there exists a unique positive semi-definite solution, $\Pi$, to the Lyapunov equation.

$$\Pi = (F^* - KQ^{*/2}G^*)\Pi(F^* - KQ^{*/2}G^*)^* + H^*R^{-1}H + K^*K.$$

We shall now prove by induction that $\Pi$ is an upper bound on the sequence $\{P_i^{a,0}\}$. The case $i = 0$ is trivial since

$$\Pi \geq 0 = P_0^{a,0}.$$

Suppose now that $\Pi \geq P_i^{a,0}$. We shall show that $\Pi \geq P_{i+1}^{a,0}$. Indeed, subtracting the Riccati recursion for $P_i^{a,0}$ from the Lyapunov recursion for $\Pi$, yields

$$\Pi-P_{i+1}^{a,0} = (F^*-KQ^{*/2}G^*)(\Pi-P_i^{a,0})(F^*-KQ^{*/2}G^*)^*+(K_{p,i}^{a,0}-KQ^{*/2})R_{e,i}^{a,0}(K_{p,i}^{a,0}-KQ^{*/2})^*.$$

Now both terms on the RHS of the above expression are positive semi-definite. This implies that

$$\Pi \geq P_{i+1}^{a,0},$$

which is our desired result.

■

We can now make some further claims regarding the asymptotic behaviour of the state transition matrix $\Phi_p^0(i,0)$.

**Lemma 8.7.2 (Convergence of $\Phi_p^0(i,0)$)** *If $\{F,H\}$ is detectable and $\{F,GQ^{1/2}\}$ is stabilizable, then*

$$\lim_{i\to\infty} \Phi_p^0(i,0) = 0. \tag{8.7.1}$$

**Proof:** The proof uses the identity

$$\Phi_p^0(i,0) = (F-K_pH)^i \left[I + PP_i^{a,0}\right],$$

established in Lemma 8.6.2. The desired result follows immediately from the fact that the $P_i^{a,0}$ are bounded from above and below,

$$0 \le P_i^{a,0} \le \Pi < \infty, \quad \forall i \ge 0,$$

and that $F - K_pH$ is stable so that

$$\lim_{i\to\infty}(F-K_pH)^i = 0.$$

■

Let us now return to the identity (8.6.1) and write it as

$$P_{i+1} - P_{i+1}^0 = \Phi_p^0(i+1,0)\left[I + \Pi_0 P_{i+1}^{a,0}\right]^{-1}\Pi_0\Phi_p^0(i+1,0)^*, \tag{8.7.2}$$

where we have also used $\mathcal{O}_i^0 = P_{i+1}^{a,0}$. Now, from Lemma 8.7.2, we know that when $\{F,H\}$ is detectable and $\{F,GQ^{1/2}\}$ is stabilizable $\lim_{i\to\infty}\Phi_p^0(i,0) = 0$. Therefore from (8.7.2) we conclude that if the matrix $\left[I+\Pi_0 P_{i+1}^{a,0}\right]^{-1}$ is uniformly bounded for all $i$, then

$$\lim_{i\to\infty} P_{i+1} = \lim_{i\to\infty} P_{i+1}^0 = P.$$

In other words a sufficient condition for the convergence of $P_i$ to $P$, the unique positive semi-definite solution to the DARE is that the matrix

$$I + \Pi_0 P_i^{a,0}, \tag{8.7.3}$$

be uniformly nonsingular for all $i$. We shall therefore continue by studying this matrix in more detail.

Let us first suppose that $\Pi_0 \geq 0$. This is, of course, a natural assumption since in estimation problems $\Pi_0$ is the covariance matrix of the random initial state vector $x_0$, and in control problems it is the inverse of the positive weight matrix in the LQ cost function associated with the initial state. In either case, we can perform the factorization $\Pi_0 = \Pi_0^{1/2}\Pi_0^{*/2}$ and write

$$\begin{aligned}\det(I + \Pi_0 P_i^{a,0}) &= \det(I + \Pi_0^{1/2}\Pi_0^{*/2}P_i^{a,0}) \\ &= \det(I + \Pi_0^{*/2}P_i^{a,0}\Pi_0^{1/2}),\end{aligned} \tag{8.7.4}$$

where we have used the identity $\det(I + AB) = \det(I + BA)$. Now since $P_i^{a,0}$ is the solution to the dual Riccati recursion (8.6.3) with zero initial value, it is a positive semi-definite matrix for all $i$. The same is therefore true of the congruent matrix $\Pi_0^{*/2}P_i^{a,0}\Pi_0^{1/2}$, from which we conclude that $I + \Pi_0^{*/2}P_i^{a,0}\Pi_0^{1/2}$ is positive *definite* for all $i$. Thus, in view of (8.7.4), $I + \Pi_0 P_i^{a,0}$ is uniformly nonsingular for all $i$ and we have the following wellknown result [Wil71b, Kuc72, Won68a, BRC72].

**Lemma 8.7.3 (Convergence for $P_0 \geq 0$)** *Consider the Riccati recursion with positive semi-definite initial condition*

$$P_{i+1} = FP_iF^* + GQG^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0 \geq 0. \tag{8.7.5}$$

*If $Q > 0$, $R > 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable then $P_i$ converges to the unique positive semi-definite matrix, $P$, that satisfies the discrete-time algebraic Riccati equation*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

Although the above result is quite satisfying, we still have not fully exploited the condition that (8.7.3) be uniformly nonsingular for all $i \geq 0$. It would therefore be interesting to explore whether it is possible to relax the nonnegativity condition on

$\Pi_0$, while still guaranteeing convergence to $P$. We should note that this is not just academic speculation since it will have certain implications for the numerical stability of the Riccati recursion. Indeed, it will show that even if the Riccati variable $P_i$ loses its positive semi-definiteness (due to say, numerical errors) it may still converge to the solution of the DARE.

When $\Pi_0$ is indefinite it no longer yields a factorization of the form described above. However, we can still factorize it as

$$\Pi_0 = \Pi_0^{1/2} J \Pi_0^{*/2},$$

where $J$ is a *signature* matrix whose diagonal elements represent the inertia (or number of positive and negative eigenvalues) of $\Pi_0$.

We now have the following sequence of equalities:

$$\begin{aligned}
\det(I + \Pi_0 P_i^{a,0}) &= \det(I + \Pi_0^{1/2} J \Pi_0^{*/2} P_i^{a,0}) \\
&= \det(I + J \Pi_0^{*/2} P_i^{a,0} \Pi_0^{1/2}) \\
&= \det(J) \cdot \det(J + \Pi_0^{*/2} P_i^{a,0} \Pi_0^{1/2}), \quad \text{since } J^2 = I.
\end{aligned}$$

Using the last of the above expressions, we see that (8.7.3) is uniformly nonsingular if, and only if, the Hermitian matrices $T_i = J + \Pi_0^{*/2} P_i^{a,0} \Pi_0^{1/2}$ are uniformly nonsingular for all $i \geq 0$. Since the matrices $\Pi_0^{*/2} P_i^{a,0} \Pi_0^{1/2}$ are non-decreasing ($P_{i+1}^{a,0} \geq P_i^{a,0}$), the matrices $T_i$ are also non-decreasing. It also follows that *all* the eigenvalues of the matrices $T_i$ are non-decreasing as well. Therefore, a sufficient condition for the $T_i$ to be uniformly nonsingular for all $i \geq 0$, is that all of the negative eigenvalues of $T_0 = J$ remain bounded above by zero as $i$ increases. This, of course, means that the matrices $T_0 = J$ and the $\{T_i\}$ should have the same inertia for all $i \geq 0$. Moreover, the $\{T_i\}$ will have the same inertia as $J$ for all $i \geq 0$ if, and only if,

$$\lim_{i \to \infty} T_i = T = J + \Pi_0^{*/2} P^a \Pi_0^{1/2}, \tag{8.7.6}$$

has the same inertia as $J$. [Note that $P^a$, the limiting solution of the dual Riccati, exists since we have assumed that $\{F, GQ^{1/2}\}$ is stabilizable.]

We have therefore found a sufficient condition for checking whether (8.7.3) is uniformly nonsingular for all $i \geq 0$, in terms of checking the inertia of a *single* matrix

$T$. To facilitate checking the inertia of $T = J + \Pi_0^{*/2} P^a \Pi_0^{1/2}$, consider the following block matrix

$$\begin{bmatrix} -I & (P^a)^{*/2}\Pi_0^{1/2} \\ \Pi_0^{*/2}(P^a)^{1/2} & J \end{bmatrix}, \tag{8.7.7}$$

where we have defined $P^a = (P^a)^{1/2}(P^a)^{*/2}$. Now two different (lower-upper and upper-lower) block triangular factorizations of the above matrix yield

$$\begin{bmatrix} -I & 0 \\ 0 & T \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -I - (P^a)^{*/2}\Pi_0(P^a)^{1/2} & 0 \\ 0 & J \end{bmatrix}. \tag{8.7.8}$$

Due to Sylvester's law of inertia, the above two congruent matrices must have the same inertia. Therefore the matrices $T$ and $J$ will have the same inertia if, and only if, the matrices $-I$ and $-I - (P^a)^{*/2}\Pi_0(P^a)^{1/2}$ have the same inertia. In other words, if, and only if,

$$I + (P^a)^{*/2}\Pi_0(P^a)^{1/2} > 0.$$

We thus have our desired result.

**Theorem 8.7.1 (Convergence of Riccati Recursion with Indefinite $P_0$)** *Consider the Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - K_{p,i}R_{e,i}K_{p,i}^*, \quad P_0 \tag{8.7.9}$$

*where $R > 0$, $Q - SR^{-1}S^* > 0$, $\{F, H\}$ is detectable and $\{F - GSR^{-1}H, GQ - GSR^{-1}S^*\}$ is stabilizable. Suppose, moreover, that the initial condition $P_0$ is such that*

$$I + (P^a)^{*/2}P_0(P^a)^{1/2} > 0, \tag{8.7.10}$$

*where $P^a$ is the unique positive semi-definite solution to the dual Riccati recursion. Then $P_i$ converges to the unique positive semi-definite matrix, $P$, that satisfies the discrete-time algebraic Riccati equation*

$$P = FPF^* + GQG^* - (FPH^* + GS)(R + HPH^*)^{-1}(FPH^* + GS)^*.$$

The above result gives a very general description of the conditions for convergence of the Riccati recursion in the positive case. Note that the detectability and stabilizability conditions are the same as those required for the existence of a unique stabilizing solution to the DARE (see Theorem 7.5.1). It is therefore not possible to weaken these conditions very much. Detectability is a necessary condition since otherwise the solution of the Riccati recursion may diverge to infinity. If stabilizability does not hold, then the DARE will have more than one positive semi-definite solution and clearly if we start the recursion with one of these solutions we will always remain at the same value. Therefore, there cannot be convergence to a unique value.

Moreover, (8.7.10) gives a rather general description of the basin of attraction of the positive semi-definite solution of the Riccati recursion. Under additional conditions (8.7.10) can be made more explicit. The result is given below. (The proof is simple and is omitted.)

**Corollary 8.7.1 (Conditions for Convergence)** *When $\{F, H\}$ is observable, the condition (8.7.10) is equivalent to*

$$P_0 > -(P^a)^{-1}. \tag{8.7.11}$$

*Moreover, in this case we have*

$$-(P^a)^{-1} = P_- \triangleq \text{the infimum over all solutions to the DARE (8.7.9).} \tag{8.7.12}$$

Before closing this section, we should mention an interesting interpretation of the condition (8.7.10) required for the convergence of the Riccati recursion from a given initial condition $P_0$.

**Corollary 8.7.2 (Interpretation of Condition (8.7.10))** *The condition*

$$I + (P^a)^{*/2} P_0 (P^a)^{1/2} > 0,$$

*sufficient for the convergence of the Riccati recursion from a given initial condition $P_0$, is equivalent to the condition that*

$$R_{e,i} = R + HP_iH^* > 0,$$

*for all $i \geq 0$, where $P_i$ is the solution to the Riccati recursion with initial condition $P_0$.*

**Proof:** The proof is similar to the proof of Theorem 8.4.1, part (b), and is omitted for brevity.

■

Note that the above result implies that we can check the positivity of the matrices $R_{e,i}$ for all $i \geq 0$, in terms of the positivity of a single matrix $I + (P^a)^{*/2} P_0 (P^a)^{1/2}$. However, more interesting is the stochastic interpretation of this result.

To this end, consider the stochastic state-space model

$$\left\{ \begin{array}{rcl} \mathbf{x}_{i+1} & = & F\mathbf{x}_i + G\mathbf{u}_i, \quad i \geq 0 \\ \mathbf{y}_i & = & H\mathbf{x}_i + \mathbf{v}_i \end{array} \right. \tag{8.7.13}$$

where the $\{\mathbf{x}_0, \mathbf{u}_i, \mathbf{v}_i\}$ are zero-mean random variables with Gramian (covariance) matrices

$$\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix}, \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix} \rangle = E \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_i \\ \mathbf{v}_i \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u}_j \\ \mathbf{v}_j \end{bmatrix}^* = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & Q & S \\ 0 & S^* & R \end{bmatrix}. \tag{8.7.14}$$

Then it is wellknown (see *e.g.*, [Kai81]) that the $R_{e,i}$ are the Gramians of the, so-called, *innovations process*, $\mathbf{e}_i$, associated with the state-space model (8.7.13-8.7.14). Therefore, the condition (8.7.10) means that we can guarantee convergence for indefinite (and even negative semi-definite) initial conditions, $P_0 = \Pi_0$, as long as the innovations process associated with the state-space model (8.7.13-8.7.14) behaves as a *true* stochastic process, *i.e.*, as one that has positive definite Gramian (covariance) matrix. Finally, since the innovations $\mathbf{e}_i$ are equivalent to the $\mathbf{y}_i$ (in a stochastic sense — see [Jaz70, AM79, Kai81]), we can guarantee convergence for all $P_0 = \Pi_0$ for which the output of the state-space model (8.7.13-8.7.14) behaves as a true stochastic process.

### 8.7.2 The Case of $Q - SR^{-1}S^* > 0$

The special of case $Q - SR^{-1}S^* > 0$ or, equivalently, of $Q > 0$, (since we shall take $S = 0$) is of interest because it is the case that arises in $H^\infty$ filtering and control. To study the convergence in this case, we can begin with Eq. (8.7.2) which we repeat here for convenience,

$$P_{i+1} - P^0_{i+1} = \Phi^0_p(i+1,0)\left[I + \Pi_0 P^{a,0}_{i+1}\right]^{-1} \Pi_0 \Phi^0_p(i+1,0)^*.$$

Using the generalized square-root of the initial condition $\Pi_0 = \Pi_0^{1/2} J \Pi_0^{*/2}$, where $J$ is a signature matrix, we may write the above equation as

$$P_{i+1} - P^0_{i+1} = \Phi^0_p(i+1,0)\Pi_0^{1/2}\left[J + \Pi_0^{*/2} P^{a,0}_{i+1} \Pi_0^{1/2}\right]^{-1} \Pi_0^{*/2} \Phi^0_p(i+1,0)^*. \qquad (8.7.15)$$

The following result is now immediate.

**Lemma 8.7.4 (Comparison with Zero-Initial-Condition Solution)** *Consider the Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - FP_iH^*(R + HP_iH^*)^{-1}HPF^*, \quad P_0 = \Pi_0$$

*where $Q > 0$ and $\{F, GQ^{1/2}\}$ is stabilizable. Moreover, suppose that the associated DARE has a unique stabilizing solution, $P$, and that the zero-initial-condition solution $P_i^0$ converges to $P$. Now if the matrices*

$$T_i = J + \Pi_0^{*/2} P^{a,0}_{i+1} \Pi_0^{1/2}$$

*are uniformly nonsingular for all $i \geq 0$, where $\Pi_0 = \Pi_0^{1/2} J \Pi_0^{*/2}$, and $P_i^{a,0}$ is the zero-initial-condition solution to the dual Riccati recursion*

$$P^{a,0}_{i+1} = F^* P^{a,0}_i F + H^* R^{-1} H - F^* P^{a,0}_i G(Q^{-1} + G^* P^{a,0}_i G)^{-1} G^* P^{a,0}_i F, \quad P^{a,0}_0 = 0,$$

*then $P_i$ will converge to $P$. Two equivalent sufficient conditions for the $T_i$ to be uniformly nonsingular for all $i \geq 0$ are that*

*(a) The matrices $T_i$ have constant inertia for all $i \geq 0$.*

*(b) The matrices $R_{e,i} = R + HP_iH^*$ have constant inertia (equal to the inertia of $R_e$) for all $i \geq 0$.*

**Proof:** To prove the first part of the Theorem we shall prove that

$$\lim_{i\to\infty} \Phi_p^0(i,0) = 0. \tag{8.7.16}$$

Once this is established, the convergence of $P_i$ to $P$ follows from (8.7.15), since $\left[J + \Pi_0^{*/2} P_{i+1}^{a,0} \Pi_0^{1/2}\right]^{-1}$ is bounded and $P_i^0$ converges to $P$.

To prove (8.7.16) we can repeatedly use (8.3.1) to write

$$P_{i+1}^0 - P_i^0 = F_{p,i}^0 (P_i^0 - P_{i-1}^0) F_{p,i-1}^{0*} = \Phi_p^0(i+1,1) P_1^0 \Phi_p^0(i,0)^* = \Phi_p^0(i+1,1) GQG^* \Phi_p^0(i,0)^*,$$

where we have used $P_1^0 = GQG^*$. Now we may write

$$(P_{i+1}^0 - P_i^0) F_{p,i}^{0*} = \Phi_p^0(i+1,1) GQG^* \Phi_p^0(i+1,0)^*,$$

which, using Lemma 8.6.1, is equivalent to

$$(P_{i+1}^0 - P_i^0) F_{p,i}^{0*} = \Phi_p^{a,0}(i+1,1)^* GQG^* \Phi_p^{a,0}(i+1,0).$$

This last expression can in turn be written as

$$F^*(P_{i+1}^0 - P_i^0) F_{p,i}^{0*} = \Phi_p^{a,0}(i+1,0)^* GQG^* \Phi_p^{a,0}(i+1,0),$$

and, using the facts that $\lim_{i\to\infty}(P_{i+1}^0 - P_i^0) = 0$ and $Q > 0$, this implies that we must have

$$\lim_{i\to\infty} Q^{*/2} G^* \Phi_p^{a,0}(i+1,0) = 0. \tag{8.7.17}$$

The quantity $\Phi_p^{a,0}(i,0) GQ^{1/2}$ is readily recognized to be the matrix that maps the initial state to the output for the linear system

$$\left\{ \begin{array}{rcl} x_{i+1} & = & (F^* - K_{p,i}^{a,o} G^*) x_i = (F^* - F^* P_i^{a,0} G (R_{e,i}^{a,0})^{-1} G^*) x_i \\ y_i & = & Q^{*/2} G^* x_i \end{array} \right. . \tag{8.7.18}$$

Therefore, (8.7.17) implies that $y_i$ tends to zero for all initial state vectors $x_0$. But (8.7.18) can be rewritten as

$$\left\{ \begin{array}{rcl} x_{i+1} & = & F^* x_i - F^* P_i^{a,0} G (R_{e,i}^{a,0})^{-1} Q^{-*/2} y_i \\ y_i & = & Q^{*/2} G^* x_i \end{array} \right. .$$

and since $\{F^*, Q^{*/2}G^*\}$ is detectable ($\{F, GQ^{1/2}\}$ is stabilizable), $y_i$ cannot tend to zero unless $x_i$ tends to zero as well. Hence, the state transition matrix

$$\Phi_p^{a,0}(i,0) = \Phi_p^0(i,0)^*,$$

tends to zero as $i \to \infty$. (This is just a reformulation of the fact that output feedback cannot destroy detectability — see *e.g.*, [Kai80].)

To prove the second part, we note that if the $T_i$ have constant inertia (equal to the inertia of $J$) for all $i \geq 0$, then they will be uniformly nonsingular as well. Finally, the equivalence of the sufficient conditions (a) and (b) is the same as the proof for Theorem 8.4.1.

■

**Remarks:**

(i) Note that in the above Lemma we still require to check the uniform nonsingularity, or constant inertia, of the sequence of matrices $T_i = J + \Pi_0^{*/2} P_{i+1}^{a,0} \Pi_0^{1/2}$. We saw in the positive case that we could replace checking the inertia of the above infinite sequence of matrices with checking the inertia of a single matrix. This was possible because we were able to show that in the positive case the sequence of matrices $P_i^{a,0}$ were nondecreasing. Since in the case where $Q > 0$, it is not generally true that the $P_i^{a,0}$ are nondecreasing, it does not seem that we can replace the checking of the inertia of the sequence of matrices $\{T_i\}$ with that of a single matrix. Indeed we have not been able to do so.

(ii) The sufficient condition (b) in Lemma 8.7.4 has an interesting interpretation since it is the exact condition that is required for the existence of $H^\infty$ filters or full information controllers (see *e.g.*, [GL95, HSK96b]). Therefore, if a time-varying $H^\infty$ filter, or full information controller, exists for all times $i \geq 0$, then the corresponding Riccati recursion converges, meaning that the time-varying $H^\infty$ filter, or controller, will converge to a time-invariant one.

Despite the result of the preceding Lemma, it is still possible to give some simple description of the basin of attraction of the Riccati recursion when $Q > 0$. However, to do so we need to take a slightly different line of attack. The approach used, follows the one presented in the proof of Theorem 8.5.2, and the result is given below.

**Theorem 8.7.2 (Conditions on $P_0$ for Convergence in the $Q > 0$ Case)** *Consider the Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - FP_iH^*(R + HP_iH^*)^{-1}HFP_iF^*, \quad P_0, \tag{8.7.19}$$

*where*

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix} \quad , \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

*$\{F, H_1\}$ is detectable, $\{F, GQ^{1/2}\}$ is stabilizable and $Q > 0$. Suppose, moreover, that the DARE*

$$P = FPF^* + GQG^* - FPH^*(R + HPH^*)^{-1}HPF^*, \tag{8.7.20}$$

*has a stabilizing solution, $P$, such that the matrices $R$ and $R_e = R + HPH^*$ have the same inertia. Then if the initial condition, $P_0$, is such that*

$$0 \leq P_0 \leq \Pi,$$

*where $\Pi$ is the unique positive (semi)definite solution of the Lyapunov equation,*

$$\Pi = F\Pi F^* + GQG^*,$$

*the sequence of matrices $\{P_i\}$ will converge to $P$.*

**Proof:** Note that due to Theorem 7.5.3, since $P$ is the stabilizing solution to the DARE it is positive semi-definite. Now since $R$ and $R_e$ have the same inertia, Theorem 7.5.3 says that the matrix $F - K_1H$ is stable, where

$$K_1 = FPH_1^*(I_{p_1} + H_1PH_1^*)^{-1}.$$

Now if we assume that $\{F, GQ^{1/2}\}$ is controllable, then the Lyapunov equation, $\Pi = F\Pi F^* + GQG^*$, equation implies that $\Pi > 0$. [We should remark that the controllability assumption is for simplicity and that the end result does not require this assumption.]

Moreover, from the proof of Theorem 8.5.2 for all times $i \geq 0$, we may write

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{\sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1, \tag{8.7.21}$$

where $\tilde{s}_i = s_i - \hat{s}_i$ is the estimation error associated with estimating $s_i$ (using some estimator that will not concern us here) from the observations $\{y_j\}_{j=-\infty}^{i}$, and where $s_i$ and $y_i$ have the state-space model

$$\left\{ \begin{array}{rcl} x_{i+1} & = & Fx_i + GQ^{1/2} \\ y_i & = & H_1 x_i + v_i \\ s_i & = & H_2 x_i \end{array} \right. \tag{8.7.22}$$

Now for all nonzero sequences $\{u_j\}$ and $\{v_j\}$, we may write (8.7.21) as

$$\left( \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 - \sum_{j=-\infty}^{-1} |\tilde{s}_j|^2 \right) + \left( \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2 - \sum_{j=0}^{\infty} |\tilde{s}_j|^2 \right) > 0. \tag{8.7.23}$$

Now let us fix an initial state $x_0$ and choose the sequence $\{u_j, v_j\}_{j=-\infty}^{-1}$ such that

$$\{u_j^m, v_j^m\}_{j=-\infty}^{-1} = \text{argmin}_{\{u_j,v_j\}} \left[ \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 \right], \tag{8.7.24}$$

where the $\{u_j, v_j, \tilde{s}_j\}$ satisfy the state-space constraints

$$x_{i+1} = Fx_i + GQ^{1/2}, \qquad -\infty < i < 0 \tag{8.7.25}$$

and generate the initial state vector $x_0$. The solution to the above minimization problem is wellknown in quadratic game theory and LQR control (see *e.g.*, [AM79, KS72, BO82]). For our purposes it suffices to know that the value of the LQ (linear quadratic) cost function at its minimum is given by

$$\sum_{j=-\infty}^{-1} |u_j^m|^2 + \sum_{j=-\infty}^{-1} |v_j^m|^2 - = \min_{\{u_j,v_j\}} \left[ \sum_{j=-\infty}^{-1} |u_j|^2 + \sum_{j=-\infty}^{-1} |v_j|^2 - \right] = x_0^* \Pi^{-1} x_0,$$

where $\Pi$ is the solution to the Lyapunov equation, $\Pi = F\Pi F^* + GQG^*$.

With this particular choice of inputs, we may write (8.7.23) as

$$x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2 - \sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2 > 0,$$

or,

$$\frac{\sum_{j=-\infty}^{\infty} |\tilde{s}_j|^2}{x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1,$$

and after discarding that portions of the numerator that involve $j > i$ and $j < 0$,

$$\frac{\sum_{j=0}^{i} |\tilde{s}_j|^2}{x_0^* \Pi^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.7.26}$$

Note that in the above inequality the $\{x_0, \{u_j\}_{j=0}^{i}, \{v_j\}_{j=0}^{i}\}$ are arbitrary. Moreover, it also readily follows that in the above relation we can replace $P^{-1}$ with any matrix $P_0^{-1}$ such that

$$P_0^{-1} \geq \Pi^{-1},$$

and write

$$\frac{\sum_{j=0}^{i} |\tilde{s}_j|^2}{x_0^* P_0^{-1} x_0 + \sum_{j=0}^{i} |u_j|^2 + \sum_{j=0}^{i} |v_j|^2} < 1. \tag{8.7.27}$$

Now from Lemma 8.A.1 in the Appendix, we can conclude that the matrices

$$R_{e,j} = R + H P_j H^*,$$

must have constant inertia for all $0 \leq j \leq i$, where $P_j$ satisfies the Riccati recursion (8.7.19). Now since the above is true for all $i \geq 0$, we conclude that with this choice of initial condition the matrices $R_{e,i}$ must have constant inertia for all $i \geq 0$. Therefore, using the result of Theorem 8.4.1, we see that for all initial conditions such that

$$0 \leq P_0 \leq \Pi,$$

the solution to the Riccati recursion, $P_i$, will converge to the unique stabilizing solution, $P$.

∎

## 8.8 Conclusion

In this chapter we studied the asymptotic behaviour of the Riccati recursion for time-invariant state-space models. The main result is a general inertia condition for the convergence of the Riccati recursion variable to the unique stabilizing solution of the associated DARE (assuming such a solution exists). We also gave some more simple and more explicit conditions in the special cases encountered in LQR and $H^\infty$ control.

## 8.A Finite-Horizon $H^\infty$ Problems

We begin by proving a Lemma that was used in the proof of Theorem 8.5.2. This Lemma is wellknown in $H^\infty$ theory (see *e.g.*, [GL95, HSK96b] and Chapter 3) and is given below.

**Lemma 8.A.1 (Finite-Horizon $H^\infty$ Filtering)** *Consider the state-space model*

$$\begin{cases} x_{j+1} &= Fx_j + GQ^{1/2}u_j, \quad 0 \le j \le i \\ y_j &= H_1 x_j + v_j \end{cases}$$

*where the $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ are unknown and the $\{y_j\}_{j=0}^i$ are known measurements. Suppose we would like to estimate the linear combination of the states,*

$$s_j = H_2 x_j,$$

*using the observations $\{y_k\}_{k=0}^j$, and designate this estimate by $\hat{s}_j$. If it is possible to choose the estimates $\{\hat{s}_j\}_{j=0}^i$ such that*

$$\frac{\sum_{j=0}^i |s_j - \hat{s}_j|^2}{x_0^* P_0^{-1} x_0 + \sum_{j=0}^i |u_j|^2 + \sum_{j=0}^i |v_j|^2} < 1, \tag{8.A.1}$$

*for all $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$, then the matrices*

$$R_{e,j} = R + HP_jH^*,$$

*will have constant inertia for all $0 \le j \le i$, where $P_j$ satisfies the Riccati recursion*

$$P_{j+1} = FP_jF^* + GQG^* - FP_jH^*(R + HP_jH^*)^{-1}HP_jF^*, \quad P_0,$$

*and*

$$R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix} \quad , \quad H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}.$$

To prove the above Lemma, it will be useful to first establish the following intermediate result.

**Lemma 8.A.2 (Minimization of Indefinite Quadratic Forms)** *Consider the quadratic form*

$$J_i = x_0^* P_0^{-1} x_0 + \sum_{j=0}^{i} u_j^* Q_j^{-1} u_j + \sum_{j=0}^{i} v_j^* R_j^{-1} v_j,$$

*where the* $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ *satisfy the state-space constraints*

$$\left\{ \begin{array}{rcl} x_{j+1} & = & Fx_j + Gu_j, \qquad x_0, \quad 0 \le j \le i \\ y_j & = & Hx_j + v_j \end{array} \right. \tag{8.A.2}$$

*and where the* $\{P_0, \{Q_j\}_{j=0}^i, \{R_j\}_{j=0}^i\}$ *are (possibly indefinite) Hermitian matrices of the appropriate dimensions. Then, if* $P_0 > 0$ *and* $Q_j > 0$, *for* $j = 0, \ldots i$, *the* $J_i$ *will recursively have minima over the free variables* $\{x_0, \{u_j\}_{j=0}^i\}$ *for all* $i \ge 0$, *if, and only if, the matrices*

$$R \quad \textit{and} \quad R_{e,i} = R + HP_iH^* \tag{8.A.3}$$

*have the same inertia for all* $i \ge 0$, *where* $P_i$ *satisfies the Riccati recursion*

$$P_{i+1} = FP_iF^* + GQG^* - FP_iH^*(R + HP_iH^*)^{-1}HP_iF^*, \quad P_0.$$

**Proof:** To prove the Lemma let us introduce the notation

$$u = \begin{bmatrix} u_0 \\ \vdots \\ u_i \end{bmatrix} \quad , \quad \begin{bmatrix} v_0 \\ \vdots \\ v_i \end{bmatrix} \quad , \quad \begin{bmatrix} y_0 \\ \vdots \\ y_i \end{bmatrix}$$

and

$$Q = Q_0 \oplus \ldots \oplus Q_i \quad , \quad R = R_0 \oplus \ldots \oplus R_i$$

along with the observability and impulse response matrices

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^i \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} 0 & & & \\ HG & 0 & & \\ HFG & HG & & \\ \vdots & \vdots & \ddots & \\ HF^{i-1}G & HF^{i-2}G & \ldots & 0 \end{bmatrix}.$$

With these definitions, we may write the state-space equations (8.A.2) in global form as

$$y = \mathcal{O}x_0 + \Gamma u + v,$$

and the quadratic cost function as

$$\begin{aligned}
J_i &= x_0^* P_0^{-1} x_0 + u^* Q^{-1} u + v^* R^{-1} v \\
&= x_0^* P_0^{-1} x_0 + u^* Q^{-1} u + (\mathcal{O}x_0 + \Gamma u - v) R^{-1} (\mathcal{O}x_0 + \Gamma u - v)^* \\
&= \begin{bmatrix} \begin{bmatrix} x_0^* & u^* \end{bmatrix} & y^* \end{bmatrix} \begin{bmatrix} \begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O} \\ \Gamma \end{bmatrix} R^{-1} \begin{bmatrix} \mathcal{O}^* & \Gamma^* \end{bmatrix} & -\begin{bmatrix} \mathcal{O} \\ \Gamma \end{bmatrix} R^{-1} \\ -R^{-1} \begin{bmatrix} \mathcal{O}^* & \Gamma^* \end{bmatrix} & R^{-1} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} x_0 \\ u \end{bmatrix} \\ y \end{bmatrix}.
\end{aligned}$$

Using this last expression, we readily see that $J_i$ has a minimum over $\{x_0, u\}$ if, and only if,

$$M = \begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O} \\ \Gamma \end{bmatrix} R^{-1} \begin{bmatrix} \mathcal{O}^* & \Gamma^* \end{bmatrix} > 0.$$

Now consider the matrix

$$T = \begin{bmatrix} \begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} & \begin{bmatrix} \mathcal{O} \\ \Gamma \end{bmatrix} \\ \begin{bmatrix} \mathcal{O}^* & \Gamma^* \end{bmatrix} & -R \end{bmatrix}.$$

Two different (upper-lower and lower-upper) block triangular factorizations of $T$ show that the matrices

$$\begin{bmatrix} M & 0 \\ 0 & -R \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} & 0 \\ 0 & -(\underbrace{R + \mathcal{O}P_0\mathcal{O}^* + \Gamma Q \mathbf{\Gamma}^*}_{R_y}) \end{bmatrix}$$

are congruent and therefore must have the same inertia. But since $\begin{bmatrix} P_0^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} > 0$, we conclude that $M > 0$ if, and only if the matrices $R$ and $R_y$ have the same inertia. Now using the theory of Kalman filtering in Krein spaces (the natural extension of classical Kalman filtering to indefinite metric spaces — see [HSK96c, HSK96b] and Chapter 2) we can perform the (block) triangular factorization of $R_y$ as follows

$$R_y = L R_e L^*,$$

where $L$ is lower triangular with unit diagonal, and

$$R_e = R_{e,0} \oplus \ldots \oplus R_{e,i},$$

with $R_{e,j}$ given by (8.A.3). Thus $M > 0$ (and $J_i$ has a minimum over $\{x_0, u\}$) if, and only if, $R_e$ and $R$ have the same inertia. But since we require that $J_i$ have a minimum over $\{x_0, \{u_j\}_{j=0}^i\}$ for all $i \geq 0$, this means that the block diagonal entries of $R_e$ and $R$ *i.e.*, the matrices $R_{e,i}$ and $R$, must have the same inertia for all $i \geq 0$.

■

**Proof of Lemma 8.A.1:** We first note that if (8.A.1) is true, then for all $i \geq 0$ we must have

$$x_0^* P_0^{-1} + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} v_j^* v_j - \sum_{j=0}^{i} (\hat{s}_j - s_j)^* (\hat{s}_j - s_j) > 0,$$

for all $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$, subject to the state-space constraints. Now the above quadratic form can be rewritten in the more familiar form

$$x_0^* P_0^{-1} + \sum_{j=0}^{i} u_j^* u_j + \sum_{j=0}^{i} \left( \begin{bmatrix} y_j \\ \hat{s}_j \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x_j \right)^* \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix} \left( \begin{bmatrix} y_j \\ \hat{s}_j \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x_j \right) > 0. \tag{8.A.4}$$

Now if the above quadratic form is to be positive for all $\{x_0, \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ it must have a minimum over the free variables $\{x_0, \{u_j\}_{j=0}^i\}$ (otherwise the $\{x_0, \{u_j\}_{j=0}^i\}$ can be chosen to make the quadratic form arbitrarily negative). By defining

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{bmatrix}$$

we can readily identify the above quadratic form with the $J_i$ of Lemma 8.A.2. Since the above quadratic form should have a minimum for all $i \geq 0$, we may apply the result of Lemma 8.A.2 to conclude that the matrices

$$R \quad \text{and} \quad R_{e,i}$$

must have the same inertia for all $i \geq 0$, which is the desired result.

■

## 8.B Proof of Lemma 8.6.1

In this section we shall provide a proof for the identity

$$\Phi_p^0(i,0)^* = \Phi_p^{a,0}(i,0). \tag{8.B.1}$$

This identity can be motivated through scattering theory [FKL76]. Here we shall give an algebraic proof that also sheds further light on the Riccati recursion and brings forth certain concepts related to scattering theory (namely, the Hamiltonian matrix).

In what follows we shall assume, without loss of generality, that $S = 0$ (see the discussion at the beginning of Sec. 8.6.1). We shall also assume, for simplicity, that $F$ is invertible, although the final result, *i.e.* (8.B.1), does not require the invertibility of $F$ (see also [FKL76]).

We begin by writing the Riccati recursion as

$$\begin{aligned} P_{i+1} &= FP_iF^* + GQG^* - FP_iH^*(R + HP_iH^*)^{-1}HP_iF^* \\ &= F\left[I - P_iH^*(R + HP_iH^*)^{-1}H\right]P_iF^* + GQG^* \\ &= F(I + P_iH^*R^{-1}H)^{-1}P_iF^* + GQG^*. \end{aligned}$$

The above equation we can write as

$$(I + P_iH^*R^{-1}H)F^{-1}(P_{i+1} - GQG^*) = P_iF^*,$$

or after some simplification

$$F^{-1}P_{i+1} + P_iH^*R^{-1}F^{-1}P_{i+1} - F^{-1}GQG^* - P_iH^*R^{-1}HF^{-1}GQG^* - P_iF^* = 0.$$

Gathering terms yields

$$\begin{bmatrix} I & -P_i \end{bmatrix} \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} P_{i+1} \\ I \end{bmatrix}. \tag{8.B.2}$$

The center matrix appearing in the above expression is referred to as the *Hamiltonian* matrix and is denoted by

$$M = \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix}. \tag{8.B.3}$$

It is convenient to expand the block row and column vectors appearing in (8.B.2) into upper triangular matrices, to obtain

$$\begin{bmatrix} I & -P_i \\ 0 & I \end{bmatrix} \begin{bmatrix} F^{-1} & -F^{-1}GQG^* \\ -H^*R^{-1}HF^{-1} & F^* + H^*R^{-1}HF^{-1}GQG^* \end{bmatrix} \begin{bmatrix} I & P_{i+1} \\ 0 & I \end{bmatrix}.$$

Due to (8.B.2), the $(1,2)$ block entry in the above resulting product is zero. Some simple algebra shows that the remaining block entries are

$$\begin{bmatrix} I & -P_i \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P_{i+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} F_{p,i}^{-1} & 0 \\ -H^*R^{-1}HF^{-1} & F_{p,i}^* \end{bmatrix}. \tag{8.B.4}$$

We now consider the product

$$\prod_{j=0}^{i} \begin{bmatrix} I & -P_j \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P_{j+1} \\ 0 & I \end{bmatrix}.$$

Since

$$\begin{bmatrix} I & -P_j \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & P_j \\ 0 & I \end{bmatrix}^{-1},$$

we readily see

$$\prod_{j=0}^{i} \begin{bmatrix} I & -P_j \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P_{j+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & -P_0 \\ 0 & I \end{bmatrix} M^{i+1} \begin{bmatrix} I & P_{i+1} \\ 0 & I \end{bmatrix}.$$

On the other hand, using (8.B.4)

$$\prod_{j=0}^{i} \begin{bmatrix} I & -P_j \\ 0 & I \end{bmatrix} M \begin{bmatrix} I & P_{j+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi_p(i+1,0)^{-1} & 0 \\ \times & \times \end{bmatrix},$$

where '$\times$' denotes irrelevant entries.

We thus have the following identity

$$\begin{bmatrix} I & -P_0 \\ 0 & I \end{bmatrix} M^{i+1} \begin{bmatrix} I & P_{i+1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi_p(i+1,0)^{-1} & 0 \\ \times & \times \end{bmatrix}.$$

In particular, if we choose $P_0 = 0$, we have

$$M^{i+1} \begin{bmatrix} I & P_{i+1}^0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \Phi_p^0(i+1,0)^{-1} & 0 \\ \times & \times \end{bmatrix},$$

which implies

$$\left[M^{i+1}\right]_{11} = \Phi_p^0(i+1,0)^{-1}, \tag{8.B.5}$$

where $[M^{i+1}]_{11}$ denotes the $(1,1)$ block entry of $M^{i+1}$.

In a similar fashion, if one starts with the dual Riccati recursion

$$P_{i+1}^a = F^* P_i^a F + H^* R^{-1} H - F^* P_i^a G (Q^{-1} + G^* P_i^a G)^{-1} G^* P_i^a F,$$

then it turns out that the dual Hamiltonian matrix $M^a$ is simply

$$M^a = M^*.$$

Moreover, the analog of (8.B.5) is

$$\left[(M^a)^{i+1}\right]_{11} = \left[(M^*)^{i+1}\right]_{11} = \left[M^{i+1}\right]_{11}^* = \Phi_p^{a,0}(i+1,0)^{-1}. \tag{8.B.6}$$

Comparing (8.B.5) and (8.B.6) yields

$$\Phi_p^{a,0}(i+1,0) = \Phi_p^0(i+1,0)^*,$$

which is the desired result.

■

## 8.C Some Global Expressions

In this section we shall give some global expressions for the solution to the Riccati recursion and for various related quantities. Although the material presented here is of interest in its own right, it is also useful in giving alternative derivations of some of the results presented in the chapter that implicitly assumed the invertibility of the state transition matrix, $F$. [We should remark that none of the results given in this chapter require this invertibility assumption. They were made primarily to streamline the arguments that appeared in the main body of the chapter.]

We also remark that the presentation given here is purposefully brief. The active (and interested) reader should be able to fill in the details.

We begin by noting that the standard state-space model can be written in global form as

$$\left\{ \begin{array}{rcl} \mathbf{x}_{i+1} & = & \Phi \mathbf{x}_0 + \mathcal{C}\mathbf{u} \ , \\ \mathbf{y} & = & \mathcal{O}\mathbf{x}_0 + \Gamma \mathbf{u} + \mathbf{v} \ , \end{array} \right. \tag{8.C.1}$$

where

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_0 \\ \vdots \\ \mathbf{y}_i \end{bmatrix} \ , \ \mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \vdots \\ \mathbf{u}_i \end{bmatrix} \ , \ \mathbf{v} = \begin{bmatrix} \mathbf{v}_0 \\ \vdots \\ \mathbf{v}_i \end{bmatrix} , \tag{8.C.2}$$

with

$$\langle \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} , \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} \rangle = \begin{bmatrix} \Pi_0 & 0 & 0 \\ 0 & \mathcal{Q} & 0 \\ 0 & 0 & \mathcal{R} \end{bmatrix} \ , \quad \begin{array}{rcl} \mathcal{Q} & = & \text{diag}(Q, \ldots Q) \ , \\ \mathcal{R} & = & \text{diag}(R, \ldots R) \ , \end{array} \tag{8.C.3}$$

and where $\Phi = F^i$ is the state transition matrix,

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{i-1} \end{bmatrix} \quad \text{and} \quad \mathcal{C} = \begin{bmatrix} F^{i-1}G & F^{i-2}G & \ldots & G \end{bmatrix} , \tag{8.C.4}$$

are the observability and controllability maps, and

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 \\ HG & 0 & 0 & 0 \\ HFG & HG & 0 & 0 \\ HF^2G & HFG & HG & 0 \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{8.C.5}$$

is the impulse response matrix.

Now using the definition of $P_{i+1}$,

$$P_{i+1} = \langle \mathbf{x}_{i+1}, \mathbf{x}_{i+1} \rangle - \langle \mathbf{x}_{i+1}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{y} \rangle^{-1} \langle \mathbf{y}, \mathbf{x}_{i+1} \rangle, \tag{8.C.6}$$

it is possible to show the following result.

**Lemma 8.C.1 (Global Expression for** $P_i$**)** *We have the following identity,*

$$P_{i+1} = \begin{bmatrix} \Phi & \mathcal{C} \end{bmatrix} \left( \begin{bmatrix} \Pi_0^{-1} & 0 \\ 0 & \mathcal{Q}^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O}^* \\ \Gamma^* \end{bmatrix} \mathcal{R}^{-1} \begin{bmatrix} \mathcal{O} & \Gamma \end{bmatrix} \right)^{-1} \begin{bmatrix} \Phi^* \\ \mathcal{C}^* \end{bmatrix}. \tag{8.C.7}$$

*In particular, when* $\Pi_0 = 0$*,*

$$P_{i+1}^0 = \mathcal{C} \left[ \mathcal{Q}^{-1} + \Gamma^* \mathcal{R}^{-1} \Gamma \right]^{-1} \mathcal{C}^*. \tag{8.C.8}$$

Similarly, using the definition $\hat{\mathbf{x}}_{i+1} = \langle \mathbf{x}_{i+1}, \mathbf{y} \rangle \langle \mathbf{y}, \mathbf{y} \rangle^{-1} \mathbf{y}$, we can show that

$$\hat{\mathbf{x}}_{i+1} = (\Phi \Pi_0 \mathcal{O}^* + \mathcal{C} \mathcal{Q} \Gamma^*)(\mathcal{R} + \mathcal{O} \Pi_0 \mathcal{O}^* + \Gamma \mathcal{Q} \Gamma^*)^{-1} (\mathcal{O} \mathbf{x}_0 + \Gamma \mathbf{u} + \mathbf{v}). \tag{8.C.9}$$

Now writing a global expression for $\tilde{\mathbf{x}}_{i+1} = \mathbf{x}_{i+1} - \hat{\mathbf{x}}_{i+1}$ one can deduce that $\Phi_p(i+1, 0)$, the mapping from $\mathbf{x}_0$ to $\tilde{\mathbf{x}}_{i+1}$, is given by the following result.

**Lemma 8.C.2 (Global Expression for** $\Phi_p(i+1, 0)$**)** *We have the following identity,*

$$\Phi_p(i+1, 0) = \Phi - (\Phi \Pi_0 \mathcal{O}^* + \mathcal{C} \mathcal{Q} \Gamma^*)(\mathcal{R} + \mathcal{O} \Pi_0 \mathcal{O}^* + \Gamma \mathcal{Q} \Gamma^*)^{-1} \mathcal{O}. \tag{8.C.10}$$

*In particular, when* $\Pi_0 = 0$*,*

$$\Phi_p^0(i+1, 0) = \Phi - \mathcal{C} \mathcal{Q} \Gamma^* (\mathcal{R} + \Gamma \mathcal{Q} \Gamma^*)^{-1} \mathcal{O}. \tag{8.C.11}$$

Using a similar (actually dual) argument allows us to write,

$$\Phi_p^{a,0}(i+1,0) = \Phi^* - \mathcal{O}^*\mathcal{R}^{-1}\Gamma(\mathcal{Q}^{-1} + \Gamma^*\mathcal{R}^{-1}\Gamma)^{-1}\mathcal{C}^*. \tag{8.C.12}$$

But the identity,

$$\mathcal{Q}\Gamma^*(\mathcal{R} + \Gamma\mathcal{Q}\Gamma^*)^{-1} = (\mathcal{Q}^{-1} + \Gamma^*\mathcal{R}^{-1}\Gamma)^{-1}\Gamma^*\mathcal{R}^{-1},$$

implies that

$$\Phi_p^{a,0}(i+1,0) = \Phi_p^0(i+1,0)^*, \tag{8.C.13}$$

which is an alternative proof of Lemma 8.6.1 that does not require the invertibility of $F$.

## 8.C.1 Alternative Proof of Lemma 8.6.3

We shall now outline the steps of a (global) alternative proof of the identity

$$P_{i+1}^{a,0} = \sum_{j=0}^{i} \Phi_p^0(j,0)^* H^* (R_{e,j}^0)^{-1} H \Phi_p^0(j,0). \tag{8.C.14}$$

To do so, we begin (with an argument similar to what led to Lemma 8.C.1) to write

$$P_{i+1}^{a,0} = \mathcal{O}^* \left[R + \Gamma Q \Gamma^*\right]^{-1} \mathcal{O}. \tag{8.C.15}$$

In order to write the above result as a sum, we can perform the triangular factorization of the output Gramian matrix

$$R_y = R + \Gamma Q \Gamma^* = L R_e^0 L^*, \tag{8.C.16}$$

with $L$ lower triangular with unit diagonal and $R_e^0$ (block) diagonal. We thus have

$$P_{i+1}^{a,0} = \mathcal{O}^* L^{-*} (R_e^0)^{-1} L^{-1} \mathcal{O}. \tag{8.C.17}$$

Moreover, it is easy to see that the nonzero entries of the $i$-th column of $L$ (where we begin counting the columns from zero) are given by

$$l_i = \begin{bmatrix} I \\ HK_{p,i}^0 \\ HFK_{p,i}^0 \\ HF^2K_{p,i}^0 \\ \vdots \end{bmatrix}. \tag{8.C.18}$$

The readily verified identity,

$$\mathcal{O} = L \begin{bmatrix} H \\ HF_{p,0} \\ HF_{p,1}F_{p,0} \\ \vdots \end{bmatrix}, \tag{8.C.19}$$

now yields the desired result.

### 8.C.2 Alternative Proof of Lemma 8.3.2

We first use Lemma 8.C.1 to write

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \begin{bmatrix} \Phi & \mathcal{C} \end{bmatrix} \Big( (A + \delta A)^{-1} - A \Big) \begin{bmatrix} \Phi^* \\ \mathcal{C}^* \end{bmatrix}, \tag{8.C.20}$$

where we have defined

$$A = \begin{bmatrix} (\Pi_0^{(1)})^{-1} & 0 \\ 0 & \mathcal{Q}^{-1} \end{bmatrix} + \begin{bmatrix} \mathcal{O}^* \\ \Gamma^* \end{bmatrix} \mathcal{R}^{-1} \begin{bmatrix} \mathcal{O} & \Gamma \end{bmatrix} \text{ and } \delta A = \begin{bmatrix} (\Pi_0^{(2)})^{-1} - (\Pi_0^{(1)})^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{8.C.21}$$

Now defining the following factorization of $\delta A$,

$$\delta A = \begin{bmatrix} B \\ 0 \end{bmatrix} J \begin{bmatrix} B^* & 0 \end{bmatrix}, \tag{8.C.22}$$

where $B$ is full rank and $J$ is an appropriate inertia matrix, allows us to write

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \begin{bmatrix} \Phi & \mathcal{C} \end{bmatrix} A^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \left( J + \begin{bmatrix} B^* & 0 \end{bmatrix} A^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \right)^{-1} \begin{bmatrix} B^* & 0 \end{bmatrix} A^{-1} \begin{bmatrix} \Phi^* \\ \mathcal{C}^* \end{bmatrix}. \tag{8.C.23}$$

Now using, the readily verified facts that

$$\begin{bmatrix} \Phi & \mathcal{C} \end{bmatrix} A^{-1} = \begin{bmatrix} \Phi_p^{(1)}(i+1,0)\Pi_0^{(1)} & \times \end{bmatrix}, \tag{8.C.24}$$

where $\times$ denotes irrelevant entries, and

$$\begin{bmatrix} B^* & 0 \end{bmatrix} A^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} = B^*(\Pi_0^{(1)} - \Pi_0^{(1)} \mathcal{O}^* R_y^{-1} \mathcal{O} \Pi_0^{(1)}) B, \tag{8.C.25}$$

where $R_y \triangleq \mathcal{R} + \mathcal{O}\Pi_0\mathcal{O}^* + \Gamma\mathcal{Q}\Gamma^*$, allows us to write,

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \Phi_p^{(1)}(i+1,0)\Pi_0^{(1)}B\left[J + B^*(\Pi_0^{(1)} - \Pi_0^{(1)}\mathcal{O}^*R_y^{-1}\mathcal{O}\Pi_0^{(1)})B\right]^{-1}B^*\Pi_0^{(1)}\Phi_p^{(1)}(i+1,0)^*,$$

and, upon further simplification,

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \Phi_p^{(1)}(i+1,0)\left[I + (\Pi_0^{(2)} - \Pi_0^{(1)})\mathcal{O}^*R_y^{-1}\mathcal{O}\right]^{-1}(\Pi_0^{(2)} - \Pi_0^{(1)})\Phi_p^{(1)}(i+1,0)^*. \tag{8.C.26}$$

Now an argument similar to what was presented in Sec. 8.C.1 allows us to conclude

$$\mathcal{O}^*R_y^{-1}\mathcal{O} = \sum_{j=0}^{i}\Phi_p^{(1)}(j,0)^*H^*(R_{e,j}^{(1)})^{-1}H\Phi_p^{(1)}(j,0). \tag{8.C.27}$$

But this is the same as the expression given for $\mathcal{O}_i^{(1)}$, given in Lemma 8.3.2, and we have therefore established the identity

$$P_{i+1}^{(2)} - P_{i+1}^{(1)} = \Phi_p^{(1)}(i+1,0)\left[I + (\Pi_0^{(2)} - \Pi_0^{(1)})\mathcal{O}_i^{(1)}\right]^{-1}(\Pi_0^{(2)} - \Pi_0^{(1)})\Phi_p^{(1)}(i+1,0)^*, \tag{8.C.28}$$

which is our desired result.

# Chapter 9

# $H^\infty$ Optimality of the LMS Algorithm

In this chapter we show that the celebrated LMS (least-mean-squares) adaptive algorithm is $H^\infty$ optimal. The LMS algorithm has been long regarded as an approximate solution to either a stochastic or a deterministic least-squares problem, and it essentially amounts to updating the weight vector estimates along the direction of the instantaneous gradient of a quadratic cost function. In this chapter we show that LMS can be regarded as the exact solution to a minimization problem in its own right. Namely, we establish that it is a minimax filter: it minimizes the maximum energy gain from the disturbances to the predicted errors, while the closely related so-called normalized LMS algorithm minimizes the maximum energy gain from the disturbances to the filtered errors. Moreover, since these algorithms are central $H^\infty$ filters, they minimize a certain exponential cost function and are thus also risk-sensitive optimal. We discuss the various implications of these results, and show how they provide theoretical justification for the widely observed excellent robustness properties of the LMS filter.

## 9.1 Introduction

Classical methods in estimation theory (such as maximum-likelihood, maximum entropy and least-squares) require a priori knowledge of the statistical properties of the exogenous signals. In many applications, however, one is faced with model uncertainties and lack of statistical information. Therefore, the introduction of the LMS (least-mean-squares) adaptive filter by Widrow and Hoff in 1960 came as a significant development for a broad range of engineering applications since the LMS adaptive linear-estimation procedure requires essentially no advance knowledge of the signal statistics [WH60]. Since this pioneering work, adaptive filtering techniques have been widely used to cope with time variations of system parameters and lack of a priori statistical information [WS85, Hay96].

The LMS algorithm was originally conceived as an approximate recursive procedure that solves the following least-squares adaptive problem: given a sequence of $1 \times n$ input row vectors $\{h_i\}$, and a corresponding sequence of desired responses $\{d_i\}$, find an estimate of an $n \times 1$ column vector of weights $w$, such that the sum of squared errors $\sum_{i=0}^{N} |d_i - h_i w|^2$ is minimized. The LMS solution recursively updates estimates of the weight vector along the direction of the instantaneous gradient of the squared error.

Algorithms that exactly minimize the sum of squared errors, for every value of $N$, are also known and are generally referred to as recursive least squares (RLS) algorithms (see, e.g., [Hay96, SK94b]). Although such exact least-squares algorithms have various desirable optimality properties (such as yielding maximum likelihood estimates) under certain statistical assumptions on the signals (such as temporal whiteness and Gaussian disturbances), they are computationally more complex, and are less robust to disturbance variation than the simple LMS algorithm. For example, it has been observed that the LMS algorithm has better tracking capabilities than the RLS algorithm in the presence of nonstationary inputs [Hay96].

In this chapter we show that the superior robustness properties of the LMS algorithm are due to the fact that it is a minimax algorithm, or more specifically an $H^\infty$ optimal algorithm. We shall define precisely what this means in Sec. 9.3. Here we note only that recently, following some pioneering work in robust control theory (see, *e.g.*, [Zam81]) there has been an increasing interest in minimax estimation (see Sec. 1.4 and Chapter 3 and the references therein) with the belief that the resulting so-called $H^\infty$ algorithms will be more robust and less sensitive to model uncertainties and parameter variations. The similarity between the objectives of adaptive filtering and $H^\infty$ estimation suggests that there should be some connection between the two, and indeed our result on the $H^\infty$ optimality of the LMS algorithm provides such a connection.

In addition to giving more insight into the inherent robustness of the LMS algorithm, and why it has found such wide applicability in a diverse range of problems, our result provides LMS with a rigorous basis and furnishes a minimization criterion that has long been missing. To be more precise, using some well-known results in $H^\infty$ estimation theory, we show that the LMS algorithm is the so-called central a priori $H^\infty$-optimal filter, while the closely related normalized LMS algorithm is the central a posteriori $H^\infty$-optimal filter.

The $H^\infty$ optimality property of LMS is a deterministic characterization of the algorithm. It is also possible to give a stochastic characterization of this algorithm under the assumptions of temporal whiteness and Gaussian disturbances. In this case, we show that LMS minimizes the expected value of a certain exponential cost function, and is therefore risk-sensitive optimal (in the sense of Whittle [Whi90]).

It is ironic that the LMS algorithm is not $H^2$ optimal, contrary to what its name suggests, but that it rather satisfies a minimax criterion. Moreover, in most $H^\infty$ problems, the optimum solution has not been determined in closed form — what is usually determined is a certain type of suboptimal solution. We show, however, that for the adaptive problem at hand, the optimum solution can be determined.

The remainder of the chapter is organized as follows. In Sec. 9.2 we introduce the problem of adaptive filtering and motivate the question of the robustness of estimators. In order to address the robustness question, we introduce the $H^\infty$ approach in Sec. 9.3 and formulate the $H^\infty$ estimation problem as one that minimizes the maximum energy gain from the disturbances to the estimation errors.[1] Sec. 9.4 closely

[1]The presentation of Secs. 9.2 and 9.3 closely follows that of Sec. 1.4, but is included to keep

follows Sec. 3.2 and studies the general problem of state-space $H^\infty$ estimation and, in particular, gives expressions for the $H^\infty$ a posteriori and a priori filters, as well as their full parametrization. The main result is given in Sec. 9.5 where we formulate the $H^\infty$ adaptive filtering problem as a state-space problem and use the results of Sec. 9.4 to show that the normalized LMS algorithm is the central a posteriori $H^\infty$ optimal adaptive filter, and that, if the learning rate is chosen appropriately, LMS is the central a priori $H^\infty$ optimal adaptive filter. In both cases, the LMS and normalized LMS algorithms guarantee that the energy of the estimation errors never exceeds the energy of the disturbances. Sec. 9.6 then considers a simple example that demonstrates the robustness of LMS compared to RLS, and also briefly discusses the merits of being $H^\infty$-optimal. In Sec. 9.7 the full parametrization of all $H^\infty$ optimal adaptive filters is given, and in Sec. 9.8 we show that LMS and normalized LMS have the additional property of being risk-sensitive optimal. Sec. 9.9 mentions some further results using the approach and ideas of this thesis and Sec. 9.10 provides the conclusion.

## 9.2 Adaptive Filtering

As shown in Fig. 9.1, suppose we observe an output sequence $\{d_i\}$ that obeys the following model:

$$d_i = h_i w + v_i, \quad i \geq 0 \tag{9.2.1}$$

where $h_i = \begin{bmatrix} h_{i1} & h_{i2} & \dots & h_{in} \end{bmatrix}$ is a known $1 \times n$ input vector, $w = \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}^T$ is an unknown $n \times 1$ weight vector that we intend to estimate, and $v_i$ is an unknown disturbance, which may also include modeling errors. We shall not make any assumptions on the noise sequence $\{v_i\}$ (such as stationarity, whiteness, Gaussian distributed, etc.). We denote the estimate of the weight vector using all the information available up to time $i$ by

$$\hat{w}_{|i} = \mathcal{F}(d_0, d_1, \dots, d_i; h_0, h_1, \dots h_i).$$

the chapter self-contained.

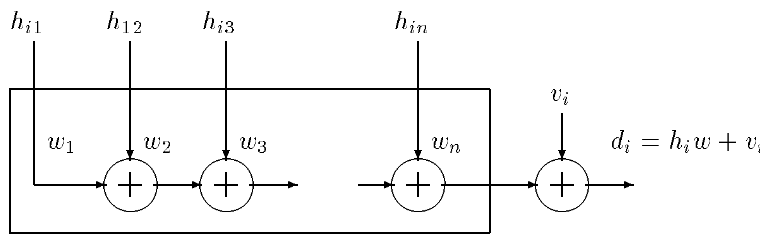


Figure 9.1: The model for adaptive filtering.

### 9.2.1 Least-Squares Methods

There are a variety of choices for $\hat{w}_{|i}$, but the most widely used estimate is one that satisfies the following least-squares (or $H^2$) criterion:

$$\min_{w} \left[ \mu^{-1} |w - \hat{w}_{|-1}|^2 + \sum_{j=0}^{i} |d_j - h_j w|^2 \right] , \tag{9.2.2}$$

where $\hat{w}_{|-1}$ is the initial estimate of $w$, and $\mu > 0$ represents the relative weight that we give to our initial estimate compared to the "sum of squared-error" term $\sum_{j=0}^{i} |d_j - h_j w|^2$.

The exact solution to the above criterion is the RLS (recursive least squares) algorithm:

$$\hat{w}_{|i} = \hat{w}_{|i-1} + k_{p,i}(d_i - h_i \hat{w}_{|i-1}) \quad , \quad \hat{w}_{|-1} \tag{9.2.3}$$

with $k_{p,i} = \frac{P_i h_i^*}{1+h_i P_i h_i^*}$ and $P_i$ satisfying the Riccati recursion

$$P_{i+1} = P_i - \frac{P_i h_i^* h_i P_i}{1 + h_i P_i h_i^*}, \quad P_0 = \mu I. \tag{9.2.4}$$

The RLS algorithm is used because under suitable stochastic assumptions it has the following two properties:[2]

(a) If $w - \hat{w}_{|-1}$ and the $\{v_j\}$ are assumed to be zero-mean, uncorrelated and, in the case of the $\{v_j\}$, temporally white random variables with variances $\mu I$ and 1,

[2]See *e.g.*, Sec. 1.3.

respectively, then the RLS algorithm minimizes the expected prediction error energy,

$$E\sum_{j=0}^{i}|h_j w - h_j w_{j-1}|^2.$$

(b) If, in addition to the assumptions of part (a), $w-\hat{w}_{|-1}$ and the $\{v_j\}$ are assumed to be jointly Gaussian, then the cost function in (9.2.2) becomes the negative of the log-likelihood function and RLS yields the maximum-likelihood estimate of the weight vector $w$.

### 9.2.2 Gradient-Based Methods

In gradient-based algorithms, instead of exactly solving the least-squares problem (9.2.2), the estimates of the weight vector are updated along the negative direction of the instantaneous gradient of the cost function appearing in (9.2.2). Two examples are the LMS (Least-Mean-Squares) [WH60]

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \mu h_i^*(d_i - h_i\hat{w}_{|i-1}) \quad , \quad \hat{w}_{|-1} \tag{9.2.5}$$

and the normalized LMS

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \frac{\mu}{1+\mu h_i h_i^*}h_i^*(d_i - h_i\hat{w}_{|i-1}) \quad , \quad \hat{w}_{|-1} \tag{9.2.6}$$

algorithms. Note that in the case of LMS the gain vector $k_{p,i}$ in RLS (which had to be computed by propagating a Riccati equation) has been simply replaced by $\mu h_i^*$. Likewise if we compare normalized LMS with the RLS algorithm, we see that the difference is that instead of propagating the matrix $P_i$ via the Riccati recursion we have simply set $P_i = \mu I$, for all $i$. For this reason the LMS and normalized LMS algorithms have long been considered to be approximate least-squares solutions and were thought to lack a rigorous basis.

We should note here that although we have introduced the LMS algorithm as an approximate deterministic least-squares solution, it is also possible to motivate it as an approximate stochastic least-squares solution (see [WS85, Hay96]).

### 9.2.3 The Question of Robustness

We saw that under suitable stochastic assumptions, the RLS algorithm has certain desirable optimality properties, namely it minimizes the expected prediction error energy and yields maximum-likelihood estimates. However, the question that begs itself is what the performance of such an estimator will be if the assumptions on the disturbances are violated, or if there are modeling errors in our model so that the disturbances must include the modeling errors? In other words

- *is it possible that* **small** *disturbances and modeling errors may lead to* **large** *estimation errors?*

Obviously, a nonrobust algorithm would be one for which the above is true, and a robust algorithm would be one for which small disturbances lead to small estimation errors. More explicitly, in the adaptive filtering problem, where we assume an FIR model, the true model may be IIR, but we neglect the tail of the filter response since its components are small. However, unless one uses a robust estimation algorithm, it is conceivable that this small modeling error may result in large estimation errors.

The problem of robust estimation is thus an important one. As we shall see in the next section, the $H^\infty$ estimation formulation is an attempt at addressing this question. The idea is to come up with estimators that minimize (or in the suboptimal case, bound) the maximum energy gain from the disturbances to the estimation errors. This will guarantee that if the disturbances are small (in energy) then the estimation errors will be as small as possible (in energy), no matter what the disturbances are. In other words the maximum energy gain is minimized over all possible disturbances. The robustness of the $H^\infty$ estimators arises from this fact. Since they make no assumption about the disturbances, they have to accommodate for all conceivable disturbances, and thus may be over-conservative.

## 9.3 The $H^\infty$ Approach

We begin with the definition of the $H^\infty$ norm of a transfer operator. As will presently become apparent, the motivation for introducing the $H^\infty$ norm is to capture the worst case behaviour of a system.[3]

**Definition 9.3.1 (The $H^\infty$ Norm)** *Let $h^2$ denote the vector space of square-summable complex-valued causal sequences with inner product $<\{f_k\},\{g_k\}> = \sum_{k=0}^{\infty} f_k^* g_k$ , where $*$ denotes complex conjugation. Let $\mathcal{T}$ be a transfer operator that maps an input sequence $\{u_i\}$ to an output sequence $\{y_i\}$. Then the $H^\infty$ norm of $\mathcal{T}$ is defined as*

$$\|\mathcal{T}\|_\infty = \sup_{u\neq 0, u\in h_2} \frac{\|y\|_2}{\|u\|_2}$$

*where the notation $\|u\|_2$ denotes the $h^2-$norm of the causal sequence $\{u_k\}$, viz., $\|u\|_2^2 = \sum_{k=0}^{\infty} u_k^* u_k$.*

Note that the $H^\infty$ norm may thus be regarded as the maximum energy gain from the input $u$ to the output $y$.

### 9.3.1 Formulation of the $H^\infty$ Adaptive Filtering Problem

Recall that $\hat{w}_{|i} = \mathcal{F}(d_0, \ldots d_i; h_0, \ldots h_i)$ denotes the estimate of the weight vector using all the information available from time 0 to time $i$. In this chapter we shall be interested in the following two estimation errors: the filtered (or a posteriori) error

$$e_{f,i} = h_i w - h_i \hat{w}_{|i}, \tag{9.3.1}$$

and the predicted (or a priori) error

$$e_{p,i} = h_i w - h_i \hat{w}_{|i-1}. \tag{9.3.2}$$

[Note that in the above errors we compare the estimates $h_i\hat{w}_{|i}$ and $h_i\hat{w}_{|i-1}$ with the uncorrupted output $h_i w$ of model (9.2.1) and not with the observation $d_i$.]

Any choice of estimation strategy $\mathcal{F}(\cdot)$ will induce transfer operators $\mathcal{T}_f(\mathcal{F})$ and $\mathcal{T}_p(\mathcal{F})$ that map the unknown disturbances $\{\mu^{-1/2}(w - \hat{w}_{|-1}), \{v_j\}_{j=0}^{\infty}\}$ to the estimation errors $\{e_{f,j}\}_{j=0}^{\infty}$ and $\{e_{p,j}\}_{j=0}^{\infty}$, respectively. See Fig. 9.2.

In the $H^\infty$ framework, robustness is ensured by minimizing the maximum energy gain from the disturbances to the estimation errors. This leads to the following problem.

[3]The material of this section follows closely that of Sec. 1.4 and Chapter 3.

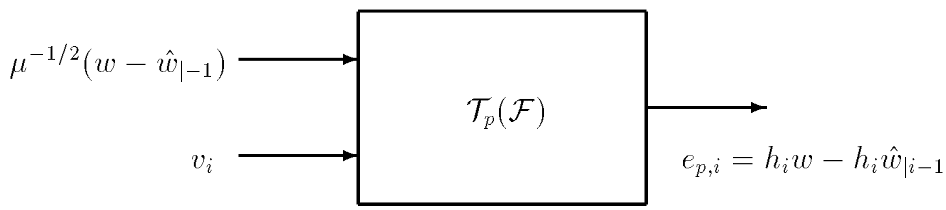


Figure 9.2: Transfer operator from the unknown disturbances $\{\mu^{-1/2}(w - \hat{w}_{|-1}), \{v_j\}_{j=0}^{\infty}\}$ to the prediction errors $\{e_{p,j}\}_{j=0}^{\infty}$. Likewise for $\mathcal{T}_f(\mathcal{F})$.

**Problem 9.3.1 ($H^\infty$ Adaptive Filtering Problem)** *Find an $H^\infty$-optimal estimation strategy $\hat{w}_{|i} = \mathcal{F}_f(d_0, \ldots, d_i; h_0, \ldots, h_i)$, that minimizes $\|\mathcal{T}_f(\mathcal{F})\|_\infty$, and an $H^\infty$-optimal strategy $\hat{w}_{|i} = \mathcal{F}_p(d_0, \ldots, d_i; h_0, \ldots, h_i)$, that minimizes $\|\mathcal{T}_p(\mathcal{F})\|_\infty$. Also obtain the resulting*

$$\gamma_{f,opt}^2 = \inf_{\mathcal{F}} \ \|\mathcal{T}_f(\mathcal{F})\|_\infty^2 = \inf_{\mathcal{F}} \ \sup_{w,v \in h^2} \ \frac{\|e_f\|_2^2}{\mu^{-1}|w - \hat{w}_{|-1}|^2 + \|v\|_2^2}, \tag{9.3.3}$$

*and*

$$\gamma_{p,opt}^2 = \inf_{\mathcal{F}} \ \|\mathcal{T}_p(\mathcal{F})\|_\infty^2 = \inf_{\mathcal{F}} \ \sup_{w,v \in h^2} \ \frac{\|e_p\|_2^2}{\mu^{-1}|w - \hat{w}_{|-1}|^2 + \|v\|_2^2}, \tag{9.3.4}$$

*where $|w - \hat{w}_{|-1}|^2 = (w - \hat{w}_{|-1})^{\mathcal{T}}(w - \hat{w}_{|-1})$.*

In order to solve the above $H^\infty$ adaptive filtering problem we shall begin by reviewing some basic results from state-space $H^\infty$ estimation theory. Although it is possible to give a "first principles" derivation of the solution to the above $H^\infty$ adaptive filtering problem (and we shall indeed do so in the Appendix), some study of the more general state-space estimation problem has its own merits, and moreover allows for various generalizations of the results presented here.

## 9.4 State-Space $H^\infty$ Estimation

We first give a brief review of some of the results in $H^\infty$ estimation theory. The presentation follows that of Chapter 3 and is included to keep this chapter self-contained. Interested readers may refer to Chapter 3, and the references therein, for earlier results and alternative approaches.

### 9.4.1 Formulation of the State-Space $H^\infty$ Problem

Consider the time-variant state-space model

$$\begin{cases} x_{i+1} = F_i x_i + G_i u_i, & x_0 \\ y_i = H_i x_i + v_i, & i \geq 0 \end{cases} \tag{9.4.1}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_i \in \mathcal{C}^{n\times m}$ and $H_i \in \mathcal{C}^{p\times n}$ are known matrices, $x_0$, $\{u_i\}$, and $\{v_i\}$ are unknown quantities and $y_i$ is the measured output. We can regard $v_i$ as a measurement noise and $u_i$ as a process noise or driving disturbance. Let $z_i$ be linearly related to the state $x_i$ via a given matrix $L_i \in \mathcal{C}^{q\times n}$, viz.,

$$s_i = L_i x_i.$$

We shall be interested in the following two cases. Let $\hat{s}_{i|i} = \mathcal{F}_f(y_0, y_1, \ldots, y_i)$ denote an estimate of $s_i$ given observations $\{y_j\}$ from time 0 up to and including time $i$, and $\hat{s}_i = \mathcal{F}_p(y_0, y_1, \ldots, y_{i-1})$ denote an estimate of $s_i$ given observations $\{y_j\}$ from time 0 to time $i-1$. We then have the filtered error

$$e_{f,i} = \hat{s}_{i|i} - L_i x_i, \tag{9.4.2}$$

and the predicted error

$$e_{p,i} = \hat{s}_i - L_i x_i. \tag{9.4.3}$$

Let $\mathcal{T}_{f,i}(\mathcal{F}_f)$ $(\mathcal{T}_{p,i}(\mathcal{F}_p))$ denote the transfer operator that maps the unknown disturbances $\{\Pi_0^{-1/2}(x_0 - \hat{x}_0), \{u_j\}_{j=0}^i, \{v_j\}_{j=0}^i\}$ to the filtered (predicted) errors $\{e_{f,j}\}_{j=0}^i$ ( $\{e_{p,j}\}_{j=0}^i$), where $\hat{x}_0$ denotes an initial guess of $x_0$, and $\Pi_0$ is a given positive definite matrix reflecting a priori knowledge of how close $x_0$ is to the initial guess $\hat{x}_0$. See Figure 9.3. The (so-called finite-horizon) $H^\infty$ estimation problem can now be stated as follows.

**Problem 9.4.1 (Optimal $H^\infty$ Problem)** *Find $H^\infty$-optimal estimation strategies $\hat{s}_{i|i} = \mathcal{F}_f(y_0, \ldots, y_i)$ and $\hat{s}_i = \mathcal{F}_p(y_0, \ldots, y_{i-1})$ that respectively minimize $\|\mathcal{T}_{f,i}(\mathcal{F}_f)\|_\infty$ and $\|\mathcal{T}_{p,i}(\mathcal{F}_p)\|_\infty$, and obtain the resulting*

$$\gamma_{f,opt}^2 = \inf_{\mathcal{F}_f} \|\mathcal{T}_{f,i}(\mathcal{F}_f)\|_\infty^2 = \inf_{\mathcal{F}_f} \sup_{x_0, u\in h^2, v\in h^2} \frac{\sum_{j=0}^i |e_{f,i}|^2}{(x_0 - \hat{x}_0)^* {\Pi_0}^{-1}(x_0 - \hat{x}_0) + \sum_{j=0}^i |u_j|^2 + \sum_{j=0}^i |v_j|^2} \tag{9.4.4}$$

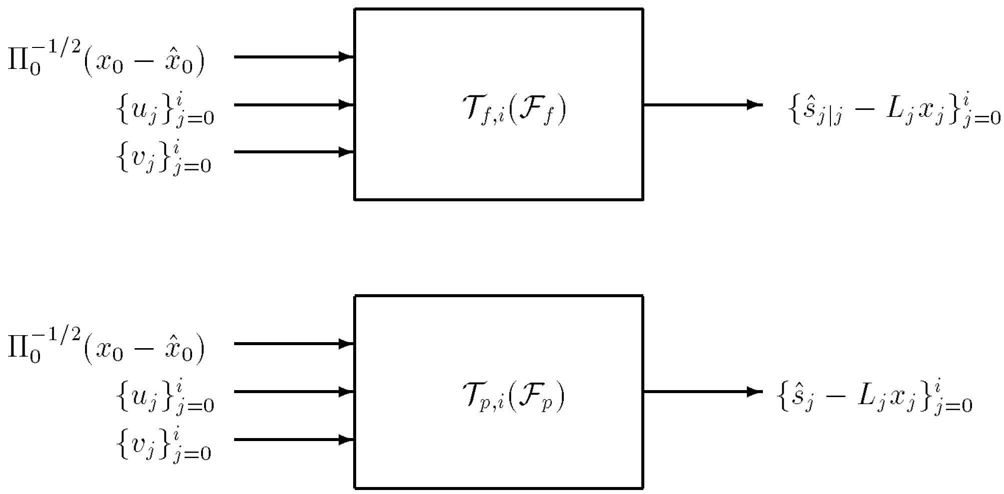


Figure 9.3: Transfer matrices from disturbances to filtered and predicted estimation errors.

*and*

$$\gamma_{p,opt}^2 = \inf_{\mathcal{F}_p} \|\mathcal{T}_{p,i}(\mathcal{F}_p)\|_\infty^2 = \inf_{\mathcal{F}_p} \sup_{x_0, u \in h^2, v \in h^2} \frac{\sum_{j=0}^i |e_{p,i}|^2}{(x_0 - \hat{x}_0)^* {\Pi_0}^{-1} (x_0 - \hat{x}_0) + \sum_{j=0}^i |u_j|^2 + \sum_{j=0}^i |v_j|^2}. \tag{9.4.5}$$

Note that the infimum in (9.4.5) is taken over all strictly causal estimators $\mathcal{F}_p$, whereas in (9.4.4) the estimators $\mathcal{F}_f$ are causal since they have additional access to $y_i$. This is relevant since the solution to the $H^\infty$ problem, as we shall see, depends on the structure of the information available to the estimator.

The above problem formulation shows that $H^\infty$ optimal estimators guarantee the smallest estimation error energy over all possible disturbances of fixed energy. $H^\infty$ estimators are thus over conservative, which reflects in a better robust behaviour to disturbance variation.

A closed form solution of the optimal $H^\infty$ problem is available only for some special cases (one of which is the adaptive filtering problem as we show here), and a simpler problem results if one relaxes the minimization condition and settles for a suboptimal solution.

**Problem 9.4.2 (Sub-optimal $H^\infty$ Problem)** *Given scalars $\gamma_f > 0$ and $\gamma_p > 0$, find estimation strategies $\hat{s}_{i|i} = \mathcal{F}_f(y_0, y_1, \ldots, y_i)$ and $\hat{s}_i = \mathcal{F}_p(y_0, y_1, \ldots, y_{i-1})$ that respectively achieve $\| \mathcal{T}_{f,i}(\mathcal{F}_f) \|_\infty < \gamma_f$ and $\| \mathcal{T}_{p,i}(\mathcal{F}_p) \|_\infty < \gamma_p$. This clearly requires checking whether $\gamma_f \geq \gamma_{f,o}$ and $\gamma_p \geq \gamma_{p,o}$.*

The above two problem formulations are for the finite horizon case. In the infinite horizon case, to guarantee that $\|\mathcal{T}_f(\mathcal{F})\|_\infty \leq \gamma_f$ and $\|\mathcal{T}_p(\mathcal{F})\|_\infty \leq \gamma_p$ we need to ensure $\|\mathcal{T}_{f,i}(\mathcal{F})\|_\infty < \gamma_f$ and $\|\mathcal{T}_{p,i}(\mathcal{F})\|_\infty < \gamma_p$ for all $i$.

## 9.4.2 The $H^\infty$ Filters

We now briefly review the solutions of the $H^\infty$ filtering problems using the notation of [HSK96c, HSK96b].

**Theorem 9.4.1 (The $H^\infty$ A Posteriori Filter)** *For a given $\gamma > 0$, if the $F_i$ are nonsingular then an estimator with $\|\mathcal{T}_{f,i}\|_\infty < \gamma$ exists if, and only if,*

$$P_j^{-1} + H_j^* H_j - \gamma^{-2} L_j^* L_j > 0, \qquad j = 0, \ldots, i \tag{9.4.6}$$

*where $P_0 = \Pi_0$, and $P_j$ satisfies the Riccati recursion*

$$P_{j+1} = F_j P_j F_j^* + G_j G_j^* - \begin{bmatrix} L_j^* & H_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j F_j^* \tag{9.4.7}$$

*with*

$$R_{e,j} = \begin{bmatrix} -\gamma^2 I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} + \begin{bmatrix} L_j \\ H_j \end{bmatrix} P_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix}.$$

*If this is the case, then one possible $H_\infty$ filter with level $\gamma$ is given by*

$$\hat{s}_{j|j} = L_j \hat{x}_{j|j},$$

*where $\hat{x}_{j|j}$ is recursively computed as*

$$\hat{x}_{j+1|j+1} = F_j \hat{x}_{j|j} + K_{f,j+1}(y_{j+1} - H_{j+1} F_j \hat{x}_{j|j}), \quad \hat{x}_{-1|-1} = \textit{initial guess} \tag{9.4.8}$$

*and*

$$K_{f,j+1} = P_{j+1}H^*_{j+1}(I + H_{j+1}P_{j+1}H^*_{j+1})^{-1}. \tag{9.4.9}$$

**Theorem 9.4.2 (The $H^\infty$ A Priori Filter)** *For a given $\gamma > 0$, if the $F_i$ are nonsingular then an estimator with $\|\mathcal{T}_{p,i}\|_\infty < \gamma$ exists if, and only if,*

$$\tilde{P}_j^{-1} = P_j^{-1} - \gamma^{-2}L_j^*L_j > 0, \qquad j = 0, \ldots, i \tag{9.4.10}$$

*where $P_j$ is the same as in Theorem 9.4.1. If this is the case, then one possible $H_\infty$ filter with level $\gamma$ is given by*

$$\hat{s}_j = L_j\hat{x}_j, \tag{9.4.11}$$

$$\hat{x}_{j+1} = F_j\hat{x}_j + K_{p,j}(y_j - H_j\hat{x}_j), \quad \hat{x}_0 = \text{initial guess} \tag{9.4.12}$$

*where*

$$K_{p,j} = F_j\tilde{P}_jH^*_j(I + H_j\tilde{P}_jH^*_j)^{-1}. \tag{9.4.13}$$

Note that the above two estimators bear a striking resemblance to the celebrated Kalman filter:

$$\left\{ \begin{array}{lcl} \hat{x}_{j+1} & = & F_j\hat{x}_j + F_jP_jH^*_j(I + H_jP_jH^*_j)^{-1}(y_j - H_j\hat{x}_j) \\ P_{j+1} & = & F_jP_jF^*_j + G_jG^*_j - F_jP_j(I + H_jP_jH^*_j)^{-1}P_jF^*_j \end{array} \right. \tag{9.4.14}$$

and that the only difference is that the $P_j$ of equation (9.4.9), and $\tilde{P}_j$ of equation (9.4.13), satisfy Riccati recursions that differ with that of (9.4.14). However, as $\gamma \to \infty$, the Riccati recursion (9.4.7) collapses to the Kalman filter recursion (9.4.14). This suggests that the $H^\infty$ norm of the Kalman filter may be quite large, indicating that it may have poor robustness properties.

It is also interesting that the structure of the $H^\infty$ estimators depends, via the Riccati recursion (9.4.7), on the linear combination of the states that we intend to estimate (*i.e.*, the $L_i$). This is as opposed to the Kalman filter, where the estimate of any linear combination of the state is given by that linear combination of the state estimate. Intuitively, this means that the $H^\infty$ filters are specifically tuned towards the linear combination $L_i x_i$.

Note also that condition (9.4.10) is more stringent than condition (9.4.6), indicating that the existence of an a priori filter of level $\gamma$ implies the existence of an a posteriori filter of level $\gamma$, but not necessarily vice versa.

We further remark that the filter of Theorem 9.4.1 (and Theorem 9.4.2) is one of many possible filters with level $\gamma$. A full parametrization of all estimators of level $\gamma$ are given by the following Theorems. (For proofs see [HSK96b] or Chapter 3).

**Theorem 9.4.3 (All $H^\infty$ A Posteriori Estimators)** *All $H^\infty$ a posteriori estimators that achieve a level $\gamma_f$ (assuming they exist) are given by*

$$\begin{aligned} \hat{s}_{j|j} &= L_j\hat{x}_{j|j} + [\gamma_f^2 I - L_j(P_j^{-1} + H_j^* H_j)^{-1} L_j^*]^{\frac{1}{2}} \\ &\quad \mathcal{S}_j\left((I + H_j P_j H_j^*)^{\frac{1}{2}}(y_j - H_j\hat{x}_{j|j}), \ldots, (I + H_0 P_0 H_0^*)^{\frac{1}{2}}(y_0 - H_0\hat{x}_{0|0})\right) \end{aligned} \tag{9.4.15}$$

*where $\hat{x}_{j|j}$ satisfies the recursion*

$$\hat{x}_{j+1|j+1} = F_j\hat{x}_{j|j} + K_{f,j+1}(y_{j+1} - H_{j+1}F_j\hat{x}_{j|j}) - K_{c,j}(\hat{s}_{j|j} - L_j\hat{x}_{j|j}) \tag{9.4.16}$$

*with $K_{f,j+1}$ the same as in Theorem 9.4.1,*

$$K_{c,j} = (I + P_{j+1}H_{j+1}H_{j+1}^*)^{-1}F_j(P_j^{-1} + H_j H_j^* - \gamma_f^{-2}L_j L_j^*)^{-1}L_j^*, \tag{9.4.17}$$

*and*

$$\mathcal{S}(a_j, \ldots, a_0) = \begin{bmatrix} \mathcal{S}_0(a_0) \\ \mathcal{S}_1(a_1, a_0) \\ \vdots \\ \mathcal{S}_j(a_j, \ldots, a_0) \end{bmatrix}$$

*is any (possibly nonlinear) contractive causal mapping, i.e.,*

$$\sum_{j=0}^{k} |\mathcal{S}_j(a_j, \ldots, a_0)|^2 < \sum_{j=0}^{k} |a_j|^2 \quad \textit{for all} \quad k = 0, 1, \ldots, i.$$

**Theorem 9.4.4 (All $H^\infty$ A Priori Estimators)** *All $H^\infty$ a priori estimators that achieve a level $\gamma_p$ (assuming they exist) are given by*

$$\begin{aligned}\hat{s}_j &= L_j\hat{x}_j + (\gamma_p^2 I - L_j P_j L_j^*)^{\frac{1}{2}} \\ &\quad \mathcal{S}_j\left((I + H_{j-1}\tilde{P}_{j-1}H_{j-1}^*)^{-\frac{1}{2}}(y_{j-1} - H_{j-1}\bar{x}_{j-1}), \ldots, (I + H_0\tilde{P}_0H_0^*)^{-\frac{1}{2}}(y_0 - H_0\bar{x}_0)\right)\end{aligned} \tag{9.4.18}$$

*where*

$$\bar{x}_k = \hat{x}_k + P_k L_k^*(-\gamma_p^2 I + L_k P_k L_k^*)^{-1}(\hat{s}_k - L_k\hat{x}_k), \tag{9.4.19}$$

*$\hat{x}_j$ satisfies the recursion*

$$\hat{x}_{j+1|j} = F_j\hat{x}_{j|j-1} + F_jP_j \begin{bmatrix} L_j^* & H_j^* \end{bmatrix} R_{e,j}^{-1} \begin{bmatrix} \hat{s}_j - L_j\hat{x}_{j|j-1} \\ y_j - H_j\hat{x}_{j|j-1} \end{bmatrix}, \tag{9.4.20}$$

*with $P_j$, $\tilde{P}_j$ and $R_{e,j}$ given by Theorem 9.4.2, and $\mathcal{S}$ is any (possibly nonlinear) contractive causal mapping.*

Note that although the filters obtained in Theorems 9.4.1 and 9.4.2 are linear, the full parametrization of all $H^\infty$ filters with level $\gamma$ is given by a nonlinear causal contractive mapping $\mathcal{S}$. The filters of Theorems 9.4.1 and 9.4.2 are known as the central filters and correspond to $\mathcal{S} = 0$. These central filters have a number of other interesting properties. They correspond, as we shall see in a subsequent section, to the risk-sensitive optimal filter [Whi90], and can be shown to be the maximum entropy filter [GM89]. (See also Sec. 1.4 and Sec. 4.2.)

## 9.5 Main Result

Let us first note that the basic equation of the adaptive filtering model (9.2.1) can be rewritten in the following state-space form:

$$\left\{\begin{array}{rcl} x_{i+1} &=& x_i \\ d_i &=& h_i x_i + v_i \end{array}\right. \qquad x_0 = w. \tag{9.5.1}$$

This is a relevant step since it reduces the adaptive filtering problem to an equivalent state-space estimation problem. This point of view has been recently proposed in

[SK94b] where a unified square-root-based derivation of exponentially-weighted RLS adaptive algorithms is obtained by reformulating the original adaptive problem as a state-space linear least-squares estimation problem and then applying various algorithms from Kalman filter theory. Here we shall instead apply the $H^\infty$ theory to the state-space model (9.5.1) and show that the optimum a priori and a posteriori $H^\infty$ filters reduce to the LMS and normalized LMS algorithms, respectively.

At this point we need one more definition.

**Definition 9.5.1 (Exciting Inputs)** *The input vectors $h_i$ are called exciting if, and only if,*

$$\lim_{N\to\infty} \sum_{i=0}^{N} h_i h_i^* = \infty$$

## 9.5.1 The Normalized LMS Algorithm

We first consider the a posteriori filter and show that it collapses to the normalized LMS algorithm.

**Theorem 9.5.1 (Normalized LMS Algorithm)** *Consider the state-space model (9.5.1), and suppose we want to minimize the $H^\infty$ norm of the transfer operator $\mathcal{T}_f(\mathcal{F})$ from the unknowns $\mu^{-1/2}(w - \hat{w}_{|-1})$ and $\{v_j\}_{j=0}^{\infty}$ to the filtered error $\{e_{f,j} = \hat{s}_{j|j} - h_j w\}_{j=0}^{\infty}$. If the input data $\{h_j\}$ is exciting, then the minimum $H^\infty$ norm is*

$$\gamma_{f,opt} = 1.$$

*In this case, the central optimal $H^\infty$ a posteriori filter is*

$$\hat{s}_{j|j} = h_j \hat{w}_{|j},$$

*where $\hat{w}_{|j}$ is given by the normalized LMS algorithm with parameter $\mu$,*

$$\hat{w}_{|j+1} = \hat{w}_{|j} + \frac{\mu h_{j+1}^*}{1 + \mu h_{j+1} h_{j+1}^*}(d_{j+1} - h_{j+1}\hat{w}_{|j}), \quad \hat{w}_{|-1} = \textit{initial guess}. \qquad (9.5.2)$$

Intuitively it is not hard to convince oneself that $\gamma_{f,opt}$ cannot be less than one. To this end, suppose that the estimator has chosen some initial guess $\hat{w}_{|-1}$. Then one may conceive of a disturbance that yields an observation that coincides with the output expected from $\hat{w}_{|-1}$, *i.e.* ,

$$h_i \hat{w}_{|-1} = h_i w + v_i = d_i.$$

In this case one expects that the estimator will not change its estimate of $w$, so that $\hat{w}_{|i} = \hat{w}_{|-1}$ for all $i$. Thus the filtered error is

$$e_{f,i} = h_i w - h_i \hat{w}_{|i} = h_i w - h_i \hat{w}_{|-1} = -v_i,$$

and the ratio in (9.3.3) becomes

$$\frac{\|v\|^2}{\mu^{-1}|w - \hat{w}_{|-1}|^2 + \|v\|^2} = \frac{\|h_i(w - \hat{w}_{|-1})\|^2}{\mu^{-1}|w - \hat{w}_{|-1}|^2 + \|h_i(w - \hat{w}_{|-1})\|^2}.$$

When the $\{h_i\}$ are exciting, for any $\epsilon > 0$, we can find a weight vector $w$ and an integer $N$ such that $\sum_{i=0}^{N} |h_i(w - \hat{w}_{|-1})|^2 \geq \frac{|w - \hat{w}_{|-1}|^2}{\epsilon\mu}$. With these choices we have

$$\frac{\sum_{i=0}^{N} |h_i(w - \hat{w}_{|-1})|^2}{\mu^{-1}|w - \hat{w}_{|-1}|^2 + \sum_{i=0}^{N} |h_i(w - \hat{w}_{|-1})|^2} \geq 1 - \epsilon,$$

so that the ratio in (9.3.3) can be made arbitrarily close to one.

The surprising fact though is that $\gamma_{f,opt}$ is exactly one and that the normalized LMS algorithm achieves it. What this means is that normalized LMS guarantees that the energy of the filtered error will never exceed the energy of the disturbances. This is not true for other estimators. For example, in the case of the recursive least-squares (RLS) algorithm, one can come up with a disturbance of small energy that will yield a filtered error of large energy (see [HK94b] and Chapter 10).

**Proof of Theorem 9.5.1:** We apply the a posteriori filter of Theorem 9.4.1 to the state-space model (9.5.1) where $F_i = I$, $G_i = 0$, $H_i = h_i$, and $L_i = h_i$. Thus the Riccati equation simplifies to

$$P_{i+1} = P_i - P_i \begin{bmatrix} h_i^* & h_i^* \end{bmatrix} \left\{ \begin{bmatrix} -\gamma^2 I & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} + \begin{bmatrix} h_j \\ h_j \end{bmatrix} P_i \begin{bmatrix} h_i^* & h_i^* \end{bmatrix} \right\}^{-1} \begin{bmatrix} h_i \\ h_i \end{bmatrix} P_i,$$

which, using the matrix inversion lemma [Kai80], implies that

$$\begin{aligned} P_{i+1}^{-1} &= P_i^{-1} + \begin{bmatrix} h_i^* & h_i^* \end{bmatrix} \begin{bmatrix} -\gamma^{-2} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} h_i \\ h_i \end{bmatrix} \\ &= P_i^{-1} + (1-\gamma^{-2}) h_i^* h_i. \end{aligned}$$

Consequently, starting with $P_0^{-1} = \mu^{-1} I$, we get

$$P_{i+1}^{-1} = \mu^{-1} I + (1-\gamma^{-2}) \sum_{j=0}^{i} h_j^* h_j. \tag{9.5.3}$$

Now we need to check the existence condition (9.4.6) and find the optimum $\gamma_{f,opt}$. It follows from the above expression for $P_{i+1}^{-1}$ that we have

$$P_{i+1}^{-1} + H_{i+1}^* H_{i+1} - \gamma^{-2} L_{i+1}^* L_{i+1} = \mu^{-1} I + (1-\gamma^{-2}) \sum_{j=0}^{i+1} h_j^* h_j. \tag{9.5.4}$$

Suppose $\gamma < 1$ so that $1 - \gamma^{-2} < 0$. Since the $\{h_j\}$ are exciting, we conclude that for some $k$, and for large enough $i$, we must have

$$\sum_{j=0}^{i+1} |h_{jk}|^2 > \frac{\mu^{-1}}{\gamma^{-2} - 1}.$$

This implies that the $k^{th}$ diagonal entry of the matrix on the right hand side of (9.5.4) is negative, viz.,

$$\mu^{-1} + (1-\gamma^{-2}) \sum_{j=0}^{i+1} |h_{jk}|^2 < 0.$$

Consequently, $P_{i+1}^{-1} + H_{i+1}^* H_{i+1} - \gamma^{-2} L_{i+1}^* L_{i+1}$ cannot be positive-definite. Therefore, $\gamma_{f,opt} \geq 1$. We now verify that $\gamma_{f,opt}$ is indeed 1. For this purpose, we note that if we consider $\gamma = 1$ then from equation (9.5.3) we have $P_i = \mu I > 0$ for all $i$ and the existence condition is satisfied. If we now write the a posteriori filter for $\gamma_{f,opt} = 1$, with $P_i = \mu I$, we get the desired so-called normalized LMS algorithm (9.5.2).

■

### 9.5.2 The LMS Algorithm

We now apply the a priori $H^\infty$-filter and show that it collapses to the LMS algorithm.

**Theorem 9.5.2 (LMS Algorithm)** *Consider the state-space model (9.5.1), and suppose we want to minimize the $H^\infty$ norm of the transfer operator $\mathcal{T}_p(\mathcal{F})$ from the unknowns $\mu^{-1/2}(w-\hat{w}_{|-1})$ and $\{v_j\}_{j=0}^{\infty}$ to the predicted error $\{e_{p,j}=\hat{s}_j-h_jw\}_{j=0}^{\infty}$. If the input data $\{h_j\}$ is exciting, and*

$$0<\mu<\inf_i \frac{1}{h_ih_i^*} \tag{9.5.5}$$

*then the minimum $H^\infty$ norm is*

$$\gamma_{p,opt}=1.$$

*In this case, the central optimal a priori $H^\infty$ filter is*

$$\hat{s}_j=h_i\hat{w}_{|j-1}$$

*where $\hat{w}_{|j-1}$ is given by the LMS algorithm with learning rate $\mu$, viz.,*

$$\hat{w}_{|j}=\hat{w}_{|j-1}+\mu h_j^*(d_j-h_j\hat{w}_{|j-1}) \quad , \quad \hat{w}_{|-1}. \tag{9.5.6}$$

**Proof:** The proof is similar to that for the normalized LMS case. For $\gamma<1$, the matrix $\tilde{P}_i$ of Theorem 9.4.2 cannot be positive-definite. For $\gamma=1$, we get $P_i=\mu I>0$ for all $i$, and

$$\begin{aligned}\tilde{P}_i^{-1} &= P_i^{-1}-L_i^*L_i\\ &= \mu^{-1}I-h_i^*h_i\end{aligned}$$

It is straightforward to see that the eigenvalues of $\tilde{P}_i^{-1}$ are

$$\{\mu^{-1},\mu^{-1},...,\mu^{-1},\mu^{-1}-h_ih_i^*\}.$$

Thus $\tilde{P}_i^{-1}$ is positive definite if, and only if, (9.5.5) is satisfied, which leads to $\gamma_{p,opt}=1$. Writing the $H^\infty$ a priori filter equations for $\gamma=1$ yields

$$\begin{aligned}\hat{w}_{|i} &= \hat{w}_{|i-1}+\tilde{P}_ih_i^*(I+h_i\tilde{P}_ih_i^*)^{-1}(d_i-h_i\hat{w}_{|i-1})\\ &= \hat{w}_{|i-1}+\tilde{P}_i(I+h_i^*h_i\tilde{P}_i)^{-1}h_i^*(d_i-h_i\hat{w}_{|i-1})\\ &= \hat{w}_{|i-1}+(\tilde{P}_i^{-1}+h_i^*h_i)^{-1}h_i^*(d_i-h_i\hat{w}_{|i-1})\\ &= \hat{w}_{|i-1}+\mu h_i^*(d_i-h_i\hat{w}_{|i-1}).\end{aligned}$$

■

The above result indicates that if the learning rate $\mu$ is chosen according to (9.5.5), then LMS ensures that the energy of the predicted error will never exceed the energy of the disturbances. It is interesting that we have obtained an upper bound on the learning rate $\mu$ that guarantees this $H^\infty$ optimality, since it is a well known fact that LMS behaves poorly if the learning rate is chosen too large. It is also interesting to compare the bound in (9.5.5) with the bounds studied in [WS85] and [Wid76].

We further note that if the input data is not exciting, then $\sum_{i=0}^{\infty} h_i^* h_i$ will have a finite limit, and the minimum $H^\infty$ norm of the a posteriori and a priori filters will be the smallest $\gamma$ that ensures

$$\mu^{-1} I + (1-\gamma^{-2}) \sum_{i=0}^{\infty} h_i^* h_i > 0.$$

This will in general yield $\gamma_{opt} < 1$, and Theorems 9.4.1 and 9.4.2 can be used to write the optimal filters for this $\gamma_{opt}$. In this case the LMS and normalized LMS algorithms will still correspond to $\gamma = 1$, but will now be suboptimal.

## 9.6 An Illustrative Example

To illustrate the robustness of the LMS algorithm we consider a special case of model (9.5.1), where $h_i$ is now a scalar that randomly takes on the values $+1$ and $-1$.

Using the LMS algorithm we can write the following state-space model for the predicted error $e_{p,i} = h_i x_i - h_i \hat{x}_i$:

$$\left\{ \begin{array}{rcl} \tilde{x}_{i+1} & = & (1-\mu|h_i|^2)\tilde{x}_i - \mu h_i^* v_i = (1-\mu)\tilde{x}_i - \mu h_i v_i \\ e_{p,i} & = & h_i \tilde{x}_i \end{array} \right. , \quad \tilde{x}_0 = w - \hat{x}_{-1} \qquad (9.6.1)$$

where $\tilde{x}_i = x_i - \hat{x}_i$, and where we have used the fact that the $h_i$ have magnitude one. Assuming we have observed $N$ points of data, we can then use (9.6.1) to write the operator, $\mathcal{T}_{lms,N}(\mu)$, that maps the disturbances $\{\mu^{-\frac{1}{2}}\tilde{x}_0, \{v_i\}_{i=0}^{N-1}\}$ to the $\{e_{p,i}\}_{i=0}^{N-1}$.

$$\begin{bmatrix} e_{p,0} \\ e_{p,1} \\ \vdots \\ e_{p,N-1} \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} \mu^{\frac{1}{2}}h_0 & 0 & 0 & \dots & 0 \\ \mu^{\frac{1}{2}}(1-\mu)h_1 & -\mu h_1 h_0 & 0 & \dots & 0 \\ \mu^{\frac{1}{2}}(1-\mu)^2 h_2 & -\mu(1-\mu)h_2 h_0 & -\mu h_2 h_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{\frac{1}{2}}(1-\mu)^{N-1}h_{N-1} & -\mu(1-\mu)^{N-2}h_{N-1}h_0 & -\mu(1-\mu)^{N-3}h_{N-1}h_1 & \dots & -\mu h_{N-1}h_{N-2} \end{bmatrix}}_{\mathcal{T}_{lms,N}(\mu)} \begin{bmatrix} \mu^{-\frac{1}{2}}\tilde{x}_0 \\ v_0 \\ \vdots \\ v_{N-2} \end{bmatrix}. \tag{9.6.2}$$

Suppose now we use the RLS algorithm (*viz.* the Kalman filter) to estimate the states in (9.5.1), *i.e.*,

$$\hat{x}_{i+1} = \hat{x}_i + k_{p,i}(d_i - h_i\hat{x}_i)$$

where $k_{p,i} = \frac{p_i h_i^*}{1+p_i|h_i|^2}$ and

$$p_{i+1} = p_i - \frac{|h_i|^2 p_i^2}{1+p_i|h_i|^2} = p_i - \frac{p_i^2}{1+p_i} = \frac{p_i}{1+p_i}, \quad p_0 = \mu. \tag{9.6.3}$$

Then we may write the following state-space model for the RLS predicted error $e'_{p,i} = h_i x_i - h_i\hat{x}_i$,

$$\left\{ \begin{array}{rcl} \tilde{x}_{i+1} & = & (1-k_{p,i}h_i)\tilde{x}_i - k_{p,i}v_i \\ e'_{p,i} & = & h_i\tilde{x}_i \end{array} \right. , \quad \tilde{x}_0 = w - \hat{x}_{-1} \tag{9.6.4}$$

Now solving (9.6.3) yields

$$p_i = \frac{\mu}{1+i\mu} \quad , \tag{9.6.5}$$

and

$$k_{p,i} = h_i p_{i+1}, \qquad 1 - k_{p,i}h_i = \frac{p_{i+1}}{p_i}. \tag{9.6.6}$$

Using (9.6.5), (9.6.6), and the state-space model (9.6.4) we can also write the transfer operator $\mathcal{T}_{rls,N}(\mu)$ that maps the disturbances to the predicted errors as follows:

$$\begin{bmatrix} e'_{p,0} \\ e'_{p,1} \\ \vdots \\ e'_{p,N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} \mu^{\frac{1}{2}}h_0 & 0 & 0 & \dots & 0 \\ \mu^{\frac{1}{2}}\frac{h_1}{1+\mu} & -\mu\frac{h_1h_0}{1+\mu} & 0 & \dots & 0 \\ \mu^{\frac{1}{2}}\frac{h_2}{1+2\mu} & -\mu\frac{h_2h_0}{1+2\mu} & -\mu\frac{h_2h_1}{1+2\mu} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu^{\frac{1}{2}}\frac{h_{N-1}}{1+(N-1)\mu} & -\mu\frac{h_{N-1}h_0}{1+(N-1)\mu} & -\mu\frac{h_{N-1}h_1}{1+(N-1)\mu} & \dots & -\mu\frac{h_{N-1}h_{N-2}}{1+(N-1)\mu} \end{bmatrix}}_{\mathcal{T}_{rls,N}(\mu)} \begin{bmatrix} \mu^{-\frac{1}{2}}x_0 \\ v_0 \\ \vdots \\ v_{N-2} \end{bmatrix}. \tag{9.6.7}$$

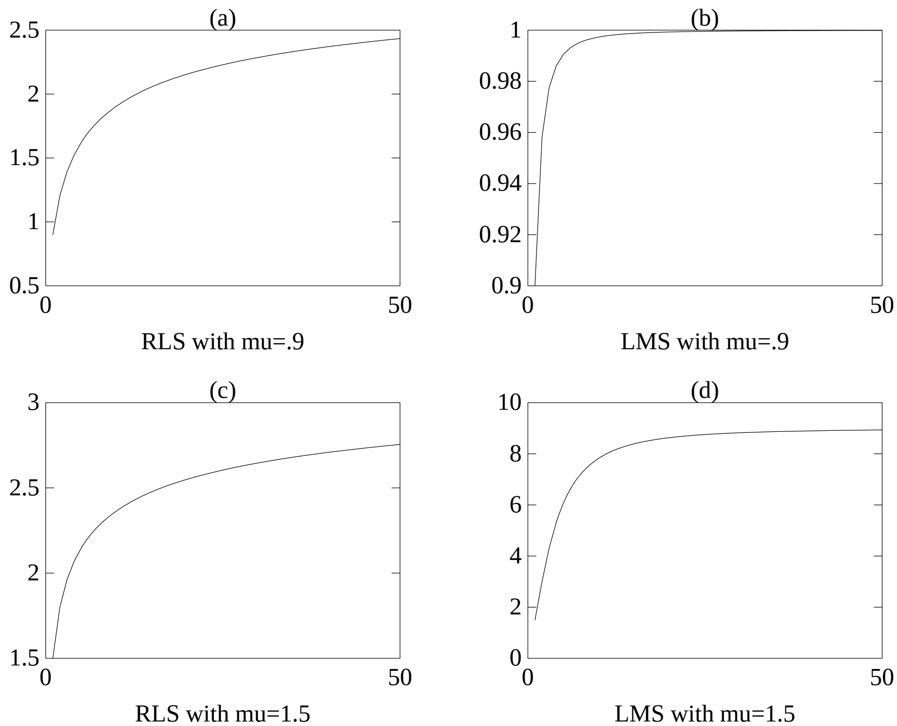


Figure 9.4: Maximum singular value of transfer operators $\mathcal{T}_{lms,N}(\mu)$ and $\mathcal{T}_{rls,N}(\mu)$ as a function of $N$ for the values $\mu = .9$ and $\mu = 1.5$.

We now study the maximum singular values of $\mathcal{T}_{lms,N}(\mu)$ and $\mathcal{T}_{rls,N}(\mu)$ as a function of $\mu$ and $N$. Note that in this special problem, condition (9.5.5) implies that $\mu$ must be less than one to guarantee the $H^\infty$ optimality of LMS. Therefore we chose the two values $\mu = .9$ and $\mu = 1.5$ (one greater and one less than $\mu = 1$). The results are illustrated in Figure 9.4 where the maximum singular values of $\mathcal{T}_{lms,N}(\mu)$ and $\mathcal{T}_{rls,N}(\mu)$ are plotted against the number of observations $N$. As expected, for $\mu = .9$ the maximum singular value of $\mathcal{T}_{lms,N}(\mu)$ remains constant at one, whereas the maximum singular value of $\mathcal{T}_{rls,N}(\mu)$ is greater than one and increases with $N$. For $\mu = 1.5$ both RLS and LMS display maximum singular values greater than one, with the performance of LMS being significantly worse.

Figure 9.5 shows the worst case disturbance signals for the RLS and LMS algorithms in the $\mu = .9$ case, and the corresponding predicted errors. These worst case disturbances are found by computing the maximum singular vectors of $\mathcal{T}_{rls,50}(.9)$ and $\mathcal{T}_{lms,50}(.9)$, respectively. The worst case RLS disturbance, and the uncorrupted

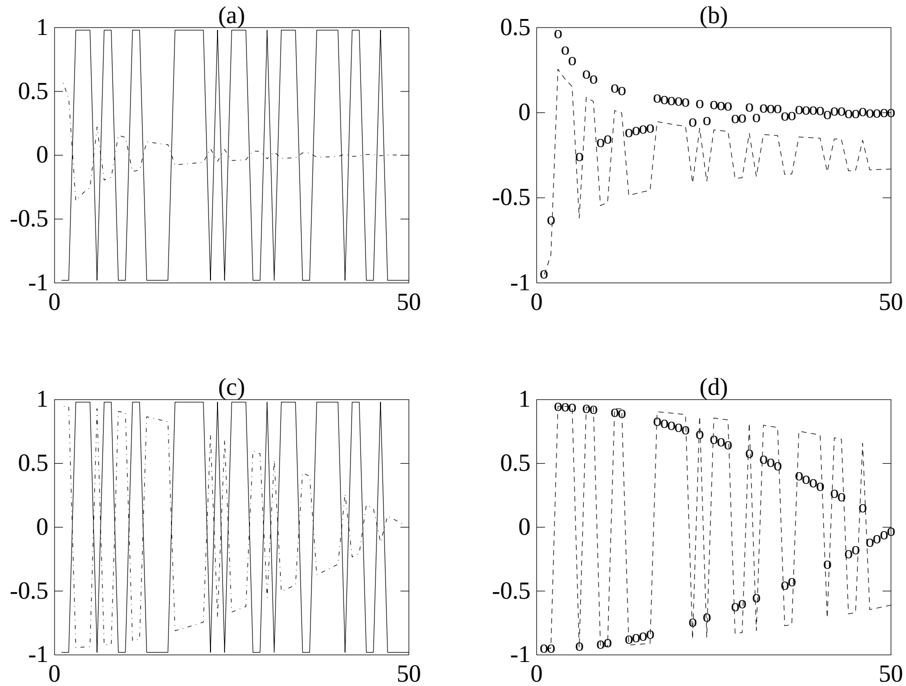


Figure 9.5: Worst case disturbances and the corresponding predicted errors for RLS and LMS. (a) The solid line represents the uncorrupted output $h_i x_i$ and the dashed line represents the worst case RLS disturbance. (b) The dashed line and the dotted line represent the RLS and LMS predicted errors, respectively, for the worst case RLS disturbance. (c) The solid line represents the uncorrupted output $h_i x_i$ and the dashed line represents the worst case LMS disturbance. (d) The dashed line and the dotted line represent the RLS and LMS predicted errors, respectively, for the worst case LMS disturbance.

output $h_i x_i$, are depicted in Figure 9.5a. As can be seen from Figure 9.5b, the corresponding RLS predicted error does not go to zero (it is actually biased), whereas the LMS predicted error does. The worst case LMS disturbance signal is given in Figure 9.5c, and as before, the LMS predicted error tends to zero, while the RLS predicted error does not. The form of the worst case disturbances (especially for RLS) are quite interesting; they compete with the true output early on, and then go to zero.

The disturbance signals considered in this example are rather contrived and may not happen in practice. However, they serve to illustrate the fact that the RLS algorithm may have poor performance even if the disturbance signals have small energy. On the other hand, LMS will have robust performance over a wide range of disturbance signals.

### 9.6.1 Discussion

In Sec. 9.5.1 we motivated the $\gamma_{f,opt} = 1$ result for normalized LMS by considering a disturbance strategy that made the observed output $d_i$ coincide with the expected output $h_i \hat{w}_{|-1}$. It is now illuminating to consider the dual strategy for the estimator.

Recall that in the a posteriori adaptive filtering problem the estimator has access to observations $d_0, d_1, \ldots, d_i$ and is required to construct an estimate of $\hat{s}_{i|i}$ of the uncorrupted output $s_i = h_i x_i$. The dual to the above mentioned disturbance strategy would be to construct an estimate that coincides with the observed output, *viz.*,

$$\hat{s}_{i|i} = d_i. \tag{9.6.8}$$

The corresponding filtered error is:

$$e_{f,i} = \hat{s}_{i|i} - h_i x_i = d_i - h_i x_i = v_i.$$

Thus the ratio in (9.3.3) can be made arbitrarily close to one, and the estimator (9.6.8) will achieve the same $\gamma_{f,opt} = 1$ that the normalized LMS algorithm does.

The fact that the simplistic estimator (9.6.8) (which is obviously of no practical use) is an optimal $H^\infty$ a posteriori filter seems to question the very merit of being $H^\infty$ optimal. A first indication towards this direction may be the fact that the $H^\infty$ estimators that achieve a certain level $\gamma$ are nonunique. In our opinion, the property of being $H^\infty$ optimal (*i.e.* , of minimizing the energy gain from the disturbances to the errors) is a desirable property in itself. The high sensitivity of the RLS algorithm to different disturbance signals, as illustrated in the example of Sec. 9.6, clearly indicates the desirability of the $H^\infty$ optimality property. However, different estimators in the set of all $H^\infty$ optimal estimators may have drastically different behaviour with respect to other desirable performance measures.

In Sec. 9.7 we develop the full parametrization of all $H^\infty$ optimal a posteriori and a priori adaptive filters, and show how to obtain (9.6.8) as a special case of this parametrization. Moreover, it can be shown (see [HHK96]) that among all $H^\infty$-optimal a posteriori filters the filter (9.6.8) has the worst $H^2$ (or, roughly speaking, average) performance. Thus it is the least desirable $H^\infty$-optimal filter with respect to an $H^2$ criterion. On the other hand, as indicated in Theorems 9.5.1 and 9.5.2, the LMS and normalized LMS algorithms correspond to the so-called central filters. These central filters have other desirable properties that we discuss in Sec. 9.8: they are risk-sensitive optimal and can also be shown to be maximum entropy.

The main problem with the estimator (9.6.8) is that it makes absolutely no use of the state-space model (9.5.1). We should note that it is not possible to come up with such a simple minded estimator in the a priori case: indeed as we shall see in the next section, the a priori estimator corresponding to (9.6.8) is highly nontrivial. The reason seems to be that since in the a priori case one deals with predicted error energy, it is inevitable that one must make use of the state-space model (9.5.1) in order to construct an optimal prediction of the next output. Thus in the a priori case, the problems arising from such unreasonable estimators such as (9.6.8) are avoided.

## 9.7 All $H^\infty$ Adaptive Filters

In Sec. 9.6.1 we came up with an alternative optimal $H^\infty$ a posteriori filter. We now use the results of Theorems 9.4.3 and 9.4.4 to parametrize all optimal $H^\infty$ a priori and a posteriori filters.

**Theorem 9.7.1 (All $H^\infty$ A Posteriori Adaptive Filters)** *If the input date $\{h_i\}$ is exciting, all $H^\infty$ optimal a posteriori adaptive filters that achieve $\gamma_{f,opt} = 1$ are given by*

$$\hat{s}_{j|j} = h_j\hat{w}_{|j} + (1+\mu h_j h_j^*)^{-\frac{1}{2}}\mathcal{S}_j\left((1+\mu h_j h_j^*)^{\frac{1}{2}}(d_j - h_j\hat{w}_{|j}), \ldots, (1+\mu h_0 h_0^*)^{\frac{1}{2}}(d_0 - h_0\hat{w}_{|0})\right) \tag{9.7.1}$$

*where $\hat{w}_{|j}$ satisfies the recursion*

$$\hat{w}_{|j+1} = \hat{w}_j + \frac{\mu h_{j+1}^*}{1+\mu h_{j+1}h_{j+1}^*}(d_{j+1} - h_{j+1}\hat{w}_{|j}) - \frac{\mu h_j^*}{1+\mu h_{j+1}h_{j+1}^*}(\hat{s}_{j|j} - h_j\hat{w}_{|j}), \quad \hat{w}_{|-1} \tag{9.7.2}$$

*and $\mathcal{S}$ is any (possibly nonlinear) contractive causal mapping.*

**Proof:** Simply restating the result of Theorem 9.4.3 for the special case $F_j = I$, $G_j = 0$, $H_j = h_j$ and $L_j = h_j$, and using the identity

$$I - h_j(P_j^{-1} + h_j^* h_j)^{-1} h_j^* = (I + h_j P_j h_j^*)^{-1},$$

along with the fact that for the $H^\infty$-optimal a posteriori adaptive filters we have $\gamma_{f,opt} = 1$ and $P_i = \mu I$, yields the desired result.

■

We can now note the significance of some special choices for the causal contraction $\mathcal{S}$.

(i) $\mathcal{S} = 0$ : This yields the normalized LMS algorithm.

(ii) $\mathcal{S} = I$ : This yields

$$\hat{s}_{j|j} = h_j \hat{w}_{|j} + (1 + \mu h_j h_j^*)^{-\frac{1}{2}} (1 + \mu h_j h_j^*)^{\frac{1}{2}} (d_j - h_j \hat{w}_{|j}) = d_j,$$

which is the simple minded estimator of Sec. 9.6.1.

(iii) $\mathcal{S} = -I$ : This yields

$$\hat{s}_{j|j} = h_j \hat{w}_{|j} - (1 + \mu h_j h_j^*)^{-\frac{1}{2}} (1 + \mu h_j h_j^*)^{\frac{1}{2}} (d_j - h_j \hat{w}_{|j}) = 2 h_j \hat{w}_{|j} - d_j,$$

so that the recursion for $\hat{w}_{|j}$ becomes

$$\hat{w}_{|j+1} = \hat{w}_j + \frac{\mu h_{j+1}^*}{1 + \mu h_{j+1} h_{j+1}^*} (d_{j+1} - h_{j+1} \hat{w}_{|j}) + \frac{\mu h_j^*}{1 + \mu h_{j+1} h_{j+1}^*} (d_j - h_j \hat{w}_{|j}), \quad \hat{w}_{|-1}.$$

**Theorem 9.7.2 (All $H^\infty$ A Priori Adaptive Filters)** *If the input data $\{h_i\}$ is exciting, and $0 < \mu < \inf_i \frac{1}{h_i h_i^*}$, then all $H^\infty$ optimal a priori adaptive filters are given by*

$$\hat{s}_j = h_j \hat{w}_{|j-1} + (1 - \mu h_j h_j^*)^{\frac{1}{2}} \mathcal{S}_j \left( (1 - \mu h_{j-1} h_{j-1}^*)^{\frac{1}{2}} (d_{j-1} - h_{j-1} \bar{w}_{|j-2}), \ldots, (1 - \mu h_0 h_0^*)^{\frac{1}{2}} (d_0 - h_0 \bar{w}_{|-1}) \right), \tag{9.7.3}$$

*where*

$$\bar{w}_{|k-1} = \hat{w}_{|k-1} + \frac{\mu h_k^*}{-1 + \mu h_k h_k^*} (\hat{s}_k - h_k \hat{w}_{|k-1}), \tag{9.7.4}$$

*$\hat{w}_{|j}$ satisfies the recursion*

$$\hat{w}_{|j} = \hat{w}_{|j-1} + \mu h_j^* (d_j - h_j \hat{w}_{|j-1}) - \mu h_j^* (\hat{s}_j - h_j \hat{w}_{|j-1}), \quad \hat{w}_{|-1} \tag{9.7.5}$$

*and $\mathcal{S}$ is any (possibly nonlinear) contractive causal mapping.*

**Proof:** Simply restating the result of Theorem 9.4.4 for the special case $F_j = I$, $G_j = 0$, $H_j = h_j$ and $L_j = h_j$, and using the fact that for the $H^\infty$-optimal a priori filter we have $\gamma_{p,opt} = 1$, $P_i = \mu I$ and $\tilde{P}_i = \mu I - h_i^* h_i$, yields the desired result. Indeed equations (9.7.3), (9.7.4) and (9.7.5) are the corresponding specializations of equations (9.4.18), (9.4.19) and (9.4.20), respectively.

∎

We once more note the consequences of some special choices of the causal contraction $\mathcal{S}$.

(i) $\mathcal{S} = 0$ : This yields the LMS algorithm.

(ii) $\mathcal{S} = I$ : This yields

$$\hat{s}_j = h_j \hat{w}_{|j-1} + (1 - \mu h_j h_j^*)^{\frac{1}{2}} (1 - \mu h_{j-1} h_{j-1}^*)^{\frac{1}{2}} (d_{j-1} - h_{j-1} \bar{w}_{|j-2}),$$

where $\bar{w}_{|j-2}$ and $\hat{w}_{|j-1}$ satisfy (9.7.4) and (9.7.5). The above filter is the a priori adaptive filter that corresponds to the simple minded estimator of Sec. 9.6.1. Note that in this case the filter is highly nontrivial.

(iii) $\mathcal{S} = -I$ : This yields

$$\hat{s}_j = h_j \hat{w}_{|j-1} - (1 - \mu h_j h_j^*)^{\frac{1}{2}} (1 - \mu h_{j-1} h_{j-1}^*)^{\frac{1}{2}} (d_{j-1} - h_{j-1} \bar{w}_{|j-2}).$$

Note that it does not seem possible to obtain a simplistic a priori estimator that achieves optimal performance.

## 9.8 Risk-Sensitive Optimality

In this section we focus on a certain property of the central $H^\infty$ filters, namely the fact that they are risk-sensitive optimal filters. This will give further insight into the LMS and normalized LMS algorithms, and in particular will provide a stochastic interpretation in the special case of disturbances that are independent Gaussian random variables.

The risk-sensitive (or exponential cost) criterion was introduced in [Jac73] and further studied in [SDJ74, Whi90, SFB92]. We begin with a brief introduction to the risk-sensitive criterion. For much more on this subject consult [Whi90].

### 9.8.1 The Exponential Cost Function

Although it is straightforward to consider the risk-sensitive criterion in the full generality of the state-space model of Sec. 9.4,[4] here we only deal with the special case of our interest. To this end, consider the state-space model corresponding to the adaptive filtering problem we have been studying:

$$\left\{\begin{array}{rcl} x_{i+1} & = & x_i \\ d_i & = & h_i x_i + v_i \end{array}\right. , \quad x_0 = w \tag{9.8.1}$$

where we now assume that $w$ and the $\{v_i\}$ are independent Gaussian random variables with means $\hat{w}_{|-1}$ and zero and covariances $\Pi_0$ and $I$, respectively. As before, we are interested in the filtered and predicted estimates $\hat{s}_{i|i} = \mathcal{F}_f(d_0, d_1, \ldots, d_i)$ and $\hat{s}_i = \mathcal{F}_p(d_0, d_1, \ldots, d_{i-1})$ of the uncorrupted output $s_i = h_i x_i$. The corresponding filtered and predicted errors are given by $e_{f,i} = \hat{s}_{i|i} - s_i$ and $e_{p,i} = \hat{s}_i - s_i$. The conventional Kalman filter is an estimator that performs the following minimization (see *e.g.* [Jaz70, AM79]):

$$\min_{\{\hat{s}_j\}} \left[ E \sum_{j=0}^{i} e_{p,j}^* e_{p,j} \right], \tag{9.8.2}$$

where the expectation is taken over the Gaussian random variables $w$ and $\{v_j\}_{j=0}^{\infty}$ whose joint conditional distribution is given by:

$$p(w, v_0, \ldots, v_i | d_0, \ldots, d_i) \propto$$

$$\exp\left[ -\frac{1}{2} \left( (w - \hat{w}_{|-1})^* \Pi_0^{-1} (w - \hat{w}_{|-1}) + \sum_{j=0}^{i} (d_j - h_j x_j)^* (d_j - h_j x_j) \right) \right],$$

and where the symbol $\propto$ stands for 'proportional to'. In the terminology of [Whi90], the filter that minimizes (9.8.2) is known as the risk-neutral filter.

[4] In fact, this is done in Sec. 4.2.

An alternative criterion that is risk-sensitive has been extensively studied in [Jac73, Whi90, SFB92] and corresponds to the following minimization problem

$$\min_{\{\hat{s}_{j|j}\}} \mu_{f,i}(\theta) = \min_{\{\hat{s}_{j|j}\}} \left( -\frac{2}{\theta} log \left[ E\exp(-\frac{\theta}{2}\mathbf{C}_{f,i}) \right] \right), \tag{9.8.3}$$

or

$$\min_{\{\hat{s}_{j}\}} \mu_{p,i}(\theta) = \min_{\{\hat{s}_{j}\}} \left( -\frac{2}{\theta} log \left[ E\exp(-\frac{\theta}{2}\mathbf{C}_{p,i}) \right] \right), \tag{9.8.4}$$

where $\mathbf{C}_{f,i} = \sum_{j=0}^{i} e_{f,i}^* e_{f,i}$ and $\mathbf{C}_{p,i} = \sum_{j=0}^{i} e_{p,i}^* e_{p,i}$. The criteria in (9.8.3) and (9.8.4) are known as the a posteriori and a priori exponential cost functions, and any filters that minimize $\mu_{f,i}(\theta)$ and $\mu_{p,i}(\theta)$ are referred to as a posteriori and a priori risk-sensitive filters, respectively. The scalar parameter $\theta$ is correspondingly called the risk-sensitivity parameter. Some intuition concerning the nature of this modified criterion is obtained by expanding $\mu_i(\theta)$ (where we have dropped the subscripts $f$ and $p$ since the argument follows for both filtered and predicted estimates) in terms of $\theta$ and writing,

$$\mu_i(\theta) = E(\mathbf{C}_i) - \frac{\theta}{4}\mathrm{Var}(\mathbf{C}_i) + O(\theta^2).$$

The above equation shows that for $\theta = 0$, we have the risk-neutral case (*i.e.*, the conventional Kalman filter). When $\theta > 0$, we seek to maximize $E\exp(-\frac{\theta}{2}\mathbf{C}_i)$, which is convex and decreasing in $\mathbf{C}_i$. Such a criterion is termed risk-seeking (or optimistic) since larger weights are on small values of $\mathbf{C}_i$, and hence we are more concerned with the frequent occurrence of moderate values of $\mathbf{C}_i$ than with the occasional large values. When $\theta < 0$, we seek to minimize $Eexp(-\frac{\theta}{2}\mathbf{C}_i)$, which is convex and increasing in $\mathbf{C}_i$. Such a criterion is termed risk-averse (or pessimistic) since large weights are on large values of $\mathbf{C}_i$, and hence we are more concerned with the occasional occurrence of large values than with the frequent occurrence of moderate ones.

The relationship between the risk-sensitive criterion and the $H^\infty$ criterion was first noted in [GD88] and has been further discussed in [Whi90, HSK96b]. It may be formally stated as follows: *In the risk-averse case $\theta < 0$, the risk-sensitive optimal filter with parameter $\theta$ is given by the central $H^\infty$ filter with level $\gamma = -\theta^{-\frac{1}{2}}$.* In particular, there is a certain smallest value of the risk-sensitivity parameter $\bar{\theta}$, after which the minimizing property of $\mu_i(\theta)$ breaks down, and it is this value that yields the optimal central $H^\infty$ filter with $\gamma_{opt} = -\bar{\theta}^{-1/2}$.

### 9.8.2 Risk-sensitive Adaptive Filtering

Using the discussion of Sec. 9.8.1, we are now in a position to state the risk-sensitive results for LMS and normalized LMS.

**Theorem 9.8.1 (Normalized LMS and Risk-sensitivity)** *Consider the state-space model (9.8.1) where the $w$ and $\{v_j\}$ are independent Gaussian random variables with means $\hat{w}_{|-1}$ and $0$, and variances $\mu I$ and $I$, respectively. The solution to the following minimization problem*

$$\min_{\{\hat{s}_{j|j}\}} \mu_f(\theta) = \min_{\{\hat{s}_{j|j}\}} \left( 2log \left[ E\, exp(\frac{1}{2}\mathbf{C}_f), \right] \right) \tag{9.8.5}$$

*where $\mathbf{C}_f = \sum_{j=0}^{\infty} e_{f,i}^* e_{f,i}$, and the expectation is taken over $w$ and $\{v_j\}$ subject to observing $\{d_0, d_1, \ldots, d_i\}$, is given by the normalized LMS algorithm*

$$\hat{s}_{i|i} = h_i \hat{w}_{|i},$$

*and*

$$\hat{w}_{|i+1} = \hat{w}_{|i} + \frac{\mu h_{i+1}^*}{1 + \mu h_{i+1} h_{i+1}^*}(d_{i+1} - h_{i+1}\hat{w}_{|i}) \quad , \quad \hat{w}_{|-1}. \tag{9.8.6}$$

**Theorem 9.8.2 (LMS and Risk-sensitivity)** *Consider the state-space model (9.8.1) where the $w$ and $\{v_j\}$ are independent Gaussian random variables with means $\hat{w}_{|-1}$ and $0$, and variances $\mu I$ and $I$, respectively. Suppose moreover, that the $\{h_i\}$ are exciting, and that*

$$0 < \mu < \inf_i \frac{1}{h_i h_i^*}.$$

*Then the solution to the following minimization problem*

$$\min_{\{\hat{s}_j\}} \mu_p(\theta) = \min_{\{\hat{s}_j\}} \left( 2log \left[ E\, exp(\frac{1}{2}\mathbf{C}_p), \right] \right) \tag{9.8.7}$$

*where $\mathbf{C}_p = \sum_{j=0}^{\infty} e_{p,i}^* e_{p,i}$, and the expectation is taken over $w$ and $\{v_j\}$ subject to observing $\{d_0, d_1, \ldots, d_{i-1}\}$, is given by the LMS algorithm*

$$\hat{s}_i = h_i \hat{w}_{i-1},$$

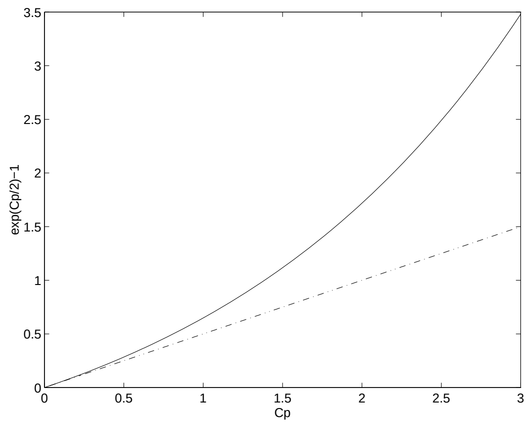


Figure 9.6: The criterion (9.8.7) is termed *risk averse* (or pessimistic) since the cost function $exp(\mathbf{C}_p/2)$ is very large for large values of $\mathbf{C}_p$. Hence we are more concerned with the occasional occurrence of large values of $\mathbf{C}_p$ than with the frequent occurrence of moderate ones. This fact corresponds well with the intuition gained from the $H^\infty$ optimality of the LMS algorithm. We have also plotted $\mathbf{C}_p/2$ (the dashed line) to compare the two cost functions, since the RLS algorithm minimizes the expected value of $\mathbf{C}_p/2$.

*and*

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \mu h_i^*(d_i - h_i\hat{w}_{|i-1}) \quad , \quad \hat{w}_{|-1}. \tag{9.8.8}$$

Before closing this section we should remark that the central $H^\infty$ filters possess other properties in addition to the one described above. In the game theoretic formulation of $H^\infty$ estimation, the central filter corresponds to the solution of the game (see *e.g.*, [BB95] and Sec. 4.3). Moreover, among all $H^\infty$ estimators that achieve a certain level $\gamma$, the central solution can be shown to be the maximum entropy [GM89] solution. However, we shall not pursue these directions here.

## 9.9 Further Remarks

In addition to yielding a new interpretation for the LMS algorithm and providing it with a rigorous basis, the results described in this chapter have lent themselves to various generalizations and have allowed the authors to obtain several new results.

We close this chapter by listing some of these ideas and results here. We should also mention that we believe the framework presented in this chapter provides a new way of looking at adaptive algorithms and should be worthy of further scrutiny.

### LMS with Time-Varying Learning Rate

In many applications one uses the LMS algorithm with time-varying stepsize (or learning rate), viz.,

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \mu_i h_i^*(d_i - h_i \hat{w}_{|i-1}), \quad \hat{w}_{|-1}. \tag{9.9.1}$$

In this case, it is straightforward to show that if the vectors $\{\mu_i^{1/2} h_i\}$ are exciting, and if $\mu_i h_i h_i^* \leq 1$ for all $i$, then the LMS algorithm with time-varying stepsize solves the following minimax problem:

$$\inf_{\mathcal{F}} \sup_{w,v \in h^2} \frac{\sum_{j=0}^{\infty} \mu_j |e_{p,j}|^2}{|w - \hat{w}_{|-1}|^2 + \sum_{j=0}^{\infty} \mu_j |v_j|^2} = 1. \tag{9.9.2}$$

### $H^\infty$ Adaptive Filtering

In this chapter we have shown that if adaptive filtering for output prediction error is considered then the central $H^\infty$-optimal adaptive filter is LMS. It is also possible to consider prediction of the filter weight vector itself, and for the purpose of coping with time-variations, to consider exponentially weighted, finite-memory and time-varying adaptive filtering. This results in some new adaptive filtering algorithms that may be useful in uncertain and non-stationary environments (see [HK94a] and Chapter 11).

### $H^\infty$ Norm Bounds for the RLS Algorithm

In order to compare the robustness of $H^2$-optimal algorithms (such as RLS) with $H^\infty$-optimal algorithms (such as LMS) it it useful to obtain $H^\infty$ norm bounds for these algorithms. This has been done for the RLS algorithm in [HK94b] (see also Chapter 10), where it is shown that unlike LMS, the $H^\infty$ norm of the RLS algorithm depends on the input data $\{h_i\}$ and, roughly speaking, grows linearly in the parameter $\mu$.

### A Time-Domain Feedback Analysis

Using some of the ideas presented here, a time-domain feedback analysis of recursive adaptive schemes, including gradient-based and Gauss-Newton filters has been developed [SR95a, SR95b], for both the FIR and IIR contexts. The analysis highlights an intrinsic feedback structure in terms of a feedforward lossless or contractive map and a feedback memoryless or dynamic map. The structure lends itself to analysis via energy conservation arguments and via standard tools in system theory such as the small gain theorem [Kha92, Vid93]. It further suggests choices for the adaptation gains (or step-sizes) in order to enforce a robust performance in the presence of disturbances (along the lines of $H^\infty$theory), as well as improve the convergence speed of the adaptive algorithms.

### Nonlinear Problems

The results presented in this chapter are for linear adaptive filters and can be somewhat generalized to nonlinear adaptive filters (such as neural networks) if one linearizes these nonlinear models around some suitable point. Using this approach it can be shown (see [HSK94a]) that, for nonlinear problems, instantaneous-gradient-based algorithms (such as backpropagation [RM86]) are locally $H^\infty$-optimal. This means that if the initial estimate of the weight vector is close enough to its true value, and if the disturbances are small enough, then the maximum energy gain from the disturbances to the output prediction errors is arbitrarily close to one. Global $H^\infty$-optimal filters can also be found in the nonlinear case, but they have the drawback of being infinite-dimensional (see [HK95b]).

## 9.10 Conclusion

We have demonstrated that the LMS algorithm is $H^\infty$ optimal. This result solves a long standing issue of finding a rigorous basis for the LMS algorithm, and also confirms its robustness. We find it quite interesting that despite the fact that there has only been recent interest in the field of $H^\infty$ estimation, there has existed an $H^\infty$

optimal estimation algorithm that has been widely used in practice for the past three decades.

## 9.A A First Principles Proof of the $H^\infty$ Optimality of LMS

In this appendix we shall outline a first principles proof of the $H^\infty$ optimality of the LMS and normalized LMS algorithms that does not require the results of Theorems 9.4.1 and 9.4.2 on $H^\infty$ filtering. The proofs rely on some easily verified inequalities. We begin with normalized LMS. (See also the last section in [SK94b] and [SR95a].)

### 9.A.1 The Normalized LMS Algorithm

Recall that in Sec. 9.5.1, after the statement of Theorem 9.5.1, we constructed a disturbance signal such that for any $\epsilon > 0$,

$$\frac{\|e_f\|^2}{\mu^{-1}|w-\hat{w}_{|-1}|^2+\|v\|^2} \geq 1-\epsilon.$$

Since this was just one special disturbance signal, we conclude that if the input vectors are exciting, we have

$$\sup_{w,v\in h^2} \frac{\|e_f\|^2}{\mu^{-1}|w-\hat{w}_{|-1}|^2+\|v\|^2} \geq 1. \tag{9.A.1}$$

We shall now show that the normalized LMS algorithm achieves one in the above inequality. This, of course, also shows that $\gamma_{f,opt} = 1$. To this end, note that the normalized LMS algorithm

$$\hat{w}_{|j} = \hat{w}_{|j-1} + \frac{h_j^*}{\mu^{-1}+h_jh_j^*}(d_j - h_j\hat{w}_{|j-1}),$$

can, after some rearrangement, be written as

$$\hat{w}_{|j-1} = \hat{w}_{|j} - \mu h_i^*(d_j - h_j\hat{w}_{|j}).$$

If we now define $\tilde{w}_{|j} = w - \hat{w}_{|j}$, the above expression allows us to write

$$\mu^{-1/2}\left[\tilde{w}_{|j-1}\right] = \mu^{-1/2}\left[\tilde{w}_{|j} + \mu h_i^*(d_j - h_j\hat{w}_{|j})\right]. \tag{9.A.2}$$

[The reason for multiplying both sides by $\mu^{-1/2}$ will become clear in a moment.] On the other hand, we may write $v_j = d_j - h_j w$ as

$$v_j = (d_j - h_j\hat{w}_{|j}) - h_j\tilde{w}_{|j}. \tag{9.A.3}$$

Squaring both sides of (9.A.2) and (9.A.3) and adding the results yields

$$\mu^{-1}|\tilde{w}_{|j-1}|^2 + |v_j|^2 = \mu^{-1}|\tilde{w}_{|j}|^2 + |h_j\tilde{w}_{|j}|^2 + (1+\mu h_j h_j^*)(d_j - h_j\hat{w}_{|j})^2. \tag{9.A.4}$$

Now since the third term on the RHS of the above expression is positive, and since $h_j\tilde{w}_{|j} = e_{f,j}$, we may write

$$\mu^{-1}|\tilde{w}_{|j-1}|^2 + |v_j|^2 \geq \mu^{-1}|\tilde{w}_{|j}|^2 + |e_{f,j}|^2. \tag{9.A.5}$$

If we now add all inequalities of the form (9.A.5) from time $j=0$ to time $j=i$, we have

$$\mu^{-1}|w - \hat{w}_{|-1}|^2 + \sum_{j=0}^{i}|v_j|^2 \geq \mu^{-1}|\tilde{w}_{|i}|^2 + \sum_{j=0}^{i}|e_{f,j}|^2 \geq \sum_{j=0}^{i}|e_{f,j}|^2, \tag{9.A.6}$$

which in turn implies

$$\frac{\sum_{j=0}^{i}|e_{f,j}|^2}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \sum_{j=0}^{i}|v_j|^2} \leq 1. \tag{9.A.7}$$

Thus, for normalized LMS, in the limit as $i \to \infty$ we have

$$\sup_{w,v\in h^2} \frac{\sum_{j=0}^{\infty}|e_{f,j}|^2}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \sum_{j=0}^{\infty}|v_j|^2} = \sup_{w,v\in h^2} \frac{\|e_f\|^2}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \|v\|^2} = 1, \tag{9.A.8}$$

which is the desired result.

## 9.A.2 The LMS Algorithm

The proof for the LMS algorithm follows the exact same lines as the one above. Eq. (9.A.2) is now replaced by

$$\mu^{-1/2}\left[\tilde{w}_{|j}\right] = \mu^{-1/2}\left[\tilde{w}_{|j-1} - \mu h_i^*(d_j - h_j\hat{w}_{|j-1})\right], \tag{9.A.9}$$

and (9.A.3) by

$$v_j = (d_j - h_j\hat{w}_{|j-1}) - h_j\tilde{w}_{|j-1}. \tag{9.A.10}$$

This time we square both sides of (9.A.9) and (9.A.10) and subtract the results to obtain

$$\mu^{-1}|\tilde{w}_{|j}|^2 - |v_j|^2 = \mu^{-1}|\tilde{w}_{|j-1}|^2 - |h_j\tilde{w}_{|j-1}|^2 - (1-\mu h_j h_j^*)(d_j - h_j\hat{w}_{|j-1})^2. \quad (9.A.11)$$

Now since we have the bound $\mu \leq \frac{1}{h_j h_j^*}$, the third term on the RHS is negative, and we can write

$$\mu^{-1}|\tilde{w}_{|j-1}|^2 + |v_j|^2 \geq \mu^{-1}|\tilde{w}_{|j}|^2 + |\underbrace{h_j\tilde{w}_{|j-1}}_{e_{p,j}}|^2. \quad (9.A.12)$$

The remainder of the proof is now identical to the normalized LMS case.

# Chapter 10

# Robustness of Least-Squares Estimators

In the previous chapter we demonstrated that instantaneous-gradient-based adaptive filters, such as the LMS and normalized LMS algorithms, minimize the maximum energy gain from the disturbances to the (prediction and filtered) estimation errors. A natural question to ask is how do other adaptive algorithms compare to these ($H^\infty$) "optimal" solutions? Needless to say, in the $H^\infty$ approach to robust adaptive filtering, given any algorithm, this comparison is achieved by computing the corresponding maximum energy gain (or $H^\infty$ norm) from the disturbances to the estimation errors.

In this chapter we shall study the robustness of the other important class of adaptive filters, namely, least-squares-based methods, such as the RLS algorithm, from the above point of view. We shall essentially obtain upper and lower bounds for the $H^\infty$ norm of the RLS algorithm (in fact, more generally, of the Kalman filter) with respect to prediction and filtered errors. The main conclusion is that, unlike LMS and normalized LMS which do not allow for any amplification of the disturbances, the RLS algorithm does allow for such amplification. This fact can be especially pronounced in the prediction error case. Moreover, it is shown that the $H^\infty$ norm for RLS is data-dependent, whereas for LMS and normalized LMS it was not so. [The $H^\infty$ norm was simply unity.] The significance of the results will also discussed.

## 10.1 Introduction

In the spirit of recent work in robust control there has been growing interest in deterministic worst-case identification. In such problems one is confronted with the task of designing identification algorithms that have robust performance in the presence of unknown but bounded noise. Likewise it is required to analyze the worst-case behaviour of various identification algorithms with respect to such disturbances. For an introduction to recent approaches in $H^\infty$ and $l_1$ identification the reader is referred to [AHN91, WL92, DTT93, AK93, Mil95, PM95] and the references therein.

In the previous chapter we showed that the LMS (normalized LMS) adaptive algorithm is $H^\infty$-optimal in the sense that it minimizes the worst-case energy gain from the disturbances to the prediction (filtered) errors. This result confirmed the robustness of LMS (normalized LMS) with respect to model uncertainties and gave it a rigorous basis which was lacking. The celebrated recursive-least-squares (RLS) algorithm [Hay96], is an also widely used adaptive algorithm that enjoys certain well known optimality properties under suitable stochastic assumptions about the exogenous noise. In Chapter 9 we also compared the robustness of the RLS algorithm to LMS, and constructed a disturbance sequence of small energy for which RLS yielded a prediction error of large energy.

In the present chapter we shall obtain upper and lower bounds on the $H^\infty$ norm of the RLS algorithm. These bounds are of interest for several reasons. First they demonstrate that unlike the LMS algorithm whose $H^\infty$ norm is unity (independent of the input-output data), the $H^\infty$ norm of the RLS algorithm depends on the input-output data, and therefore it may be more robust or less robust with respect to different data sets. Moreover, the exact calculation of the $H^\infty$ norm for RLS requires the calculation of the induced two-norm of a linear time-varying operator, which can be quite cumbersome, and, in addition, needs all the input-output data, which may not be available in real-time scenarios. The $H^\infty$ bounds we obtain only require simple a priori knowledge of the data, and may therefore be used as a simple check to verify whether RLS has the desired robustness with respect to a given application.

A brief outline of the chapter is as follows. In Sec. 10.2 we give general upper and lower bounds for the $H^\infty$ norm of the Kalman filter, with respect to prediction and filtered errors. The proofs of the upper bounds are given in Sec. 10.3 and are based on certain minimization properties of least-squares estimators. The proofs of the lower bounds are given in 10.4 and are essentially based on computing the energy gains for suitably chosen disturbances. Sec. 10.5 specializes the general results of Sec. 10.2 to the adaptive filtering problem, discusses its various implications and provides a simple example to illustrate the results. The chapter concludes with Sec. 10.6.

## 10.2 A General $H^\infty$ Bound

In this section we shall derive general upper and lower bounds for the $H^\infty$ norm of the Kalman filter with respect to prediction and filtered errors. These results will then be specialized to the adaptive filtering problem to obtain corresponding $H^\infty$ norm bounds for the RLS algorithm.

To this end, consider, once more, the standard state-space model

$$\begin{cases} x_{i+1} &= F_i x_i + G_i u_i, \qquad x_0 \\ y_i &= H_i x_i + v_i, \qquad i \geq 0 \end{cases} \tag{10.2.1}$$

where $F_i \in \mathcal{C}^{n\times n}$, $G_i \in \mathcal{C}^{n\times m}$ and $H_i \in \mathcal{C}^{p\times n}$ are known matrices, $x_0$, $\{u_i\}$, and $\{v_i\}$ are unknown quantities and $y_i$ is the measured output. Moreover, $v_i$ can be regarded as measurement noise and $u_i$ as process noise or driving disturbance.

Recall from Chapter 2 that the Kalman filter for computing the predicted estimates of the states, denoted by $\hat{x}_i$, (*i.e.*, $\hat{x}_i$ is the least-squares estimate of $x_i$, given $\{y_j, j < i\}$) is given by

$$\hat{x}_{i+1} = F_i \hat{x}_i + K_{p,i}(y_i - H_i \hat{x}_i), \tag{10.2.2}$$

where

$$K_{p,i} = F_i P_i H_i^* R_{e,i}^{-1} \quad \text{and} \quad R_{e,i} = R_i + H_i P_i H_i^* \tag{10.2.3}$$

and where $P_i$ satisfies the Riccati recursion,

$$P_{i+1} = F_i P_i F_i^* + G_i Q_i G_i^* - K_{p,i} R_{e,i} K_{p,i}^*, \quad P_0 = \Pi_0. \tag{10.2.4}$$

[Note here that $\{Q_i, R_i\}$ and $\Pi_0$ are given positive definite weighting matrices.]

There is also a filtered form of the Kalman filter recursions for computing, $\hat{x}_{i|i}$, the least-squares estimate of $x_i$, given $\{y_j, j \leq i\}$, which is given below,

$$\hat{x}_{i+1|i+1} = F_i \hat{x}_{i|i} + K_{f,i+1}(y_{i+1} - H_{i+1} F_i \hat{x}_{i|i}), \tag{10.2.5}$$

where

$$K_{f,i} = P_i H_i^* R_{e,i}^{-1}. \tag{10.2.6}$$

Now using $\hat{x}_i$ and $\hat{x}_{i|i}$ the predicted and filtered estimation errors of the uncorrupted output, $s_i = H_i x_i$, are given by

$$e_{p,i} = H_i x_i - H_i \hat{x}_i \triangleq H_i \tilde{x}_i, \tag{10.2.7}$$

and

$$e_{f,i} = H_i x_i - H_i \hat{x}_{i|i} \triangleq H_i \tilde{x}_{i|i}. \tag{10.2.8}$$

Note that both these estimation errors are different from the innovations (the prediction errors for estimating $y_i$),

$$e_i = y_i - H_i \hat{x}_i. \tag{10.2.9}$$

Indeed it is straightforward to see that we have

$$e_{p,i} = e_i - v_i, \tag{10.2.10}$$

and

$$e_{f,i} = R_i R_{e,i}^{-1} e_i - v_i. \tag{10.2.11}$$

The latter equality is justified below.

$$\begin{aligned}
e_{f,i} &= H_i x_i - H_i \hat{x}_{i|i} \\
&= y_i - v_i - H_i(\hat{x}_i + P_i H_i^* R_{e,i}^{-1} e_i) \\
&= e_i - H_i P_i H_i^* R_{e,i}^{-1} e_i - v_i \\
&= (R_{e,i} - H_i P_i H_i^*) R_{e,i}^{-1} e_i - v_i = R_i R_{e,i}^{-1} e_i - v_i.
\end{aligned}$$

We shall have the occasion to make use of both identities (10.2.10) and (10.2.11).

We can now state the main result of this chapter.

**Theorem 10.2.1 (Bounds for the $H^\infty$ Norm of the Kalman Filter)** *Consider the standard state-space model (10.2.1) and the predicted and filtered forms of the Kalman filter recursions, (10.2.2) and (10.2.5). Then for any $N$, we have the following results,*

$$(\sqrt{\bar{r}}-1)^2 \le \sup_{x_0,u,v\in h^2} \frac{\sum_{i=0}^{N}(H_i x_i - H_i\hat{x}_i)^* R_i^{-1}(H_i x_i - H_i\hat{x}_i)}{x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} \le (\sqrt{\bar{r}}+1)^2 \quad (10.2.12)$$

*and*

$$(\sqrt{1/\underline{r}}-1)^2 \le \sup_{x_0,u,v\in h^2} \frac{\sum_{i=0}^{N}(H_i x_i - H_i\hat{x}_{i|i})^* R_i^{-1}(H_i x_i - H_i\hat{x}_{i|i})}{x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} \le (\sqrt{1/\underline{r}}+1)^2 \quad (10.2.13)$$

*where we have defined,*

$$\bar{r} = \sup_i \ \bar{\sigma}\left(R_i^{-1/2} R_{e,i} R_i^{-*/2}\right) \quad (10.2.14)$$

*and*

$$\underline{r} = \inf_i \ \underline{\sigma}\left(R_i^{-1/2} R_{e,i} R_i^{-*/2}\right) \quad (10.2.15)$$

*and $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$ denote the maximum and minimum singular values of the matrix $A$, respectively.*

**Remarks:**

(i) Note that the quantities

$$\sup_{x_0,u,v\in h^2} \frac{\sum_{i=0}^{N}(H_i x_i - H_i\hat{x}_i)^* R_i^{-1}(H_i x_i - H_i\hat{x}_i)}{x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i}$$

and

$$\sup_{x_0,u,v\in h^2} \frac{\sum_{i=0}^{N}(H_i x_i - H_i\hat{x}_{i|i})^* R_i^{-1}(H_i x_i - H_i\hat{x}_{i|i})}{x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i}$$

in the inequalities (10.2.12) and (10.2.13) are simply the maximum energy gains from the (normalized) disturbances $\{\Pi_0^{-1/2}x_0, \{Q_i^{-1/2}u_i, R_i^{-1/2}v_i\}_{i=0}^{N}\}$ to the (normalized) prediction and filtered estimation errors $\{R_i^{-1/2}e_{p,i}\}_{i=0}^{N}$ and $\{R_i^{-1/2}e_{f,i}\}_{i=0}^{N}$, respectively. Thus (10.2.12) and (10.2.13) yield upper and lower bounds on the $H^\infty$ norm of the Kalman filter for prediction and filtered errors, respectively.

(ii) Note, moreover, that the upper and lower bounds on the $H^\infty$ norms, as given by Theorem 10.2.1 are relatively tight (especially for large values of $\bar{r}$ and $1/\underline{r}$). Indeed the upper and lower bounds differ only by two, since,

$$(\sqrt{\bar{r}}+1)-(\sqrt{\bar{r}}-1)=2 \quad \text{and} \quad (\sqrt{1/\underline{r}}+1)-(\sqrt{1/\underline{r}}-1)=2.$$

(iii) Note that Theorem 10.2.1 bounds the $H^\infty$ norm of the Kalman filter by quantities related to the maximum and minimum singular values of the normalized innovations variance, $R_i^{-1/2}R_{e,i}R_i^{-*/2}$. Intuitively, (10.2.12) suggests that the larger $R_{e,i}$ is, the larger $\bar{r}$ is, and hence the less robust the Kalman filter is with respect to prediction errors. On the other hand, (10.2.13) suggests that the smaller $R_{e,i}$ is, the larger $1/\underline{r}$ is, and hence the less robust the Kalman filter is with respect to filtered errors.

In particular, note that

$$R_i^{-1/2}R_{e,i}R_i^{-*/2}=I_p+R_i^{-1/2}H_iP_iH_i^*R_i^{-*/2}\geq I_p, \tag{10.2.16}$$

so that

$$\bar{r}\geq 1\geq 1/\underline{r}. \tag{10.2.17}$$

But using (10.2.13) this means that

$$\sup_{x_0,u,v\in h^2}\frac{\sum_{i=0}^{N}(H_ix_i-H_i\hat{x}_{i|i})^*R_i^{-1}(H_ix_i-H_i\hat{x}_{i|i})}{x_0^*\Pi_0^{-1}x_0+\sum_{i=0}^{N}u_i^*Q_i^{-1}u_i+\sum_{i=0}^{N}v_i^*R_i^{-1}v_i}\leq 4, \tag{10.2.18}$$

which is a very explicit, and quite surprising, bound. Thus the Kalman filter guarantees that the energy gain from the disturbances to the filtered errors never exceeds four.

(iv) The bounds of Theorem 10.2.1 are true for any value of $N$, and, in fact, they are also true when the upper limits of the sums in (10.2.12) and (10.2.13) are infinite. In other words, it is true that,

$$(\sqrt{\bar{r}}-1)^2\leq\sup_{x_0,u,v\in h^2}\frac{\sum_{i=0}^{\infty}(H_ix_i-H_i\hat{x}_i)^*R_i^{-1}(H_ix_i-H_i\hat{x}_i)}{x_0^*\Pi_0^{-1}x_0+\sum_{i=0}^{\infty}u_i^*Q_i^{-1}u_i+\sum_{i=0}^{\infty}v_i^*R_i^{-1}v_i}\leq(\sqrt{\bar{r}}+1)^2 \tag{10.2.19}$$

and

$$(\sqrt{1/\underline{r}}-1)^2 \leq \sup_{x_0,u,v\in h^2} \frac{\sum_{i=0}^{\infty}(H_ix_i - H_i\hat{x}_{i|i})^* R_i^{-1}(H_ix_i - H_i\hat{x}_{i|i})}{x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{\infty} u_i^*Q_i^{-1}u_i + \sum_{i=0}^{\infty} v_i^*R_i^{-1}v_i} \leq (\sqrt{1/\underline{r}}+1)^2 \tag{10.2.20}$$

where, as before

$$\bar{r} = \sup_i \ \bar{\sigma}\left(R_i^{-1/2}R_{e,i}R_i^{-*/2}\right) \quad \text{and} \quad \underline{r} = \inf_i \ \underline{\sigma}\left(R_i^{-1/2}R_{e,i}R_i^{-*/2}\right).$$

In particular, in the time-invariant (doubly) infinite-horizon case we have

$$(\sqrt{\bar{r}}-1)^2 \leq \sup_{x_0,u,v\in h^2} \frac{\sum_{i=-\infty}^{\infty}(H_ix_i - H_i\hat{x}_i)^* R^{-1}(H_ix_i - H_i\hat{x}_i)}{x_0^*\Pi_0^{-1}x_0 + \sum_{i=-\infty}^{\infty} u_i^*Q_i^{-1}u_i + \sum_{i=-\infty}^{\infty} v_i^*R_i^{-1}v_i} \leq (\sqrt{\bar{r}}+1)^2 \tag{10.2.21}$$

and

$$(\sqrt{1/\underline{r}}-1)^2 \leq \sup_{x_0,u,v\in h^2} \frac{\sum_{i=-\infty}^{\infty}(Hx_i - H\hat{x}_{i|i})^* R^{-1}(Hx_i - H\hat{x}_{i|i})}{\sum_{i=-\infty}^{\infty} u_i^*Q^{-1}u_i + \sum_{i=-\infty}^{\infty} v_i^*R^{-1}v_i} \leq (\sqrt{1/\underline{r}}+1)^2 \tag{10.2.22}$$

where now

$$\bar{r} = \bar{\sigma}\left(R{-}1/2R_eR^{-*/2}\right) \quad \text{and} \quad \underline{r} = \underline{\sigma}\left(R^{-1/2}R_eR^{-*/2}\right), \tag{10.2.23}$$

and $R_e = R + HPH^*$, with $P$ the unique positive semidefinite and stabilizing solution of the DARE

$$P = FPF^* + GQG^* - FPH^*(R+HPH^*)^{-1}HPF^*.$$

We will now proceed with the proof of Theorem 10.2.1.

## 10.3 Proof of the Upper Bounds

To prove the upper bounds of Theorem 10.2.1 we need the following three facts.

**Lemma 10.3.1 (Minimization of a Quadratic Form)** *We have*

$$\min_{x_0,\{u_i,v_i\}} \ x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N} u_i^*Q_i^{-1}u_i + \sum_{i=0}^{N} v_i^*R_i^{-1}v_i = \sum_{i=0}^{N} e_i^*R_{e,i}^{-1}e_i, \tag{10.3.1}$$

*where the minimization is taken subject to the state-space constraints (10.2.1), and where $e_i = y_i - H_i\hat{x}_i$ is the innovations.*

**Proof:** The lemma is simply a special case of Theorem 2.7.4.

■

**Lemma 10.3.2 (Simple Inequality)** *For any vectors $a$, $b$, and any matrix $M > 0$, we have*

$$(a+b)^*M(a+b) \geq (1-\frac{1}{\alpha})a^*Ma + (1-\alpha)b^*Mb, \quad \forall \alpha > 0. \tag{10.3.2}$$

**Proof:** Follows from

$$\begin{aligned}(a+b)^*M(a+b) - (1-\frac{1}{\alpha})a^*Ma + (1-\alpha)b^*Mb &= \frac{1}{\alpha}a^*Ma + a^*Mb + b^*Ma + \alpha b^*Mb \\ &= (\frac{1}{\sqrt{\alpha}}a + \sqrt{\alpha}b)^*M(\frac{1}{\sqrt{\alpha}}a + \sqrt{\alpha}b) \\ &\geq 0.\end{aligned}$$

■

**Lemma 10.3.3 (A Simple Minimization)** *For all $\beta > 0$, we have*

$$\min_{\alpha>1} \frac{\alpha^2 + (\beta-1)\alpha}{\alpha-1} = (1+\sqrt{\beta})^2, \tag{10.3.3}$$

*and*

$$\arg\min_{\alpha>1} \frac{\alpha^2 + (\beta-1)\alpha}{\alpha-1} = 1+\sqrt{\beta}. \tag{10.3.4}$$

**Proof:** Readily verified via differentiation.

■

We shall first prove the upper bound in (10.2.12) for the prediction error case. To this end, define

$$J = x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^{N} u_i^*Q_i^{-1}u_i + \sum_{i=0}^{N} v_i^*R_i^{-1}v_i. \tag{10.3.5}$$

Now using Lemma 10.3.1 it is obvious that

$$J \geq \min_{x_0,\{u_i,v_i\}} J = \sum_{i=0}^{N} e_i^*R_{e,i}^{-1}e_i. \tag{10.3.6}$$

Thus we may write,

$$\begin{aligned}
J &\geq \sum_{i=0}^{N} e_i^* R_{e,i}^{-1} e_i \\
&= \sum_{i=0}^{N} e_i^* R_i^{-*/2} R_i^{*/2} R_{e,i}^{-1} R_i^{1/2} R_i^{-1/2} e_i \\
&\geq \sum_{i=0}^{N} \underline{\sigma}\left(R_i^{*/2} R_{e,i}^{-1} R_i^{1/2}\right) e_i^* R_i^{-1} e_i \\
&\geq \left[\inf_i \; \underline{\sigma}^2\left(R_i^{*/2} R_{e,i}^{-1} R_i^{1/2}\right)\right] \sum_{i=0}^{N} e_i^* R_i^{-1} e_i \\
&= \left[\sup_i \; \frac{1}{\bar{\sigma}^2\left(R_i^{-1/2} R_{e,i} R_i^{-*/2}\right)}\right] \sum_{i=0}^{N} e_i^* R_i^{-1} e_i \\
&= \frac{1}{\bar{r}} \sum_{i=0}^{N} e_i^* R_i^{-1} e_i \\
&= \frac{1}{\bar{r}} \sum_{i=0}^{N} (e_{p,i} + v_i)^* R_i^{-1} (e_{p,i} + v_i) \quad \text{using (10.2.10)}
\end{aligned}$$

Now using Lemma (10.3.2) with $a = e_{p,i}$, $b = v_i$ and $M = R_i^{-1}$, we have

$$x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i \geq \frac{1}{\bar{r}} \sum_{i=0}^{N} \left[(1-\alpha) v_i^* R_i^{-1} v_i + (1 - \frac{1}{\alpha}) e_{p,i}^* R_i^{-1} e_{p,i}\right], \tag{10.3.7}$$

for any $\alpha > 0$. Now rearranging terms, we can write

$$(1 - \frac{1}{\alpha}) \sum_{i=0}^{N} e_{p,i}^* R_i^{-1} e_{p,i} \leq \bar{r} x_0^* \Pi_0^{-1} x_0 + \bar{r} \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \bar{r}(1 - \frac{1-\alpha}{\bar{r}}) \sum_{i=0}^{N} v_i^* R_i^{-1} v_i, \tag{10.3.8}$$

so that, assuming $\alpha > 1$, we have

$$\begin{aligned}
\sum_{i=0}^{N} e_{p,i}^* R_i^{-1} e_{p,i} &\leq \frac{\bar{r}}{1 - \frac{1}{\alpha}} \left[x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i\right] + \frac{\bar{r}}{1 - \frac{1}{\alpha}} (1 - \frac{1-\alpha}{\bar{r}}) \sum_{i=0}^{N} v_i^* R_i^{-1} v_i \\
&= \frac{\bar{r}}{1 - \frac{1}{\alpha}} \left[x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i\right] + \frac{\alpha^2 + \alpha(\bar{r} - 1)}{\alpha - 1} \sum_{i=0}^{N} v_i^* R_i^{-1} v_i .
\end{aligned}$$

To obtain the "tightest" possible bound on $\sum_{i=0}^{N} e_{p,i}^* R_i^{-1} e_{p,i}$, let us minimize, over $\alpha > 1$, the coefficient of $\sum_{i=0}^{N} v_i^* R_i^{-1} v_i$ on the RHS of the above inequality. But from

Lemma 10.3.3, we have

$$\min_{\alpha>1} \frac{\alpha^2+(\bar{r}-1)\alpha}{\alpha-1} = (1+\sqrt{\bar{r}})^2 \quad \text{and} \quad \arg\min_{\alpha>1} \frac{\alpha^2+(\bar{r}-1)\alpha}{\alpha-1} = 1+\sqrt{\bar{r}}. \quad (10.3.9)$$

Therefore

$$\begin{aligned}
\sum_{i=0}^{N} e_{p,i}^* R_i^{-1} e_{p,i} &\leq \frac{\bar{r}}{1-\frac{1}{1+\sqrt{\bar{r}}}} \left[ x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i \right] + (1+\sqrt{\bar{r}})^2 \sum_{i=0}^{N} v_i^* R_i^{-1} v_i \\
&= \sqrt{\bar{r}}(1+\sqrt{\bar{r}}) \left[ x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i \right] + (1+\sqrt{\bar{r}})^2 \sum_{i=0}^{N} v_i^* R_i^{-1} v_i \\
&\leq (1+\sqrt{\bar{r}})^2 \left[ x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i \right] + (1+\sqrt{\bar{r}})^2 \sum_{i=0}^{N} v_i^* R_i^{-1} v_i.
\end{aligned}$$

Therefore, we have

$$\frac{\sum_{i=0}^{N} e_{p,i}^* R_i^{-1} e_{p,i}}{x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} \leq (1+\sqrt{\bar{r}})^2, \quad (10.3.10)$$

which is the desired result.

To prove the upper bound of (10.2.13) for the filtered estimation errors we need to proceed as follows.

$$\begin{aligned}
J &\geq \sum_{i=0}^{N} e_i^* R_{e,i}^{-1} e_i \\
&= \sum_{i=0}^{N} (e_{f,i}+v_i)^* R_i^{-1} R_{e,i} R_{e,i}^{-1} R_{e,i} R_i^{-1} (e_{f,i}+v_i) \quad \text{using (10.2.11)} \\
&= \sum_{i=0}^{N} (e_{f,i}+v_i)^* R_i^{-*/2} R_i^{-1/2} R_{e,i} R_i^{-*/2} R_i^{-1/2} (e_{f,i}+v_i) \\
&\geq \sum_{i=0}^{N} \underline{\sigma}\left(R_i^{-1/2} R_{e,i} R_i^{-*/2}\right) (e_{f,i}+v_i)^* R_i^{-1} (e_{f,i}+v_i) \\
&\geq \left[\inf_i \ \underline{\sigma}^2\left(R_i^{-1/2} R_{e,i} R_i^{-*/2}\right)\right] \sum_{i=0}^{N} (e_{f,i}+v_i)^* R_i^{-1} (e_{f,i}+v_i) \\
&= \underline{r} \sum_{i=0}^{N} (e_{f,i}+v_i)^* R_i^{-1} (e_{f,i}+v_i).
\end{aligned}$$

Proceeding now with an argument similar to what was done in the predicted case, leads ro the desired result,

$$\frac{\sum_{i=0}^{N} e_{f,i}^* R_i^{-1} e_{f,i}}{x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^{N} u_i^* Q_i^{-1} u_i + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} \leq (1+\sqrt{1/\underline{r}})^2. \quad (10.3.11)$$

■

## 10.4 Proof of the Lower Bounds

Perhaps the most general way of computing the a lower bound for the $H^\infty$ norm of RLS, or any other algorithm for that matter, is to compute the energy gain for some particular choice of disturbance signal, $\{x_0, \{u_i, v_i\}_{i=0}^N\}$.[1] We shall presently see that the special choice of disturbance signal that yields the lower bound of Theorem 10.2.1 is that disturbance signal that minimizes the quadratic form

$$J = x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^N u_i^* Q_i^{-1} u_i + \sum_{i=0}^N v_i^* R_i^{-1} v_i, \tag{10.4.1}$$

subject to the state-space constraints (10.2.1). To facilitate the presentation of the proof it will be convenient to make the following definitions.

$$y \triangleq \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_N \end{bmatrix}, \quad u \triangleq \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix}, \quad v \triangleq \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_N \end{bmatrix} \tag{10.4.2}$$

and

$$e \triangleq \begin{bmatrix} y_0 - H_0\hat{x}_0 \\ y_1 - H_1\hat{x}_1 \\ \vdots \\ y_N - H_N\hat{x}_N \end{bmatrix}, \quad e_p \triangleq \begin{bmatrix} H_0x_0 - H_0\hat{x}_0 \\ H_1x_1 - H_1\hat{x}_1 \\ \vdots \\ H_Nx_N - H_N\hat{x}_N \end{bmatrix}, \quad e_f \triangleq \begin{bmatrix} H_0x_0 - H_0\hat{x}_{0|0} \\ H_1x_1 - H_1\hat{x}_{1|1} \\ \vdots \\ H_Nx_N - H_N\hat{x}_{N|N} \end{bmatrix}. \tag{10.4.3}$$

Moreover, from Chapter 2 we know that the innovations, $e$, can be found via

$$e = L^{-1}y, \tag{10.4.4}$$

where $L$ is a lower triangular matrix with unit diagonal that is given from the unique (block) LDU decomposition,

$$R_y = LR_eL^*, \tag{10.4.5}$$

[1] The resulting energy gain will be less than or equal to the maximum energy gain and hence a lower bound to the (squared) $H^\infty$ norm.

where

$$R_e = \operatorname{diag}(R_{e,0}, R_{e,1}, \ldots, R_{e,N}), \tag{10.4.6}$$

and $R_y$, the output Gramian, is given by

$$R_y = \mathcal{O}\Pi_0\mathcal{O}^* + \Gamma Q \Gamma^* + R, \tag{10.4.7}$$

where

$$\mathcal{O} = \begin{bmatrix} H_0 \\ H_1 F_0 \\ \vdots \\ H_N F_{N_1} \ldots F_0 \end{bmatrix}, \tag{10.4.8}$$

is the observability map, and

$$\Gamma = \begin{bmatrix} 0 & & & \\ H_1 G_0 & 0 & & \\ H_2 F_1 G_0 & H_2 G_1 & 0 & \\ H_3 F_2 F_1 G_0 & H_3 F_2 G_1 & H_3 G_2 & . \\ . & . & . & . \; . \end{bmatrix}, \tag{10.4.9}$$

is the impulse response matrix, and

$$Q \triangleq \operatorname{diag}(Q_0, Q_1, \ldots, Q_N) \quad \text{and} \quad R \triangleq \operatorname{diag}(R_0, R_1, \ldots, R_N). \tag{10.4.10}$$

Finally, we need to note the global relation,

$$y = \mathcal{O}x_0 + \Gamma u + v, \tag{10.4.11}$$

and, using (10.2.10) and (10.2.11), the relations

$$e_p = e - v = L^{-1}y - v, \tag{10.4.12}$$

and

$$e_f = RR_e^{-1}e - v = RR_e^{-1}L^{-1}y - v. \tag{10.4.13}$$

Now note that, using a completion of squares argument, we can write

$$\begin{aligned} J &= x_0^*\Pi_0^{-1}x_0 + u^*Q^{-1}u + (y - \mathcal{O}x_0 - \Gamma u)^* R^{-1}(y - \mathcal{O}x_0 - \Gamma u) \\ &= \begin{bmatrix} x_0^* - \hat{x}_{0|N}^* & u^* - \hat{u}_{|N}^* \end{bmatrix} \begin{bmatrix} \Pi_0^{-1} & \\ & Q^{-1} \end{bmatrix} \begin{bmatrix} x_0 - \hat{x}_{0|N} \\ u - \hat{u}_{|N} \end{bmatrix} + y^* R_y^{-1} y, \end{aligned}$$

where

$$\begin{bmatrix} \hat{x}_{0|N} \\ \hat{u}_{|N} \end{bmatrix} = \begin{bmatrix} \Pi_0 & \\ & Q \end{bmatrix} \begin{bmatrix} \mathcal{O}^* \\ \Gamma^* \end{bmatrix} R_y^{-1} y. \tag{10.4.14}$$

With the above choice of disturbance (for $x_0$ and $u$) we have

$$J = \hat{x}_{0|N}^* \Pi_0^{-1} \hat{x}_{0|N} + \hat{u}_{|N}^* Q^{-1} \hat{u}_{|N} + v^* R^{-1} v = y^* R_y^{-1} y, \tag{10.4.15}$$

and

$$\begin{aligned} e_p &= L^{-1}y - v && \text{using (10.4.12)} \\ &= L^{-1}y - (y - \mathcal{O}\hat{x}_{0|N} - \Gamma \hat{u}_{|N}) && \text{using (10.4.11)} \\ &= L^{-1}y - \Big(y - (\mathcal{O}\Pi_0\mathcal{O}^* + \Gamma^* Q \Gamma) R_y^{-1} y\Big) && \text{using (10.4.14)} \\ &= L^{-1}y - \Big(y - (R_y - R) R_y^{-1} y\Big) && \text{using (10.4.7)} \\ &= L^{-1}y - R R_y^{-1} y. \end{aligned}$$

[Note that we have not yet specified our choice of disturbance $v$, or equivalently, of observations $y$.]

Now if we define the notation, $\|a\|^2 = \sum_{i=0}^N a_i^* a_i$, we can write the maximum energy gain from disturbances to estimation errors as

$$\sup_{x_0,u,v} \frac{\sum_{i=0}^N (H_i x_i - H_i \hat{x}_i)^* R_i^{-1} (H_i x_i - H_i \hat{x}_i)}{x_0^* \Pi_0^{-1} x_0 + \sum_{i=0}^N u_i^* Q_i^{-1} u_i + \sum_{i=0}^N v_i^* R_i^{-1} v_i} = \sup_{x_0,u,v} \frac{\|R^{-1/2} e_p\|^2}{x_0^* \Pi_0^{-1} x_0 + \|Q^{-1/2} u\|^2 + \|R^{-1/2} v\|^2}.$$

Now, with our choice of disturbance signal, we can lower bound the maximum energy gain as follows,

$$\begin{aligned} \sup_{x_0,u,v} \frac{\|R^{-1/2} e_p\|^2}{x_0^* \Pi_0^{-1} x_0 + \|Q^{-1/2} u\|^2 + \|R^{-1/2} v\|^2} &\geq \sup_y \frac{\|R^{-1/2} e_p\|^2}{y^* R_y^{-1} y} \\ &= \sup_y \frac{\|R^{-1/2}(L^{-1} - R R_y^{-1}) y\|^2}{R_e^{-1/2} L^{-1} y} \\ &= \sup_x \frac{\|R^{-1/2}(L^{-1} - R R_y^{-1}) L R_e^{1/2} x\|^2}{\|x\|^2} \\ &= \sup_x \frac{\|(R^{-1/2} R_e^{1/2} - R^{*/2} L^{-*} R_e^{-*/2}) x\|^2}{\|x\|^2} \\ &= \left[\bar{\sigma}\left(R^{-1/2} R_e^{1/2} - R^{*/2} L^{-*} R_e^{-*/2}\right)\right]^2. \end{aligned}$$

Now the triangular inequality for the maximum singular value yields

$$\sup_{x_0,u,v} \frac{\|R^{-1/2} e_p\|^2}{x_0^* \Pi_0^{-1} x_0 + \|Q^{-1/2} u\|^2 + \|R^{-1/2} v\|^2} \geq \left[\bar{\sigma}(R^{-1/2} R_e^{1/2}) - \bar{\sigma}(R^{*/2} L^{-*} R_e^{-*/2})\right]^2, \tag{10.4.16}$$

Moreover, note that we have

$$
\begin{aligned}
\bar{\sigma}(R^{-1/2}R_e^{1/2}) &= \sqrt{\bar{\sigma}(R^{-1/2}R_eR^{-*/2})} \\
&= \sqrt{\bar{\sigma}\left(\text{diag}(R_0^{-1/2}R_{e,0}R_0^{-*/2},\ldots R_N^{-1/2}R_{e,N}R_N^{-*/2})\right)} \\
&= \sqrt{\sup_i \ \bar{\sigma}(R_i^{-1/2}R_{e,i}R_i^{-*/2})} \\
&= \sqrt{r}, \qquad (10.4.17)
\end{aligned}
$$

and

$$
\begin{aligned}
(R^{*/2}L^{-*}R_e^{-*/2})(R_e^{-1/2}L^{-1}R^{1/2}) &= R^{*/2}R_y^{-1}R^{1/2} \\
&= (I + R^{-1/2}\mathcal{O}\Pi_0\mathcal{O}^*R^{-*/2} + R^{-1/2}\Gamma Q\Gamma^*R^{-*/2})^{-1} \\
&\leq I,
\end{aligned}
$$

so that

$$\bar{\sigma}(R^{*/2}L^{-*}R_e^{-*/2}) \leq 1. \qquad (10.4.18)$$

Combining Eqs. (10.4.16), (10.4.17) and (10.4.18) yields the desired result,

$$\sup_{x_0,u,v} \frac{\sum_{i=0}^N (H_ix_i - H_i\hat{x}_i)^* R_i^{-1}(H_ix_i - H_i\hat{x}_i)}{x_0^*\Pi_0^{-1}x_0 + \sum_{i=0}^N u_i^*Q_i^{-1}u_i + \sum_{i=0}^N v_i^*R_i^{-1}v_i} \geq (\sqrt{r}-1)^2. \qquad (10.4.19)$$

The proof of the lower bound for filtered errors is very similar. The only difference is that here we need to compute $e_f$ in terms of $y$ (for our choice of disturbances as given by (10.4.14)). In this case

$$e_f = RR_e^{-1}e - v = RR_e^{-1}L^{-1}y - RR_y^{-1}y. \qquad (10.4.20)$$

Thus we now have

$$
\begin{aligned}
\sup_{x_0,u,v} \frac{\|R^{-1/2}e_f\|^2}{x_0^*\Pi_0^{-1}x_0 + \|Q^{-1/2}u\|^2 + \|R^{-1/2}v\|^2} &\geq \sup_y \frac{\|R^{-1/2}e_f\|^2}{y^*R_y^{-1}y} \\
&= \sup_y \frac{\|R^{-1/2}(RR_e^{-1}L^{-1} - RR_y^{-1})y\|^2}{R_e^{-1/2}L^{-1}y} \\
&= \sup_x \frac{\|R^{-1/2}(RR_e^{-1}L^{-1} - RR_y^{-1})LR_e^{1/2}x\|^2}{\|x\|^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_x \frac{\|(R^{*/2}R_e^{-*/2} - R^{*/2}L^{-*}R_e^{-*/2})x\|^2}{\|x\|^2} \\
&= \left[\bar{\sigma}\left(R^{*/2}R_e^{-*/2} - R^{*/2}L^{-*}R_e^{-*/2}\right)\right]^2 \\
&\geq \left[\bar{\sigma}(R^{*/2}R_e^{-*/2}) - \bar{\sigma}(R^{*/2}L^{-*}R_e^{-*/2})\right]^2 \\
&\geq \left[\sqrt{1/\underline{r}} - 1\right]^2,
\end{aligned}
$$

where in the last step we used the (readily verifiable) fact that

$$\bar{\sigma}(R^{*/2}R_e^{-*/2}) = \sqrt{1/\underline{r}}. \tag{10.4.21}$$

But this is our desired result.

■

## 10.5 RLS Adaptive Filtering

We are now in a position to specialize the result of Theorem 10.2.1 to the case of adaptive filtering. Recall that in adaptive filtering the model is given by

$$d_i = h_i^T w + v_i, \tag{10.5.1}$$

where $d_i$ is the observation, $h_i^T = \left[\begin{array}{cccc} h_{i1} & h_{i2} & \ldots & h_{in} \end{array}\right]$ is a known $1 \times n$ input vector, $w$ is the unknown weight vector that we intend to estimate, and $v_i$ is an unknown disturbance signal.

As mentioned earlier in Secs. 1.2.1 and Sec. 9.5, the adaptive filtering problem is a special case of a state-space estimation problem (see the state-space model (10.2.1)) resulting from the parameters,

$$F_i = I_n \quad , \quad G_i = 0 \quad , \quad H_i = h_i^T \quad , \quad R_i = I_p. \tag{10.5.2}$$

The RLS (recursive-least-squares) algorithm is essentially the Kalman filter corresponding to a state-space model with the above system matrices. Thus the least-squares estimates $\hat{w}_{|i}$ (of the weight vector $w$, using the observations, $\{d_j, j \leq i\}$) obey the following recursions,

$$\hat{w}_{|i} = \hat{w}_{|i-1} + k_{p,i}(d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{\text{-}1} \tag{10.5.3}$$

where

$$k_{p,i} = \frac{P_i h_i}{1 + h_i^T P_i h_i}, \tag{10.5.4}$$

and $P_i$ satisfies the Riccati recursion,

$$P_{i+1} = P_i - \frac{P_i h_i h_i^T P_i}{1 + h_i^T P_i h_i}, \qquad P_0 = \mu I. \tag{10.5.5}$$

It is also useful to remark that at each time instant, $i$, the above RLS algorithm solves the following least-squares problem,

$$\min_w \left[ \mu^{-1} |w - \hat{w}_{|-1}|^2 + \sum_{j=0}^{i} |d_j - h_j^T w|^2 \right], \tag{10.5.6}$$

where $\mu^{-1}|w - w_{-1}|^2$ is a possible regularization term that reflects a priori knowledge as to how close $w$ is to the initial estimate $w_{-1}$. The special case where $mu = \infty$, so that the first term in the cost function of (10.5.6) disappears, is referred to as a *pure* least-squares problem. We will shortly have more to say about such problems.

As before let us define the following prediction and filtered estimation errors,

$$e_{p,i} = h_i^T w - h_i^T \hat{w}_{|i-1}, \tag{10.5.7}$$

and

$$e_{f,i} = h_i^T w - h_i^T \hat{w}_{|i-1}. \tag{10.5.8}$$

It will also be useful to define the following smoothed estimation error (since we know from Sec. 3.4 that least-squares smoothers are $H^\infty$ optimal),

$$e_{s,i} = h_i^T w - h_i^T \hat{w}_{|N}, \qquad i = 0, 1 \ldots N. \tag{10.5.9}$$

The following result is now immediate.

**Theorem 10.5.1 (Bounds for the $H^\infty$ Norm of the RLS Algorithm)** *Consider the adaptive filtering model (10.5.1) and the least-squares estimates $\hat{w}_{|i}$, given by the RLS algorithm (10.5.3). Then for any $N$, we have the following results,*

$$(\sqrt{\bar{r}} - 1)^2 \le \sup_{w,v \in h^2} \frac{\sum_{i=0}^{N} (h_i^T w - h_i^T \hat{w}_{|i-1})^* (h_i^T w - h_i^T \hat{w}_{|i-1})}{\mu^{-1}|w - \hat{w}_{|-1}|^2 + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} \le (\sqrt{\bar{r}} + 1)^2 \tag{10.5.10}$$

$$(\sqrt{1/\underline{r}}-1)^2 \leq \sup_{w,v\in h^2} \frac{\sum_{i=0}^{N}(h_i^T w - h_i^T \hat{w}_{|i})^*(h_i^T w - h_i^T \hat{w}_{|i})}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} \leq (\sqrt{1/\underline{r}}+1)^2 \quad (10.5.11)$$

*and*

$$\sup_{w,v\in h^2} \frac{\sum_{i=0}^{N}(h_i^T w - h_i^T \hat{w}_{|N})^*(h_i^T w - h_i^T \hat{w}_{|N})}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} = 1 \quad (10.5.12)$$

*where we have defined,*

$$\bar{r} = \sup_i \left[1 + h_i P_i h_i^T\right], \quad (10.5.13)$$

*and*

$$\underline{r} = \inf_i \left[1 + h_i P_i h_i^T\right]. \quad (10.5.14)$$

**Proof:** The proof of the bounds (10.5.10) and (10.5.11) follow immediately from Theorem 10.2.1 when the system matrices are appropriately specialized. Eq. (10.5.12) follows from the $H^\infty$ optimality of least-squares smoothers, as given by Sec. 3.4, and by the results of Chapter 9 on $H^\infty$-optimal adaptive filters.

■

In the RLS algorithm the it is easy to solve (10.5.5) to obtain $P_i = (\mu^{-1}I + \sum_{j=0}^{i-1} h_j h_j^T)^{-1}$, which implies that the $P_i$ are a monotonically decreasing sequence of matrices. If we assume that the input vectors $h_i$ have equal magnitude (*i.e.* $h_i^T h_i =$ const.) then we have the following result.

**Corollary 10.5.1 (Constant Magnitude Inputs)** *If the input vectors have constant magnitude $h_i^T h_i = \bar{h}^2$ then*

$$(\sqrt{1+\mu\bar{h}^2}-1)^2 \leq \sup_{w,v\in h^2} \frac{\sum_{i=0}^{N}(h_i^T w - h_i^T \hat{w}_{|i-1})^*(h_i^T w - h_i^T \hat{w}_{|i-1})}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} \leq (\sqrt{1+\mu\bar{h}^2}+1)^2. \quad (10.5.15)$$

The following is also straightforward.

**Corollary 10.5.2** *If $\bar{h}^2 \triangleq \sup_i h_i^T h_i$ and $\underline{h}^2 \triangleq \inf_i h_i^T h_i$ then*

$$(\sqrt{1+\mu\underline{h}^2}-1)^2 \leq \sup_{w,v\in h^2} \frac{\sum_{i=0}^{N}(h_i^T w - h_i^T \hat{w}_{|i-1})^*(h_i^T w - h_i^T \hat{w}_{|i-1})}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \sum_{i=0}^{N} v_i^* R_i^{-1} v_i} \leq (\sqrt{1+\mu\bar{h}^2}+1)^2. \quad (10.5.16)$$

Finally, the following result for filtered errors is also immediate and follows readily from the expression $P_i = (\mu^{-1}I + \sum_{j=0}^{i-1} h_j h_j^T)^{-1}$.

**Corollary 10.5.3** *We have*

$$\lim_{i\to\infty} \underline{r} = 1, \tag{10.5.17}$$

*so that*

$$\sup_{w,v\in h^2} \frac{\sum_{i=0}^N (h_i^T w - h_i^T \hat{w}_{|i})^*(h_i^T w - h_i^T \hat{w}_{|i})}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \sum_{i=0}^N v_i^* R_i^{-1} v_i} \le 4. \tag{10.5.18}$$

**Remarks:**

(i) Corollary 10.5.2 has an interesting interpretation: for large values of $\mu$, the RLS algorithm is less robust with respect to prediction errors. In fact, we see that the (upper and lower bounds of the) $H^\infty$ norm grows as $\sqrt{\mu}$. This is reminiscent of the robustness properties of LMS, where, as shown in Chapter 9, the learning rate $\mu$ had to be small enough to guarantee $H^\infty$ optimality.

More importantly, Corollary 10.5.2 shows that the pure least-squares problem (corresponding to $\mu = \infty$) can be highly nonrobust with respect to prediction errors.

(ii) From Corollary 10.5.3, for filtered errors, the RLS algorithm yields,

$$\sup_{w,v\in h_2} \frac{\|e_f\|_2^2}{\mu^{-1}|w-\hat{w}_{|-1}|^2 + \|v\|_2^2} \le 4,$$

Note that, as with the normalized LMS algorithm, the $H^\infty$ norm does not depend on $\mu$.

The above result shows that, for filtered errors, least-squares algorithms are at most four times worse than $H^\infty$-optimal algorithms (where $\gamma_{opt} = 1$). This demonstrates an intermediate stage between the smoothed error case (which has access to all the observations, and where the $H^\infty$ and $H^2$ optimal filters coincide) and the prediction error case (which does not have access to current observations, and where the performance of LMS and RLS can be drastically different).

### 10.5.1 An Alternative Lower Bound

As mentioned earlier, the most natural method for finding lower bounds to the $H^\infty$ norm is to calculate the ratio $\frac{\|e\|_2^2}{\mu^{-1}|w-w_{-1}|^2+\|v\|_2^2}$ for a particular choice of disturbance $w - w_{-1}$ and $\{v_i\}$.

Since the RLS algorithm is considerably simpler than the Kalman filter (in its full generality), it is much easier to compute the energy gain for various disturbances in the RLS case than it is for the Kalman filter. The particular choice of disturbances,

$$w - w_{-1} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \quad \text{and} \quad v_i = \frac{-1}{(1+i\mu\underline{h}^2)(1+(i+1)\mu\underline{h}^2)} \tag{10.5.19}$$

where $\underline{h}^2 = \inf_i h_i^T h_i$, leads to the following result.

**Lemma 10.5.1 (Alternative Lower Bound)** *Consider the adaptive filtering model (10.5.1) and the least-squares estimates $\hat{w}_{|i}$, given by the RLS algorithm (10.5.3). A lower bound for the maximum energy gain from the disturbances, $\{\mu^{-1/2}(w - \hat{w}_{|-1}), \{v_i\}_{i=0}^N\}$, to the prediction errors, $\{h_i^T w - h_i^T \hat{w}_{|i-1}\}_{i=0}^N$, is given by*

$$\underline{h}^2 \frac{(\frac{1}{\underline{h}^2}+\sqrt{\mu})^2 S_2(\mu\underline{h}) - \frac{2}{\underline{h}^2}(\frac{1}{\underline{h}^2}+\sqrt{\mu})S_3(\mu\underline{h}) + \frac{1}{\underline{h}^4}S_4(\mu\underline{h})}{1 - \frac{1}{\mu^2\underline{h}^4}(1+\frac{1}{\mu\underline{h}^2}) + \frac{2}{\mu^2\underline{h}^4}S_2(\mu\underline{h})}, \tag{10.5.20}$$

*where*

$$S_i(x) = \sum_{j=0}^{N} \frac{1}{(1+jx)^i}, \qquad i = 2, 3, 4.$$

**Proof:** The proof involves the algebraic computation of the energy gain and will be omitted for brevity. Similar results can be established for filtered errors.

■

### 10.5.2 Example

In this section we shall consider a simple example where the $h_i$ are scalars that randomly take on the values $+1$ and $-1$. Thus in this example $\bar{h}^2 = \underline{h}^2 = 1$. We have plotted the results of the $H^\infty$ norm of RLS as a function of the number of iterations for different values of $\mu$. The upper and lower bounds of Theorem 10.5.1 and Lemma 10.5.1 are also given.

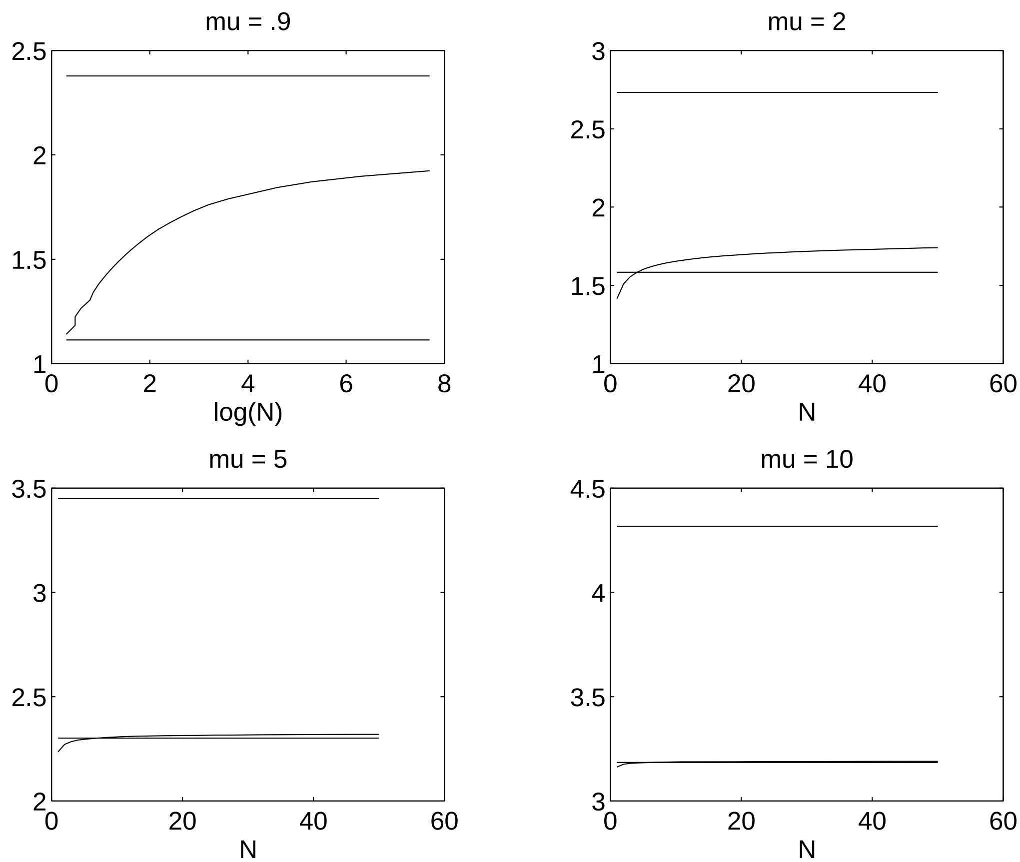


Figure 10.1: $H^\infty$ norm of RLS as a function of the number of data points and as a function of $\mu$. The upper and lower bounds of Theorem 10.5.1 and Lemma 10.5.1 are given by the horizontal lines. As can be seen from the figures, in this example, the lower bounds of Lemma 10.5.1 seem quite accurate for large $\mu$.

## 10.6 Conclusion

In this chapter we obtained upper and lower bounds for the $H^\infty$ norm of the RLS algorithm. These bounds may be used to study the robustness of RLS in different applications. Our results show that the $H^\infty$ norm of RLS depends on the input-output data (viz. on the $\{h_i\}$), as opposed to the LMS and normalized LMS algorithms where the $H^\infty$ norm is independent of the data. The bounds further show that, for prediction errors, the $H^\infty$ norm of RLS grows as the square-root of $\mu$ (where $\mu^{-1}I$ is the regularization term in least-squares problems that reflects a priori knowledge of the weight vector), whereas for filtered errors, the $H^\infty$ norm of RLS (and the Kalman filter) is bounded by two. Simulations show that the bounds are quite reasonable compared to the true computed $H^\infty$ norms.

# Chapter 11

# $H^\infty$ Adaptive Filtering

The previous two chapters were more or less concerned with the study of adaptive filters from an $H^\infty$ point of view. Chapter 9 studied the problems of output prediction and output filtered estimation in adaptive filtering, using an $H^\infty$ approach, and demonstrated the $H^\infty$ optimality of the LMS and normalized LMS algorithms with respect to these two estimation criteria. Chapter 10 studied the robustness of least-squares-based adaptive filters, such as the RLS algorithm, using the $H^\infty$ framework. A reading of Chapters 9 and 10 indicates that there may be great promise in the interplay of adaptive filtering and $H^\infty$ estimation theory.

To round up the story, in this chapter, we shall present a preliminary study of the design of adaptive filters using an $H^\infty$ criterion. The strength of $H^\infty$-optimal adaptive filters lies in the fact that they guarantee the smallest possible estimation error energy over all possible disturbances of fixed energy, and are therefore robust with respect to model uncertainties and lack of statistical information on the exogenous signals. Specifically, in this chapter we study the problem of prediction of the weight vector itself, and for the purpose of coping with time-variations, exponentially weighted, finite-memory and time-varying adaptive filtering. This results in some new adaptive filtering algorithms that may be useful in uncertain and non-stationary environments.

We should mention that the presentation of this chapter is deliberately brief. The main reasons are that the results given here are preliminary, and that we believe the area is worthy of much more scrutiny and study. Indeed, we have just scratched the surface. The main goal, therefore, has been to hint at the possibilities.

## 11.1 Introduction

Since its inception in the early 1960's [WH60, GWW61] adaptive filtering has been widely used to cope with time-variations of system parameters and lack of a priori knowledge of the statistical properties of the input data. This is in contrast to Wiener and Kalman filter theory which require a priori statistical information. Early successes of adaptive filtering were in channel equalization for digital communications [Luc65] and in adaptive antennas [How65, App66, Wid67]. Currently, recent advances in ASIC's (application-specific processors) and sensor technology have spurred an increasing range of applications from biomedical engineering to wireless communications, so that there is still great interest in the field.

Although there have been several successful applications of adaptive filtering, it is fair to say that most of the results are empirical and that there still is not an adequate theoretical basis for the analysis of the performance of these algorithms — limits of performance, sensitivity, optimality, etc. (The basic reason is that adaptive systems are inherently time-variant and nonlinear). Much work has been done in this area in control theory, especially in adaptive control, though many questions still remain.

Due to the similarity between the objectives of adaptive filtering (coping with time-variations, insensitivity to lack of statistical knowledge, etc.) and $H^\infty$ estimation (robustness with respect to uncertainties in the underlying model and/or statistics), it is expected that there should be some connection between the two. Indeed, we have seen in Chapter that the celebrated LMS algorithm [WH60], which is widely used in adaptive filtering, is $H^\infty$ optimal. This result gives more insight into the inherent robustness of the LMS algorithm and why it has found such wide applicability in such a diverse range of problems. As a matter of fact, we believe that the $H^\infty$ approach yields a new way of looking at adaptive systems, and has several ramifications, as well as suggesting directions for further research.

In this chapter we further pursue the connections between adaptive filtering and $H^\infty$ estimation. In particular, we consider predicting the complete filter weight vector which leads to a new adaptive algorithm. We also develop a host of $H^\infty$ algorithms to deal with time-variations and non-stationary signals. Two of these algorithms are based on exponentially weighted and finite memory windows, respectively, and a third algorithm allows for general time variations in the underlying filter weight vector. The goal of this chapter is to outline the use of the $H^\infty$ criterion in the design adaptive filter algorithms. There are, no doubt, a wide variety of other $H^\infty$ adaptive algorithms (not considered here) that could be worthy of further scrutiny.

**Important Remark**

At this point we should make an important remark on the contribution of this chapter. As is wellknown, once an upper bound on the value of $\gamma$ that ensures the existence of an $H^\infty$ filter is given, the structure of the adaptive filters readily follow from the standard solution to the $H^\infty$ estimation problem (see *e.g.*, Chapter 3). Therefore, since adaptive filtering is just a special case of state-space estimation, that has been solved (in its full generality) in the $H^\infty$ framework, it appears that there is not much more to be done or said.

However, we should note that finding the optimum value of $\gamma$ essentially amounts to finding the maximum singular value of a linear time-varying operator. Bounds on $\gamma$ can be found by checking for the positivity of the solution of a certain time-varying discrete-time Riccati recursion. Although both approaches can be used in principle, they require knowledge of *all* the input data vectors $\{h_i\}$.

Since in adaptive filtering problems we are given, and are forced to process, the data in real time, we do not have the luxury of storing all the data and computing explicit bounds for $\gamma$ using the aforementioned methods. Therefore the main effort in $H^\infty$ adaptive filtering is to obtain bounds on $\gamma$ that use simple a priori knowledge of the $\{h_i\}$ and not their explicit values. This is what is done in the results given below.

## 11.2 Full Weight Vector Estimation

As mentioned in earlier chapters, in adaptive filtering we assume that we observe an output sequence $\{d_i\}$ that obeys the following linear filter model

$$d_i = h_i^T w + v_i, \tag{11.2.1}$$

where $h_i^T = \begin{bmatrix} h_{i1} & h_{i2} & \dots & h_{in} \end{bmatrix}$ is a known input vector, $w$ is the unknown filter weight vector that we intend to estimate, and $\{v_i\}$ is an unknown disturbance sequence that may include modeling errors. Let $\hat{w}_{|i} = \mathcal{F}_i(d_0, d_1, \dots, d_i)$ denote the estimate of the weight vector given the inputs $\{h_j\}$ and the outputs $\{d_j\}$ from time 0 up to and including time $i$.

In $H^\infty$ estimation the structure of the estimator depends on the linear combination of the state (in our case the linear combination of the weight vector) that we intend to estimate. This is as opposed to $H^2$ estimation where the estimate of any linear combination of the state is simply that linear combination of the state estimate.

In Chapters 9 and 10 our concern was prediction of the filter output, which corresponds to the linear combination $h_i^T w$. Although in many applications predicting the output of the filter suffices, in some applications (especially those concerned with identification) we are interested in estimating the weight vector itself. In this case we define the weight vector estimation error as

$$\tilde{w}_{|i} = w - \hat{w}_{|i}, \tag{11.2.2}$$

and we are concerned with the transfer operator $\mathcal{T}_{s,N}$, from the disturbances $\{w - \hat{w}_{|-1}, \{v_i\}_{i=0}^N\}$ to the weight estimation errors $\{\tilde{w}_{|i}\}_{i=0}^N$. The $H^\infty$ full weight vector estimation problem may now be formulated as follows, where we have defined $\| \tilde{w} \|_2^2 = \sum_{i=0}^N \tilde{w}_{|i}^T \tilde{w}_{|i}$.

**Problem 11.2.1 (Optimal Weight Estimation Problem)** *Find an $H^\infty$-optimal estimation strategy $\hat{w}_{|i} = \mathcal{F}_i(d_0, d_1, \dots, d_i)$ that minimizes $\|\mathcal{T}_{s,N}\|_\infty$, and obtain the resulting*

$$\gamma_s^2 = \inf_{\mathcal{F}} \ \|\mathcal{T}_s\|_\infty^2 = \inf_{\mathcal{F}} \ \sup_{w,v\in h^2} \frac{\|\tilde{w}\|_2^2}{\mu^{-1}|w - w_{-1}|^2 + \|v\|_2^2} \tag{11.2.3}$$

*where* $|w - w_{-1}|^2 = (w - w_{-1})^T(w - w_{-1})$, *and* $\mu$ *is a positive constant that reflects a priori knowledge as to how close* $w$ *is to the initial guess* $w_{-1}$.

Before stating the solution to the above problem we need to define the sample covariance matrix of the input vectors $\{h_j\}_{j=0}^{i-1}$ as follows

$$R_i = \frac{1}{i} \sum_{j=0}^{i-1} h_j h_j^T. \tag{11.2.4}$$

**Theorem 11.2.1 (Full Weight Estimator)** *Consider the model (11.2.1), and suppose we wish to minimize the* $H^\infty$ *norm of the transfer operator,* $\mathcal{T}_{s,N}$, *from the unknowns* $w - w_{-1}$ *and* $\{v_i\}_{i=0}^N$ *to the weight vector estimation errors* $\tilde{w}_{|i}$. *Then*

$$\gamma_s = \sup_i \sqrt{\frac{1}{\frac{1}{\mu i} + \underline{\sigma}(R_i)}}, \tag{11.2.5}$$

*where* $\underline{\sigma}(R_i)$ *denotes the minimum singular value of* $R_i$. *The central optimal* $H^\infty$ *estimator is given by*

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \frac{P_i h_i}{1 + h_i^T P_i h_i}(d_i - h_i^T \hat{w}_{|i-1}), \quad w_{-1} \tag{11.2.6}$$

*where* $P_i$ *satisfies the recursion*

$$P_{i+1}^{-1} = P_i^{-1} + h_i h_i^T - \gamma_s^{-2} I, \qquad P_0^{-1} = \mu I. \tag{11.2.7}$$

**Proof:** The proof is a direct application of Theorem 3.2.1 on a posteriori state-space $H^\infty$ estimators for the special state-space model

$$F_i = I \ \ , \ \ G_i = 0 \ \ , \ \ H_i = h_i^T \ \ , \ \ L_i = I_n \tag{11.2.8}$$

which corresponds to the adaptive filtering model (11.2.1). Note that we have taken $L_i = I_n$ since the desired combination of the weights is the identity matrix (*i.e.*, we want to estimate $w$ itself). Now from Theorem 3.2.1 if an estimator of level $\gamma_s$ exists, then the solution is given by (11.2.6) and (11.2.7). Therefore all that remains to be shown is the condition for the existence of the filter.

But according to Theorem 3.2.1 this condition is

$$P_{i+1}^{-1} = P_i^{-1} + h_i h_i^T - \gamma_s^{-2} I > 0, \quad i = 0, 1, \ldots N. \tag{11.2.9}$$

Now the above Riccati equation can be solved to yield,

$$\begin{aligned} P_{i+1}^{-1} &= \mu^{-1} I + \sum_{j=0}^{i} h_j h_j^T - (i+1)\gamma_s^{-2} I, \\ &= \mu^{-1} I + (i+1) R_{i+1} - (i+1)\gamma_s^{-2} I, \end{aligned}$$

so that the condition for $P_{i+1}^{-1} > 0$ becomes

$$R_{i+1} > \gamma_s^{-2} I - \frac{1}{\mu(i+1)} I,$$

or

$$\underline{\sigma}(R_{i+1}) > \gamma_s^{-2} - \frac{1}{\mu(i+1)}. \tag{11.2.10}$$

Rearranging the above inequality yields,

$$\gamma_s > \sqrt{\frac{1}{\frac{1}{\mu i} + \underline{\sigma}(R_i)}}, \quad i = 0, 1, \ldots N \tag{11.2.11}$$

from which we obtain the desired result (11.2.5).

■

**Remarks:**

(i) It is interesting to compare the algorithm of Theorem 11.2.1 with the RLS algorithm of Chapter 10. The only difference is that the covariance update in RLS can be written as $P_{i+1}^{-1} = P_i^{-1} + h_i h_i^T$. Due to this covariance update, $P_i$ (and hence the gain vector) may approach zero for large $i$. However, in the algorithm of Theorem 11.2.1 the covariance update is more conservative and $P_i$ does not tend to zero meaning that we always have a non-zero gain vector. This is similar to some ad-hoc schemes that are employed with RLS to guarantee that the gain vector does not go to zero (see [Hay96]).

(ii) It is interesting to note that $\underline{\sigma}(R_i)$, the minimum singular value of $R_i$, appears in the bound for $\gamma_s$. Indeed if $\underline{\sigma}(R_i) = 0$, then $\gamma_s$ grows unbounded as time progresses to infinity. This makes perfect sense, since $\underline{\sigma}(R_i) = 0$ means that the sample covariance matrix $R_i = \frac{1}{i}\sum_{j=0}^{i-1} h_j h_j^T$ is singular, *i.e.*, that there are certain directions which the input vectors $\{h_i\}$ do not (persistently) excite. In such a case it will not be possible to estimate $w$ along those directions, and hence $\gamma_s$ will tend to infinity.

(iii) If the components of the input vectors $\{h_i\}$ are independent random variables with variance $\sigma^2$, then provided $\mu\sigma < 1$, for large $i$ the algorithm of Theorem 11.2.1 reduces to

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \frac{\mu}{1-\mu\sigma}.\frac{h_i}{1+\frac{\mu}{1-\mu\sigma}h_i^T h_i}(d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{|-1} \tag{11.2.12}$$

which is the so-called normalized LMS algorithm with parameter $\frac{\mu}{1-\mu\sigma}$.

## 11.3 Time-Variation

The $H^\infty$ adaptive filters developed in Chapter 9 and the previous section are robust with respect to model uncertainties and lack of statistical information on the exogenous signals. However, since we have assumed the underlying model to be stationary (we have taken the unknown weight vector as time-invariant) these algorithms may not have desirable tracking properties in the face of time variations.

In this section we shall incorporate provisions to cope for time variations in the underlying model. The methods used are similar to those used in the usual $H^2$ setting, and here we generalize them to the $H^\infty$ setting. Although this can be done for both the output estimation case and the full weight vector case, for brevity we shall only consider the former.

### 11.3.1 Exponentially-Windowed Adaptive Filtering

One method to account for time-variations is to introduce a forgetting factor $0 < \lambda < 1$, and to exponentially weight the energies with this forgetting factor. In this manner the more recent data will be given larger weights than the earlier ones. This will allow the algorithms to track the time-variations of the underlying models. In particular, the prediction error and disturbance energies are computed as:

$$\sum_{j=0}^{N} \lambda^{-i}|e_{p,i}|^2 \quad \text{and} \quad \sum_{j=0}^{N} \lambda^{-i}|v_i|^2. \tag{11.3.1}$$

The forgetting factor $0 < \lambda < 1$ is chosen based upon a priori knowledge of how fast the weight vector varies with time.

Now for any choice of estimator $\mathcal{F}$, we shall denote by $\mathcal{T}_{\lambda,N}(\mathcal{F})$ the transfer operator from the disturbances $\{\mu^{-\frac{1}{2}}(w - \hat{w}_{|-1}), \{\lambda^{-\frac{i}{2}}v_i\}_{i=0}^{N}\}$ to the prediction errors $\{\lambda^{-\frac{i}{2}}e_{p,i}\}_{i=M}^{i}$. The $H^\infty$ output prediction problem with exponential weighting can thus be formulated as follows.

**Problem 11.3.1 (Exponentially-Windowed Problem)** *Find an $H^\infty$-optimal estimation strategy $w_i = \mathcal{F}(d_0, d_1, \ldots, d_i)$ that minimizes $\|\mathcal{T}_{\lambda,N}\|_\infty$, and obtain the resulting*

$$\gamma_\lambda^2 = \inf_{\mathcal{F}} \ \|\mathcal{T}_{\lambda,N}\|_\infty^2 = \inf_{\mathcal{F}} \ \sup_{w,v\in h^2} \frac{\sum_{i=0}^{N}|e_{p,i}|^2\lambda^{-i}}{\mu^{-1}|w - w_{-1}|^2 + \sum_{i=0}^{N}|v_i|^2\lambda^{-i}} \tag{11.3.2}$$

*where $|w - w_{-1}|^2 = (w - w_{-1})^T(w - \hat{w}_{|-1})$, and $\mu$ is a positive constant that reflects a priori knowledge as to how close $w$ is to the initial guess $\hat{w}_{|-1}$.*

Before giving the solution to the above problem we need the following definitions

$$\bar{h} \triangleq \sup_i h_i^T h_i \quad , \quad \underline{h} \triangleq \inf_i h_i^T h_i \tag{11.3.3}$$

and

$$R_{\lambda,i}\Delta{=}\lambda^i \sum_{j=0}^{i-1} \lambda^{-j} h_j h_j^T. \tag{11.3.4}$$

**Theorem 11.3.1 (Exponentially-Windowed Algorithm)** *Consider the model (11.2.1), and suppose we wish to minimize the $H^\infty$ norm of the transfer operator $T_{\lambda,N}$ of Problem 11.3.1. Then*

$$\gamma_\lambda^2 \le \sup_i \frac{\bar{h} + \underline{\sigma}(R_{\lambda,i})}{\frac{\lambda^i}{\mu} + \underline{\sigma}(R_{\lambda,i})} \quad i = 0, 1, \ldots N \tag{11.3.5}$$

*and the central $H^\infty$-optimal estimator is given by*

$$\hat{w}|i = \hat{w}_{|i-1} + \frac{\bar{P}_i h_i}{1 + h_i^T \bar{P}_i h_i}(d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{|-1} \tag{11.3.6}$$

*where $\bar{P}_i$ satisfies the recursion*

$$\bar{P}_{i+1}^{-1} = \lambda \bar{P}_i^{-1} + \lambda h_i h_i^T - \gamma_\lambda^{-2} h_{i+1} h_{i+1}^T, \qquad \bar{P}_0^{-1} = \mu^{-1} I - \gamma_\lambda^{-2} h_0 h_0^T. \tag{11.3.7}$$

**Proof:** The proof is a direct application of Theorem 3.2.2 on a priori $H^\infty$ estimation as applied to a Krein space state-space model with

$$F_i = I \ , \quad H_i = \begin{bmatrix} h_i^T \\ h_i^T \end{bmatrix} \ , \quad G_i = 0 \ , \quad \Pi_0 = \mu I \ , \quad R_i = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma_\lambda^2 \end{bmatrix} \lambda^i \tag{11.3.8}$$

and with $x_0 = w$. Therefore, using Theorem 3.2.2, assuming a solution exists, the solution is given by

$$\hat{x}_{i+1} = \hat{x}_i + \tilde{P}_i h_i (\lambda^i + h_i^T \tilde{P}_i h_i)^{-1} (d_i - h_i^T \hat{x}_i), \quad \hat{x}_0 = \hat{w}_{|-1} \tag{11.3.9}$$

or, since $x_i = w$, for all $i$

$$\hat{w}_{|i} = \hat{w}_{|i-1} + \tilde{P}_i h_i (\lambda^i + h_i^T \tilde{P}_i h_i)^{-1} (d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{|-1} \tag{11.3.10}$$

where

$$\tilde{P}_i^{-1} = P_i^{-1} - \gamma_\lambda^{-2} \lambda^{-i} h_i h_i^T, \tag{11.3.11}$$

and where $P_i$ satisfies the Riccati recursion

$$P_{i+1} = P_i - P_i \begin{bmatrix} h_i & h_i \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -\gamma_\lambda^2 \end{bmatrix} \lambda^i + \begin{bmatrix} h_i^T \\ h_i^T \end{bmatrix} P_i \begin{bmatrix} h_i & h_i \end{bmatrix} \right\}^{-1} \begin{bmatrix} h_i^T \\ h_i^T \end{bmatrix} P_i, \quad P_0 = \mu I. \tag{11.3.12}$$

Therefore we have

$$P_{i+1}^{-1} = P_i^{-1} + (1 - \gamma_\lambda^{-2}) \lambda^{-i} h_i h_i^T, \quad P_0^{-1} = \mu^{-1} I \tag{11.3.13}$$

so that

$$\tilde{P}_{i+1}^{-1} = \tilde{P}_i^{-1} + \lambda^{-i} h_i h_i^T - \gamma_\lambda^{-2} \lambda^{-i-1} h_{i+1} h_{i+1}^T, \quad \tilde{P}_0^{-1} = \mu^{-1} I - \gamma_\lambda^{-2} h_0 h_0^T. \tag{11.3.14}$$

Now if we define

$$\bar{P}_i = \lambda^{-i}\tilde{P}_i, \tag{11.3.15}$$

we see that the recursion for $\hat{w}_{|i}$ becomes

$$\hat{w}|i = \hat{w}_{|i-1} + \frac{\bar{P}_i h_i}{1 + h_i^T \bar{P}_i h_i}(d_i - h_i^T \hat{w}_{|i-1}), \quad \hat{w}_{|-1} \tag{11.3.16}$$

which is precisely (11.3.6), and, moreover, $\bar{P}_i$ satisfies the recursion

$$\bar{P}_{i+1}^{-1} = \lambda \bar{P}_i^{-1} + \lambda h_i h_i^T - \gamma_\lambda^{-2} h_{i+1} h_{i+1}^T, \qquad \bar{P}_0^{-1} = \mu^{-1} I - \gamma_\lambda^{-2} h_0 h_0^T. \tag{11.3.17}$$

which is precisely (11.3.7).

Therefore all that remains to be shown is the existence condition for the filter. But from Theorem 3.2.2 this existence condition is

$$\tilde{P}_i > 0, \quad i = 0, 1, \ldots N. \tag{11.3.18}$$

Now the recursion (11.3.14) can be solved for $\tilde{P}_i$ to yield

$$\begin{aligned} \tilde{P}_{i+1}^{-1} &= \mu^{-1} I + \sum_{j=0}^{i} (1 - \gamma_\lambda^{-2}) \lambda^{-j} h_j h_j^T - \gamma_\lambda^{-2} \lambda^{-i-1} h_{i+1} h_{i+1}^T \\ &= \mu^{-1} I + (1 - \gamma_\lambda^{-2}) \lambda^{-i-1} R_{\lambda,i+1} - \gamma_\lambda^{-2} \lambda^{-i-1} h_{i+1} h_{i+1}^T. \end{aligned}$$

Therefore the condition $\tilde{P}_{i+1} > 0$ implies

$$\mu^{-1} I + (1 - \gamma_\lambda^{-2}) \lambda^{-i-1} R_{\lambda,i+1} > \gamma_\lambda^{-2} \lambda^{-i-1} h_{i+1} h_{i+1}^T,$$

or, equivalently,

$$(1 - \gamma_\lambda^{-2}) R_{\lambda,i+1} > \gamma_\lambda^{-2} h_{i+1} h_{i+1}^T - \frac{\lambda^{i+1}}{\mu} I. \tag{11.3.19}$$

A sufficient condition for the above inequality is that

$$\begin{aligned} (1 - \gamma_\lambda^{-2}) \underline{\sigma}(R_{\lambda,i+1}) &> \gamma_\lambda^{-2} \bar{\sigma}(h_{i+1} h_{i+1}^T) - \frac{\lambda^{i+1}}{\mu} \\ &= \gamma_\lambda^{-2} \bar{h} - \frac{\lambda^{i+1}}{\mu}. \end{aligned}$$

Rearranging this last expressions leads to

$$\gamma_\lambda^2 > \frac{\bar{h} + \underline{\sigma}(R_{\lambda,i+1})}{\frac{\lambda^{i+1}}{\mu} + \underline{\sigma}(R_{\lambda,i+1})} \tag{11.3.20}$$

which yields the desired result (11.3.5).

■

**Remarks:**

(i) It is interesting to note that $\underline{\sigma}(R_{\lambda,i})$, the minimum singular value of $R_{\lambda,i}$, appears in the bound for $\gamma_\lambda$. Moreover, it follows from (11.3.5) that if $\underline{\sigma}(R_{\lambda,i}) = 0$, then $\gamma_\lambda$ grows unbounded as time progresses to infinity.

(ii) For large $i$ the bound in (11.3.5) becomes

$$\gamma_\lambda^2 \leq 1 + \frac{\bar{h}}{\underline{\sigma}(R_{\lambda,i})}, \tag{11.3.21}$$

Note that for $\lambda = 1$, we have $\underline{\sigma}(R_{\lambda,i}) \to \infty$ so that we are reduced to the bound $\gamma = 1$ of $H^\infty$ output prediction adaptive filtering (that was solved by LMS). Therefore the extra term $\frac{\bar{h}}{\underline{\sigma}(R_{\lambda,i})}$ represents the increase in the $H^\infty$ norm we must incur due to the time-varying nature of the problem.

(iii) There is a similar exponentially-windowed a posteriori solution that can be constructed for filtered errors. However, for brevity we shall not do so here.

### 11.3.2 Finite-Memory Adaptive Filtering

Another approach for dealing with time-variations is the so-called sliding or finite-memory window. In this case one only considers data over a finite window of length $L$. Therefore as each new data is observed, the least recent data point is discarded so that we have a memory of constant size $L$. Due to the fact that old data is discarded, this method has the promise to cope with time variations in the underlying model.

In this framework, the prediction error and disturbance energies are computed as

$$\sum_{j=i-L+1}^{i} |e_{p,j}|^2 \quad \text{and} \quad \sum_{j=i-L+1}^{i} |v_j|^2, \tag{11.3.22}$$

respectively.

The $H^\infty$ finite memory adaptive filtering problem may now be stated as follows.

**Problem 11.3.2 (Finite-Memory Problem)** *Find an $H^\infty$-optimal estimation strategy $w_i = \mathcal{F}(d_0, d_1, \ldots, d_i)$ that achieves*

$$\gamma_L^2 = \inf_{\mathcal{F}} \sup_{w,v \in h^2} \frac{\sum_{j=i-L+1}^{i} |e_{p,j}|^2}{\mu^{-1}|w - w_{-1}|^2 + \sum_{j=i-L+1}^{i} |v_j|^2}, \quad i = 0, 1, \ldots N \tag{11.3.23}$$

*where $|w - w_{-1}|^2 = (w - w_{-1})^T(w - w_{-1})$, and $\mu$ is a positive constant that reflects a priori knowledge as to how close $w$ is to the initial guess $w_{-1}$.*

To give the solution to the above problem we need to define the following finite-memory covariance matrix

$$R_i^L = \sum_{j=i-L+1}^{i} h_j h_j^T. \tag{11.3.24}$$

**Theorem 11.3.2 (Finite-Memory Algorithm)** *Consider the model (11.2.1), and suppose we would like to solve the adaptive problem 11.3.2. Then*

$$\gamma_L^2 \leq \sup_i \frac{\bar{h} + \underline{\sigma}(R_i^L)}{\frac{1}{\mu} + \underline{\sigma}(R_i^L)}, \quad i = 0, 1, \ldots N \tag{11.3.25}$$

*and the central optimal $H^\infty$ estimator is given by the following equations*

- *For downdating*

$$w_{i-1}^d = w_{i-1} + \frac{P_i^d h_{i-L}}{-1 + h_{i-L}^T P_i^d h_{i-L}}(d_{i-L} - h_{i-L}^T w_{i-1}) \tag{11.3.26}$$

*with*

$$(P_i^d)^{-1} = P_i^{-1} - (1 - \gamma_m^{-2}) h_{i-L} h_{i-L}^T.$$

- *For updating*

$$w_i = w_{i-1}^d + \frac{P_i^d h_i}{1 + h_i^T P_i h_i}(d_i - h_i^T w_{i-1}^d) \tag{11.3.27}$$

*with*

$$P_{i+1}^{-1} = (P_i^d)^{-1} + (1 - \gamma_m^{-2}) h_{i+1} h_{i+1}^T.$$

**Proof:** The proof is another application of the a priori $H^\infty$ filtering result of Theorem 3.2.2. Since the proof is almost identical to the proof of Theorem 11.3.1 we shall omit it for brevity.

■

**Remarks:**

(i) Note, once more, that a minimum singular value, here $\underline{\sigma}(R_i^L)$ has entered the bound for $\gamma_L$. Now, however, if $\underline{\sigma}(R_i^L) = 0$, $\gamma_L$ does not become unbounded. [Indeed it is bounded by $\mu\bar{h}$.

(ii) Note, from Theorem 11.3.2, that if $\mu\bar{h} < 1$ then $\gamma_L < 1$, and that if $\mu\bar{h} > 1$ then $\gamma_L > 1$. However, the case $\mu\bar{h} = 1$ deserves special attention since it leads to the following LMS-type finite-memory algorithm.

**Corollary 11.3.1 (Finite Memory LMS)** *Suppose that $\mu\bar{h} = 1$. Then $\gamma_L = 1$, and an $H^\infty$ optimal estimator is given by the following LMS-type algorithm*

$$w_{i-1}^d = w_{i-1} - \mu h_{i-L}(d_{i-L} - h_{i-L}^T w_{i-1}) \tag{11.3.28}$$

*for "downdating", and*

$$w_i = w_{i-1}^d + \mu h_i(d_i - h_i^T w_{i-1}^d) \tag{11.3.29}$$

*for "updating".*

(iii) There is also a similar finite-memory a posteriori solution that can be constructed for filtered errors. However, for brevity we shall not do so here.

### 11.3.3 General Time-Variation

In this section we shall consider a time-varying filter model of the form

$$d_i = h_i^T x_i + v_i, \tag{11.3.30}$$

where $\{d_i\}$ is the observed output sequence, $\{h_i\}$ is the known input vector, $\{x_i\}$ is the unknown time-varying weight vector that we intend to estimate, and $\{v_i\}$

is an unknown disturbance that may include modeling errors. We shall denote by $\hat{x}_i = \mathcal{F}(d_0, d_1, \ldots, d_{i-1})$ the estimate of the weight vector $x_i$ given observations $\{d_j\}$ from time 0 up to and including time $i-1$. The prediction error will therefore be

$$e_{p,i} = h_i^T x_i - h_i^T \hat{x}_i. \tag{11.3.31}$$

Note that since the time variation in the weight vector $x_i$, viz.,

$$\delta x_i = x_{i+1} - x_i, \tag{11.3.32}$$

is *unknown*, we shall consider it as a disturbance. Thus for every choice of estimator $\mathcal{F}$ we will have a transfer operator from the disturbances $\{\mu^{-\frac{1}{2}}(x_0 - \hat{x}_0), \{v_i\}_{i=0}^N, \{\delta x_i\}_{i=0}^N\}$ to the prediction errors $\{e_{p,i}\}_{i=0}^N$, that we shall denote by $\mathcal{T}_{g,N}(\mathcal{F})$. We are thus immediately led to the following problem.

**Problem 11.3.3 (Time-Varying Problem)** *Find an $H^\infty$-optimal estimation strategy $\hat{x}_i = \mathcal{F}(d_0, d_1, \ldots, d_{i-1})$ that minimizes $\|\mathcal{T}_{g,N}\|_\infty$, and obtain the resulting*

$$\gamma_g^2 = \inf_{\mathcal{F}} \ \|\mathcal{T}_{g,N}\|_\infty^2 = \inf_{\mathcal{F}} \sup_{x_0, v, \delta x \in h^2} \frac{\|e_p\|_2^2}{\mu^{-1}|x_0 - \hat{x}_0|^2 + \|v\|_2^2 + q^{-1}\|\delta x\|_2^2} \tag{11.3.33}$$

*where $|x_0 - \hat{x}_0|^2 = (x_0 - \hat{x}_0)^T (x_0 - \hat{x}_0)$, and $\mu$ and $q$ are positive constants that respectively reflect a priori knowledge as to how close $x_0$ is to the initial guess $\hat{x}_0$, and as to how rapid the weight vector $x_i$ varies with time.*

Note that for a filter that varies slowly with time $q$ will typically be very small.

**Theorem 11.3.3 (Time-Varying Algorithm)** *Consider the model (11.3.30), and suppose we wish to minimize the $H^\infty$ norm of the transfer operator $\mathcal{T}_{g,N}$ from the disturbances $\{x_0 - \hat{x}_0, \{v_i, \delta x_i\}_{i=0}^N\}$ to the prediction errors $\{e_{p,i}\}_{i=0}^N$. Then*

$$\gamma_q^2 \leq \sup_i \ \left[q + \frac{1}{h_i^T h_i}\right] h_{i+1}^T h_{i+1}, \quad i = 0, 1 \ldots N \tag{11.3.34}$$

*and the central $H^\infty$-optimal estimator is given by*

$$\hat{x}_{i+1} = \hat{x}_i + \frac{\tilde{P}_i h_i}{1 + h_i^T \tilde{P}_i h_i}(d_i - h_i^T \hat{x}_i), \quad \hat{x}_0 \tag{11.3.35}$$

*where* $\tilde{P}_i^{-1} = P_i^{-1} - \gamma_g^{-2} h_i h_i^T$, *and*

$$P_{i+1} = \left[ P_i^{-1} + (1-\gamma_g^{-2}) h_i h_i^T \right]^{-1} + qI, \quad P_0 = \mu I. \tag{11.3.36}$$

**Proof:** This is another application of the a priori $H^\infty$ filter of Theorem 3.2.2 where now the Krein space model is,

$$F_i = I_n \ , \ G_i = I_n \ , \ H_i = \begin{bmatrix} h_i^T \\ h_i^T \end{bmatrix} \ , \ \Pi_0 = \mu I_n \ , \ Q_i = qI_n \ , \ R_i = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma_g^2 \end{bmatrix}. \tag{11.3.37}$$

This implies that the central solution (assuming a solution of level $\gamma$ exists) is given by (11.3.35), where $\tilde{P}_i^{-1} = P_i^{-1} - \gamma_g^{-2} h_i h_i^T$ and $P_i$ satisfies the recursion

$$P_{i+1} = P_i - P_i \begin{bmatrix} h_i & h_i \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -\gamma_g^2 \end{bmatrix} + \begin{bmatrix} h_i^T \\ h_i^T \end{bmatrix} P_i \begin{bmatrix} h_i & h_i \end{bmatrix} \right\}^{-1} \begin{bmatrix} h_i^T \\ h_i^T \end{bmatrix} P_i, \quad P_0 = \mu I. \tag{11.3.38}$$

Two consecutive applications of the matrix inversion lemma yield the recursions (11.3.36).

Therefore all that remains to be shown is the condition for existence. But according to Theorem 3.2.2 this is given by

$$\tilde{P}_{i+1}^{-1} = P_{i+1}^{-1} - \gamma_g^{-2} h_{i+1} h_{i+1}^T > 0, \quad i = 0, 1, \ldots N. \tag{11.3.39}$$

Now this last condition is equivalent to

$$\begin{aligned} \gamma_g^2 &> h_{i+1}^T P_{i+1} h_{i+1} \\ &= h_{i+1}^T \left[ qI + \left( P_i^{-1} + (1-\gamma_g^{-2}) h_i h_i^T \right)^{-1} \right] h_{i+1} \\ &= q h_{i+1}^T h_{i+1} + h_{i+1}^T \left( \tilde{P}_i^{-1} + h_i h_i^T \right)^{-1} h_{i+1}. \end{aligned}$$

A “sufficient” condition fot the above to happen is that

$$\gamma_g^2 > q h_{i+1}^T h_{i+1} + h_{i+1}^T h_{i+1} \bar{\sigma} \left[ (\tilde{P}_i^{-1} + h_i h_i^T)^{-1} \right]. \tag{11.3.40}$$

But

$$h_i^T(\tilde{P}_i^{-1} + h_i h_i^T)h_i = h_i^T \tilde{P}_i^{-1} h_i + (h_i^T h_i)^2 \leq (h_i^T h_i)^2,$$

since $\tilde{P}_i^{-1} > 0$ (by the assumption of the existence of an $H^\infty$ filter). But the above inequality implies

$$\underline{\sigma}(\tilde{P}_i^{-1} + h_i h_i^T) \geq h_i^T h_i, \qquad (11.3.41)$$

and therefore

$$\bar{\sigma}\left[(\tilde{P}_i^{-1} + h_i h_i^T)^{-1}\right] \leq \frac{1}{h_i^T h_i}. \qquad (11.3.42)$$

This implies that a sufficient condition for the existence of a filter is that

$$\gamma_g^2 > q h_{i+1}^T h_{i+1} + \frac{h_{i+1}^T h_{i+1}}{h_i^T h_i}, \qquad (11.3.43)$$

from which the desired result (11.3.34) follows.

■

## 11.4 Simulation Results

For brevity we shall only describe one typical simulation result here. To this end, consider the model (11.3.30) where the weight vector $x_i$ is now a scalar. To reflect time-variation we chose $\delta x_i = .02$, and to reflect modeling error,

$$v_i = .1 * (h_i x_i)^3 + n_i, \qquad (11.4.1)$$

where $n_i$ is a zero-mean Gaussian random variable with variance $\sigma^2 = .04$. We chose $x_0 = -1$ and considered 100 time samples so that $x_{100} = 1$. We predicted the output of the filter using various $H^\infty$ and $H^2$ adaptive algorithms and computed the prediction error energy for each. The resulting prediction error energies were averaged over 50 independent runs, and the results are given in Tables 11.1 and 11.2. The $H^\infty$ algorithms considered were LMS and the algorithms of Theorems 11.3.1 and 11.3.3, and the $H^2$ algorithms were RLS, exponentially-weighted RLS (denoted by $\lambda$-RLS) and the Kalman filter (denoted by KF). Note that the prediction error energies for the $H^\infty$ algorithms are virtually identical, and that although the exponentially-weighted

RLS algorithm performs significantly better than RLS and the Kalman filter, it does not perform as well as the $H^\infty$ algorithms. (The parameters used in this simulation were $\mu = .9$, $\lambda = .9$ and $q = .0004$.)

| | LMS | Thm. 11.3.1 | Thm. 11.3.3 |
|---|---|---|---|
| $\sum_{j=0}^{100} \lvert e_j\rvert^2$ | 1.48 | 1.54 | 1.48 |

Table 11.1: The $H^\infty$ algorithms.

| | RLS | $\lambda$-RLS | KF |
|---|---|---|---|
| $\sum_{j=0}^{100} \lvert e_j\rvert^2$ | 6.00 | 2.08 | 5.11 |

Table 11.2: The $H^2$ algorithms.

## 11.5 Conclusion

We close this chapter, and our study of $H^\infty$ adaptive filtering, with a brief remark on the general relevance of the $H^\infty$ approach to estimation.

It is important to note that if one has a priori knowledge of the underlying statistics and distributions of the signals, one is always best served by considering algorithms that are specifically tuned for the situation at hand. On the other hand, if one does not have such a priori knowledge and uses an algorithm that makes specific assumptions about the disturbances, then the algorithm may perform poorly if these assumptions are not met. $H^\infty$ optimal algorithms will therefore be most applicable in uncertain environments where there may be modeling errors, and where the statistics and/or distributions of the disturbances are not known (or are too expensive to obtain).

# Chapter 12

# Conclusions and Future Work

Having reached the end of a dissertation of such length, it is probably best to conclude with some brief remarks on various directions for future research that are suggested by the methods and results presented in this thesis. For convenience, we have divided the discussion into two parts — the first deals with extensions of some of the results studied in the earlier chapters, and the second deals with what we believe is a very important and promising approach to estimation and control, namely the mixed $H^2/H^\infty$ approach.

## 12.1 Various Extensions

We begin with various extensions suggested by the studies and results of the earlier chapters.

### Continuous-time Krein Space Theory

Perhaps the first direction for future work that is suggested by this thesis is the development of a continuous-time counterpart to the discrete-time Krein space estimation theory given in Chapter 2. Although the author, and his coworkers, have not yet fully embarked on such an endeavor, preliminary investigations suggest that this should be possible without too much difficulty. Here the Krein space projections will be related to stationarizing certain indefinite quadratic integral costs, and the conditions for a minimizing solution will be given in terms of the positivity of certain integral operators.[1] When one has state-space structure, the invertibility (and positivity) of the integral operators can be checked via the existence (and properties) of solutions to certain Riccati differential equations. Moreover, the projections can be computed via the continuous-time Krein space Kalman filter, a natural generalization of the conventional continuous-time Kalman filter. We also remark that such studies should have ramifications to continuous-time $H^\infty$ estimation and control, and to quadratic differential games, as well as other areas.

## Suboptimal Recursive Total Least-Squares Algorithms

We mentioned in Chapter 4 that another application of Krein space estimation is in the development of recursive algorithms for suboptimal total least-squares problems. Preliminary studies towards this goal have been performed in [SHK96a], where it is shown that the total least-squares problem (or errors-in-variables method) can be reformulated as

$$\min_x \left[-\bar{\sigma}_{n+1}^2 + (b - Ax)^*(b - Ax)\right],$$

where $\bar{\sigma}_{n+1}^2$, $b$ and $A$ are known quantities that need not concern us here. The important fact is that the above problem is an indefinite quadratic minimization problem that is amenable to Krein space methods. Although it is very unlikely to be possible to give recursive solutions to the exact total least-squares problem (since the solution involves a singular value decomposition which cannot be performed recursively), the Krein space theory may suggest certain suboptimal variants of this exact solution that will allow for efficient recursive computations. Moreover, we believe that there are probably other areas of least-squares theory that may benefit from the Krein space approach.

[1] Of course, in dealing with integral operators, one must take care as to whether these operators are invertible, bounded, etc. However, the condition for the existence and uniqueness of continuous-time Krein space projections is that the corresponding integral operators be invertible (very much like the discrete-time case where the associated indefinite Gramian had to be invertible).

## Numerical Analysis of $H^\infty$ Square-Root and Chandrasekhar Algorithms

As mentioned in Chapter 5, the conventional square-root array algorithms are preferred because of their numerical stability, which is largely due to the fact that the dynamic range of the variables are reduced (thus also reducing the condition numbers), and because the computations involve unitary transformations that are well-known to not amplify numerical (or roundoff) errors. Since the $H^\infty$ square-root array algorithms (and the $H^\infty$ Chandrasekhar recursions) are the direct analogs of their conventional counterparts, it seems plausible that they may be more attractive for the numerical implementation of $H^\infty$ filters (and controllers). However, since $J$-unitary, rather than unitary, operations are involved, further investigation is needed to determine what the numerical performances of these algorithms are, and what the best way to implement the $J$-unitary transformations is. This should be an area of much practical significance.

## Further Study of the Indefinite DARE

In Chapter 7 we studied the discrete-time algebraic Riccati equation (DARE) with possibly indefinite coefficient matrices. We should mention that similar results can be obtained for the continuous-time algebraic Riccati equation (CARE), and indeed that it is quite easier to do so for the CARE than it is for the DARE.[2] Now in Chapter 7 the existence of solutions to the DARE were shown to be equivalent to the existence of certain proper factorizations of the associated Popov functions, and also equivalent to certain properties of the invariant subspaces of the associated Hamiltonian matrices. However, more explicit conditions on the Popov function for the existence of such factorizations would be desirable, and further research is necessary to see whether this can be done. More generally, some basic questions on the factorization of para-Hermitian rational transfer matrices, such as the existence of canonical, but possibly nonproper, factorizations, $J$-spectral factorizations, connections to algebraic Riccati

[2] In fact, the continuous-time algebraic Riccati equation is much easier to analyze than its discrete-time counterpart and, consequently, significantly more is known about the CARE than the DARE.

equations, or possibly some other form of algebraic equations, are open and worthy of further scrutiny.

We should also mention that the approach of Chapter 7 can be used to study the existence of solutions to algebraic Riccati equations with non-Hermitian coefficient matrices. In this framework, it can be shown that solutions to such equations exist if, and only if, certain proper factorizations of the associated Popov function (which is now a non-para-Hermitian rational transfer matrix) exist. However, it is not clear to this author, at this time, what benefit such an investigation might entail.

## Further Study of the Asymptotic Behaviour of the $H^\infty$ Riccati Recursion

Chapter 8 studied the asymptotic behaviour of the Riccati recursion with possibly indefinite coefficient matrices. The main result was that, if certain inertia conditions are satisfied at all time time instants, then the solution to the Riccati recursion (exponentially) converges to the stabilizing solution of the associated DARE (assuming such a solution exists). Although the aforementioned inertia conditions need to be recursively checked, in the special case where the coefficient matrices of the Riccati recursion are positive semi-definite, they can be replaced by a single inertia condition, which results in a very explicit requirement on the initial condition. It would therefore be interesting to study whether it is possible to find other special cases where one can replace the (infinitely many) recursive inertia conditions, required for convergence, by a single inertia condition. In particular, it would be beneficial if one could do so for the special case that arises in $H^\infty$ estimation and control.

## $H^\infty$ Adaptive Filtering

Chapter 11 provided a preliminary study of the use of the $H^\infty$ criterion in adaptive filtering. We believe that further study of this approach, and especially issues such as the relationship between robustness and convergence and robustness and tracking (of time-variations), has great merit and may bear considerable fruit. Since its objectives are fully compatible with those of adaptive filtering, the $H^\infty$ approach may, in the long run, compete with least-squares approaches to adaptive filtering in terms of their influence on the field.

## 12.2 Mixed $H^2/H^\infty$ Estimation and Control

$H^\infty$ optimal algorithms (be it in estimation, control, or signal processing) are most applicable in situations where the exact models and statistics of the underlying disturbances are not available (or are too expensive to obtain). Since they make no assumption about the disturbances, they have to accommodate for all conceivable disturbances, and are thus over-conservative. In many applications one may have some notion of what the statistics of the signals may be, and so one would somehow like to incorporate that knowledge while still guaranteeing the robustness of the algorithm. The mixed $H^2/H^\infty$ formulation to estimation and control is an attempt in this direction.

To illustrate and motivate the approach, consider the adaptive filtering problem of Chapter 9 where the goal was to design an adaptive filtering algorithm for output prediction. Recall, in that case, that the $H^2$ and $H^\infty$ optimal solutions were given by the RLS and LMS algorithms, respectively. Moreover, let us denote the transfer operators that map the disturbances to prediction errors for each of these algorithms by $\mathcal{T}_{p,rls}$ and $\mathcal{T}_{p,lms}$, respectively. Since RLS and LMS are linear filters, (for finite horizon problems) $\mathcal{T}_{p,rls}$ and $\mathcal{T}_{p,lms}$ will be finite matrices.

Fig. 12.2 shows the (squared) singular values of $\mathcal{T}_{p,rls}$ and $\mathcal{T}_{p,lms}$ for $N = 50$ (where $N$ is the number of observed data points) and $\mu = .9$, for a simple one-dimensional (*i.e.*, single-tap) adaptive filtering problem. As can be seen the (squared) maximum singular value (which is just the maximum energy gain or (squared) $H^\infty$ norm) for $\mathcal{T}_{p,lms}$ is unity, whereas for $\mathcal{T}_{p,rls}$ it is much larger. On the other hand, (under appropriate statistical assumptions) the RLS algorithm has the best average performance (which can be represented as the sum of the squared singular values and is just the area under the singular value curve), whereas the average performance of LMS is significantly worse. Thus the RLS algorithm will have better average performance than LMS, although its worst-case performance is significantly worse.

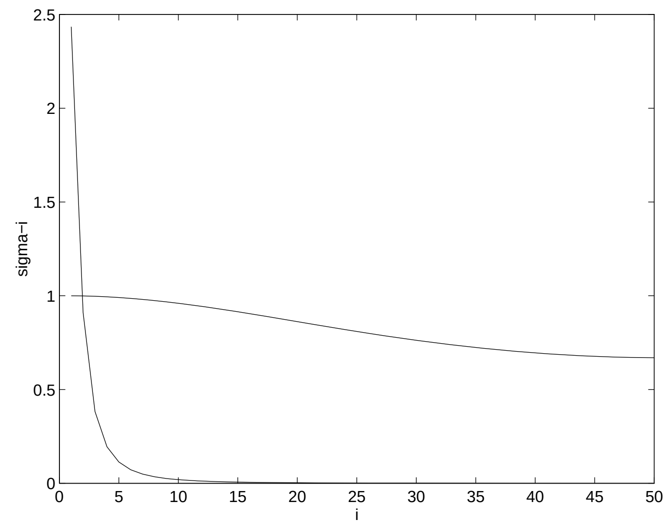


Figure 12.1: Singular values for $\mathcal{T}_{p,rls}$ and $\mathcal{T}_{p,lms}$ for $N = 50$ and $\mu = .9$.

Note, moreover, that although the LMS algorithm does not allow any amplification of the disturbances, it does not provide significant suppression of the disturbances, either. (The smallest squared singular value for $\mathcal{T}_{p,lms}$ which represents the minimum energy gain is roughly 0.65.) Since the $H^\infty$ optimal filters are not unique (LMS is only the central solution), it is very interesting to study the possibility of choosing other $H^\infty$ optimal filters to further reduce the sum of the (squared) singular values of the transfer operator. This will result in algorithms that have the best possible average behaviour, while at the same time having the best possible worst-case performance. This is the essence of the mixed $H^2/H^\infty$ framework.

## Problem Formulation

We should note that classical methods in estimation theory (such as least-squares, maximum-likelihood, and maximum entropy) and the more recent robust methods in estimation theory (such as $H^\infty$) can be regarded as two extremes in terms of their requirements regarding the statistical properties of the exogenous signals, as well as in terms of their goals. In classical estimation methods optimality of the average (or expected) performance of the estimator under some assumptions regarding the statistical nature of the signals is the key issue and hence their performance heavily depends upon the validity of these assumptions. On the other hand, robust estimation methods, or so-called minimax estimation strategies, safeguard against the worst-case disturbances and therefore make no assumptions on the (statistical) nature of the signals.

The mixed estimation (control) problem was introduced as a compromise between these two extreme point of views [BH89, YBC92, KR92, LA94, ZGBD94a, ZGBD94b, Meg94, FFT94, FFL95, HHK96]. The mixed $H^2/H^\infty$ problem allows one to trade off between the best average performance of the $H^2$ estimator (controller) and the best guaranteed worst-case performance of the $H^\infty$ estimator (controller). As a result, the optimal mixed $H^2/H^\infty$ estimators (controllers) achieve the best average performance, not over the set of all estimators (controllers), but over a restricted set of estimators (controllers) that achieve a certain worst case performance bound. We note that the suboptimal (and even optimal) $H^\infty$ estimators (controllers) are highly non-unique and the mixed $H^2/H^\infty$ approach attempts to exploit this non-uniqueness by choosing the one that has the best average performance. Unlike optimal $H^2$ and suboptimal $H^\infty$ problems, the question of finding the optimal mixed solution is still open.

Recall from Chapter 1 that the objectives in $H^2$ estimation and control are to solve the following problems,

$$\min_{\text{causal } \mathcal{K}} \|\mathcal{T}_{\mathcal{K}}\|_2 \quad \text{and} \quad \min_{\text{causal } \mathcal{K}} \|\mathcal{T}^c_{\mathcal{K}}\|_2 \tag{12.2.1}$$

where $\mathcal{T}_{\mathcal{K}}$ is the transfer operator from the disturbances to the estimation errors, $\mathcal{T}^c_{\mathcal{K}}$ is the transfer operator from the disturbances to the regulated and control signals, and $\mathcal{K}$ is, depending on the problem, the causal estimator or controller.

Likewise, in (suboptimal) $H^\infty$ estimation and control, for a given $\gamma > 0$, we are required to find causal estimator and controllers that achieve

$$\|\mathcal{T}_{\mathcal{K}}\|_\infty < \gamma \quad \text{and} \quad \|\mathcal{T}^c_{\mathcal{K}}\|_\infty < \gamma. \tag{12.2.2}$$

In view of the above arguments, the mixed $H^2/H^\infty$ problems can be formulated as follows,

$$\left\{ \begin{array}{c} \min_{\text{causal } \mathcal{K}} \|\mathcal{T}_{\mathcal{K}}\|_2 \\ \text{subject to } \|\mathcal{T}_{\mathcal{K}}\|_\infty < \gamma \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{c} \min_{\text{causal } \mathcal{K}} \|\mathcal{T}^c_{\mathcal{K}}\|_2 \\ \text{subject to } \|\mathcal{T}^c_{\mathcal{K}}\|_\infty < \gamma \end{array} \right. \tag{12.2.3}$$

Recall that $\mathcal{T}_\mathcal{K}$ and $\mathcal{T}^c_\mathcal{K}$ are afine in $\mathcal{K}$. For example, in estimation

$$\mathcal{T}_\mathcal{K} = \left[ \begin{array}{cc} \mathcal{L} - \mathcal{K}\mathcal{H} & -\mathcal{K} \end{array} \right], \tag{12.2.4}$$

where $\mathcal{L}$ and $\mathcal{H}$ are known.

The mixed $H^2/H^\infty$ problem (12.2.3) has turned out to be surprisingly difficult, and satisfactory solutions are not yet known. Some recent results include [BH89, YBC92, KR92, LA94, ZGBD94a, ZGBD94b, Meg94, FFT94, FFL95, HHK96]. Some preliminary studies show that there may be some merit in departing from linear estimators and controllers in (12.2.3), and instead trying to solve the mixed problem with (possibly) nonlinear ones [HK96].

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